OBERSEMINAR WINTER 2014/15: MODULI SPACES OF HIGH-DIMENSIONAL MANIFOLDS, AFTER GALATIUS AND RANDAL-WILLIAMS

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The goal of this seminar is to understand the recent work [2] by Galatius and Randal-Williams which is described in the sequel. Let $W_{g,1} := \sharp^g (S^n \times S^n) \setminus D^{2n}$ be the g-fold connected sum of $S^n \times S^n$, minus a disc. Let $\mathrm{Diff}_\partial(W_{g,1})$ be the topological group of all diffeomorphisms which fix the boundary and $B\mathrm{Diff}_\partial(W_{g,1})$ be its classifying space. There are maps $B\mathrm{Diff}_\partial(W_{g,1}) \to B\mathrm{Diff}_\partial(W_{g+1,1})$ given by extending the diffeomorphisms. The goal of the theory is to understand the homology of

$$\mathrm{hocolim}_{g\to\infty} B\mathrm{Diff}_\partial(W_{g,1}).$$

The answer is given in terms of stable homotopy theory. Let $\Theta : BO(2n)\langle n\rangle \to BO(2n)$ be the n-connective cover and $\gamma_{2n} \to BO(2n)$ be the universal vector bundle. The Madsen-Tillmann-Weiss spectrum $\mathrm{MT}\Theta(2n)$ is by definition the Thom spectrum of the additive inverse $-\Theta^*\gamma_{2n}$. There is a map

$$\alpha : \mathrm{hocolim}_{g\to\infty} B\mathrm{Diff}_\partial(W_{g,1}) \to \Omega^\infty_0 \mathrm{MT}\Theta(2n)$$

obtained from a Pontrjagin-Thom type construction. The main result is

Theorem 1. (Galatius, Randal-Williams [2]) If $n \geq 3$, then α is a homology equivalence.

Theorem 1 is also true if $n = 1$ and even $n = 0$ (correctly interpreted). The $n = 0$ case is the classical Barratt-Priddy-Quillen-Segal theorem, which asserts that a certain map

$$\mathbb{Z} \times B\Sigma_\infty \to \Omega^\infty \mathbb{S}^\infty$$

is a homology equivalence. The $n = 1$ case was first proven by Madsen and Weiss [6] and proves a conjecture by Mumford on the homology of the moduli space of Riemann surface. In fact, the proof by Madsen and Weiss initiated the development of mathematical ideas that culminates in Theorem 1. In this seminar, we will concentrate on the high-dimensional case (even though we will prove almost everything needed for the low-dimensional cases).

The proof of Theorem 1 will be in the framework of cobordism categories and many steps work for manifolds of arbitrary (also odd and low) dimension and more general tangential structures. Let $\theta : B \to BO(d)$ be a map; one defines the spectrum $\mathrm{MT}\theta(d)$ in a similar way as above. Let $\mathcal{C}_\theta(\mathbb{R}^N)$ be the cobordism category of $d-1$-dimensional θ submanifolds of $\mathbb{R}^{N-1}$ and cobordisms in $\mathbb{R}^N$ (a θ-manifold is a manifold together with a lift of its Gauss map to B). By passing to the colimit $N \to \infty$, one obtains the cobordism category $\mathcal{C}_\theta$ of θ-manifolds. The cobordism category has a classifying space $B\mathcal{C}_\theta$, and in the first part of the seminar, we will discuss the proof of the following result (which by now is almost classical).

Theorem 2. (Galatius, Madsen, Tillmann, Weiss [3]) There is a homotopy equivalence $B\mathcal{C}_\theta \simeq \Omega^{\infty-1}\mathrm{MT}\theta(d)$.

We will not follow the original proof, but a newer and simpler one which was given by Galatius and Randal-Williams [1]. Let $\Psi^\theta_d(\mathbb{R}^N)$ be the space of all d-dimensional θ-manifolds inside $\mathbb{R}^N$. The topology on this space is quite intricate and is defined in talk 2. Let $\psi^\theta_d(N,k)$ be the subspace of manifolds which are contained in $(-1,1)^k \times \mathbb{R}^{N-k}$. The global structure of the proof of Theorem 2 will be as follows:

$$B\mathcal{C}_\theta(\mathbb{R}^N) \overset{Talk3}{\simeq} \psi_\theta(N,1) \overset{Talk4}{\simeq} \Omega^{N-1}\psi_\theta(N,N) \overset{Talk2}{\simeq} \Omega^{N-1}\mathrm{MT}\theta(d)_N.$$

Theorem 2 follows by passing to the colimit $N \to \infty$.

