# ÜBUNGEN ZUR VORLESUNG TOPOLOGIE II

Aufgabenblatt 4
Abgabe: Mittwoch, 13.5.2009 in der Vorlesung.

**Exercise 4.1.** Give an example of two chain complexes $C_\bullet$ and $B_\bullet$ and a chain map $C_\bullet \to B_\bullet$ which is a quasiisomorphism but not a chain homotopy equivalence.

**Exercise 4.2.** Let $R$ be a commutative ring and let $C_\bullet, B_\bullet$ be two chain complexes of $R$-modules; the differentials of both are denoted by the symbol $\partial$. We define a new chain complex $\mathrm{Hom}(C_\bullet, B_\bullet)_\bullet$ of $R$-modules. The $n$th group is $\mathrm{Hom}(C_\bullet, B_\bullet)_n := \prod_{m\in\mathbb{Z}} \mathrm{Hom}_R(C_m, B_{m+n})$ (here $\mathrm{Hom}_R(C_m, B_{m+n})$ is the $R$-module of $R$-linear maps; this is the same as $\mathsf{Mor}_{R-\mathbf{Mod}}(C_m, B_{m+n})$). The differential is defined by ($f \in \mathrm{Hom}(C_\bullet, B_\bullet)_n$) $D_n(f) := \partial\circ f - (-1)^n f\circ\partial$. Show the following statements:

(1) $\mathrm{Hom}(C_\bullet; B_\bullet)_\bullet$, with the differential $D$, is a chain complex of $R$-modules.
(2) A 0-cycle of $\mathrm{Hom}(C_\bullet, B_\bullet)$ is a chain map and two 0-cycles are homologous if and only if they are chain homotopic. The zeroeth homology $H_0(\mathrm{Hom}(C_\bullet, B_\bullet))$ is isomorphic to the group of chain homotopy classes of chain maps from $C$ to $B$.

**Exercise 4.3.** (the Hurewicz homomorphism) Let $X$ be a path-connected space and $x \in X$ be a basepoint. In this exercise, a homomorphism $h : \pi_1(X, x) \to H_1(X)$ is constructed. Let $c : [0, 1] \to X$ be a path. We can interprete $c$ as a singular 1-simplex in $X$. This defines a map $F : \{c : [0, 1] \to X\} \to C_1(X)$. Show:

(1) If $c$ is closed, then $F(c)$ is a cycle. If $c$ is constant, then $F(c)$ is the boundary of a singular 2-chain.
(2) Let $c^{-1}$ be the path $c$ with the opposite direction. Then $F(c^{-1})+F(c)$ is null-homologous.
(3) Let $c, d$ be two path with $c(1) = d(0)$ and let $c * d$ be the composition of the two paths. Then $F(c * d) - F(c) - F(d)$ is nullhomologous.
(4) Let $c, d$ be two paths with $c(0) = d(0)$ and $c(1) = d(1)$ and assume that $c$ and $d$ are homotopic relative to the endpoints. Then $F(c) - F(d)$ is nullhomologous.
(5) For a closed path $c$ based at $x$, let $[[c]] \in \pi_1(X, x)$ be its homotopy class. The map $[[c]] \mapsto [F(c)]$ is a well defined homomorphism $h : \pi_1(X, x) \to H_1(X)$, the *Hurewicz homomorphism*. Show that the homomorphisms $h$ assemble to a natural transformation of covariant functors $\mathbf{Top}_* \to \mathbf{Gr}$ from the category of pointed spaces to the category of groups and homomorphisms.
(6) Show, by an example, that $h$ is not an isomorphism in general.

**Exercise 4.4.** Let $R$ be a field and $C_\bullet$ be a chain complex of $R$-vector spaces. Let $H_\bullet$ be the chain complex whose $n$th module is $H_n(C_\bullet)$ and all of whose differentials are zero. Show: $C_\bullet$ and $H_\bullet$ are chain homotopy equivalent. Hint: any vector space has a basis.

**Exercise 4.5.** Let $\mathcal{I}$ be a small category and let $C : \mathcal{I} \to \mathbf{Ch}$ be a functor. Show that the colimit $\mathrm{colim}_\mathcal{I} C$ exists. Hint: show first that colimits of $R$-modules exist; furthermore the $n$th group of $\mathrm{colim}_\mathcal{I} C$ will be the colimit of the $n$th groups of $C$. Show that there is a natural homomorphism $\Phi : \mathrm{colim}_\mathcal{I} H_*(C) \to H_*(\mathrm{colim}_\mathcal{I} C)$.

We say that $\mathcal{I}$ is *directed* if the following conditions are satisfied:

- For $i, j \in \mathsf{Ob}(\mathcal{I})$, there exists at most one morphism from $i$ to $j$.
- For any two objects $i$ and $j$ of $\mathcal{I}$, there exists another object $k$ and morphisms $i \to k$, $j \to k$.

(The standard example of a directed small category is the totally ordered set $\mathbb{N}$: The objects are natural numbers and there exists a unique morphism from $n$ to $m$ if and only if $n \leq m$; the composition of morphisms is determined by these properties.)

Now assume that $\mathcal{I}$ is a directed small category and $C : \mathcal{I} \to \mathbf{Ch}$ a functor. Show that the homomorphism $\Phi : \operatorname{colim}_{\mathcal{I}} H_*(C) \to H_*(\operatorname{colim}_{\mathcal{I}} C)$ is an isomorphism.