For each closed θ-manifold M, there is a tautological map $B\mathrm{Diff}^\theta(M) \to \Omega B\mathcal{C}_\theta$ and Theorem 1 states that an appropriate stabilized version of this map is a homology isomorphism, under suitable conditions. In order to make the comparison between diffeomorphism groups and cobordism categories work, one needs to show that certain subcategories of $\mathcal{C}_\theta$ have have the same (up to homotopy) classifying space as $\mathcal{C}_\theta$ itself.

Namely, one considers the subcategories $\mathcal{C}_\theta^{\kappa,l} \subset \mathcal{C}_\theta$. The letter κ indicates that each morphism is κ-connected with respect to its outgoing boundary, and the letter l indicates that the structural map $M \to B$ given by the θ-structure on M is l-connected. There are inclusions

$$\mathcal{C}_\theta^{\kappa,-1} \subset \mathcal{C}_\theta^{-1,-1} = \mathcal{C}_\theta;\ \mathcal{C}_\theta^{\kappa,l} \subset \mathcal{C}_\theta^{\kappa,-1}$$

and for κ and l in an appropriate range, we have

$$B\mathcal{C}_\theta^{\kappa,l} \simeq B\mathcal{C}_\theta^{\kappa,-1} \overset{Talks5,6,7}{\simeq} B\mathcal{C}_\theta^{-1,-1} = B\mathcal{C}_\theta. \tag{1}$$

Both results are proven by a surgery method, and one can in fact view both homotopy equivalences as a parametrized version of the classical result on surgery below the middle dimension. The proof of the first equivalence will not be discussed in detail in the seminar; the proof is similar to that for the second equivalence and the next step, which is the key of the whole argument and involves surgery in the middle dimension. This will require that $d = 2n$ is even, that the manifolds are high-dimensional ($d \geq 6$) and that θ has a special property ("spherical"). By the above results, one ontains that $B\mathcal{C}_\theta^{n-1,n-2} \simeq B\mathcal{C}_\theta$. Let $\mathcal{A}$ be a set of objects in the cobordism category $\mathcal{C}_\theta^{n-1,n-2}$ such that each object in $\mathcal{C}_\theta^{n-1,n-2}$ is cobordant to one object in $\mathcal{A}$. Under all these assumptions, there is homotopy equivalence

$$B\mathcal{C}_\theta^{n-1,\mathcal{A}} \overset{Talks8and9}{\simeq} B\mathcal{C}_\theta^{n-1,n-2} \simeq \Omega^{\infty-1}\mathrm{MT}\theta(2n).$$

An appropriate choice of θ, $\mathcal{A}$ and an application of the "group-completion" theorem will finish the proof of Theorem 1, in talk 10. If we find enough speakers, we discuss in the last talks 11 and 11 a generalization of Theorem 1.

An important advice to the speakers. The paper [2] is both highly technical and marvellously written in the sense that every necessary detail is spelled out (the same applies, to a lesser extent, to [1]). Nevertheless, preparing the talks will be a special challenge quite different from other advanced seminar topics, for a couple of reasons.

(1) One characteristics of these papers is that they mainly use fairly simple-minded tools (knowing the transversality theorem, the isotopy extension theorem, the homotopy excision theorem and basic obstruction theory suffices to understand 80 % of the argument). But they use these results in a very special and very original way. Speakers will have few opportunities to hide behind the quotation of hard results. In order to convey the way of thinking of the authors, you will have to familiarize yourself with the general type of arguments used in the papers. The best advice I can give is to read [2], §2, §6.2 and one of the chapters §3–5, together with the relevant part of §6, as a whole.
(2) Please read and take serious Bill Thurston's answer to a mathoverflow question. For the preparation of the talks, you will have to translate the formal language in [2] into geometric intuition in order to communicate the argument to the audience. Giving the talks by verbally repeating parts of [2] will almost surely result in an embarrassment (Talk 10 might be an exception).
(3) The papers [1] and [2] spell out the arguments in great detail. This makes the papers relatively easy to read as a whole, but not locally. Each talk will have a number of pages to cover, and it is important to isolate the key ideas, in every lemma and proposition of

the paper and you will have to condense the material considerably (as opposed to working out details).

(4) The published version of [2] is 120 pages long (in large font). It is unreasonable to expect than we can cover the whole material in one semester. You will have to make shortcuts, but do it in a meaningful way. For example: it is not a good idea to spell out all definitions in say §4.3 [2] in detail! To each talk description, I indicated which aspects should be emphasized more than others.

As a corollary, *early* preparation of the talks will be *crucial*. I will feel responsible to coach speakers two or three weeks ahead of the talk, but certainly not Friday afternoon before the talk.

1. Overview and background

Talk 1. (Overview and a non-technical introduction, Johannes Ebert) Cobordism categories, diffeomorphism groups. Semisimplicial spaces and classifying spaces of categories. The Pontrjagin-Thom construction. Thom spectra. Statement of the main theorems. Cohomology of the infinite loop space, characteristic classes of manifold bundles. Failure in odd dimensions. Overview of the proof.

2. The Galatius-Madsen-Tillmann-Weiss theorem

Talk 2. (Spaces of manifolds, Paul Bubenzer) In this talk, the space of d-dimensional submanifolds $\Psi_d(U)$ of a manifold U is introduced. The basic reference for this material is [1], §2. Define the topology on this space, but do not attempt to prove everything in detail. The important statements are the "Theorems" in §2 loc. cit. Also, discuss the version with tangential structures $\Psi_\theta(U)$. The main player of the next couple of talks is the subspace $\psi_\theta(n,k) \subset \Psi_\theta(\mathbb{R}^n)$ of manifolds which are contained in $\mathbb{R}^k \times (-1,1)^{N-k}$ ("noncompact in k directions"), see Definition 3.5 of [1]. The main result of this talk is Theorem 3.22 of [1], which identifies the space of all submanifolds $\psi_\theta(n,n)$ with a suitable Thom space. Moreover, you should discuss Proposition 2.16 of [2] which gives a good example explaining how one works with the topology on the space of all manifolds. Hint: in the context of this seminar, the properties of the topology and the way one works with it are much more important than the verification of these properties. Therefore, the focus should definitely be on the proof of Theorem 3.22 [1].

Talk 3. (The homotopy type of the cobordism category, Arthur Bartels) First, you should introduce the embedded cobordism category [1], §3.2 and mention the connection to classifying spaces of diffeomorphism groups as in the introduction to [2]. The main results of this talk are Theorems 3.9 and 3.10 [1], and you should cover the proofs of these results. Then mention Theorem 3.13 (whose proof is given in the next talk) and show how these results together prove Theorem 2. Hint: What is really needed is a subcategory $\mathcal{C}_{\theta,L} \subset \mathcal{C}_\theta$, see [2], §2.6 and 2.7 and this forces some modifications to the proof. The ambitious speaker should try to include these modifications. Theorem 3.10 is much more important than Theorem 3.9. As an exercise, try to prove it using Theorem 6.2 of [2]. Probably the speaker of this talk will feel less time-pressure than the other speakers. If you do not need the full 90 minutes to convey the above results, a good use for the extra time would be to introduce some of the tools about (semi-)simplicial spaces which are often used in the following talks. One is Lemma 2.4 [2], another one is Lemma 3.14 [1] (important for the next talk), both originally due to Segal [8]. Please resist the temptation to discuss anything related to higher category in this talk.

Talk 4. (The scanning principle, Ulrich Pennig) In this talk, the proof of the main result of [3] is completed, following the alternative proof [1] instead. The main result is Theorem 3.13 of [1]. It states that a certain very geometric map $\psi_\theta(N,k) \to \Omega\psi_\theta(N,k+1)$ is a weak homotopy equivalence if $k \geq 1$. This makes essential use of the fact that the components $\pi_0(\psi_\theta(N,k))$ form an abelian group ($1 \leq k \leq N-1$), Proposition 3.6 and Corollary 3.11 loc. cit., so give at least the idea how these groups are expressed as suitable cobordism groups.

3. Surgery in cobordism categories

Talk 5. (Surgery below the middle dimension I, Michael Joachim) Begin by stating Theorems 3.1 and 4.1 [2]. The proof of Theorem 3.1 is similar in spirit to that of Theorem 4.1 and won't be discussed. Try to give the main idea of the proof (of 4.1), as explained in the introduction to §4 [2]. Then discuss tangential surgery [2], §4.2 and the "standard family" §4.3 [2].

Talk 6. (Surgery below the middle dimension II, Michael Weiss) Define the space of surgery data §4.3 [2]. State Theorem 4.5 (which is proven in the next talk) and give the proof of Theorem 4.1 (§4.4). Hint: the proof of Theorem 4.1 requires a more flexible model for the cobordism category, which is introduced in §2.8.

Talk 7. (Surgery below the middle dimension III, Christoph Winges) This talk proves Theorem 4.5 [2] and thereby completes the proof of 4.1. References: §6.1, 6.2 6.4. Hint: Theorem 6.2 is the main tool and will be used once more, so it needs to be carefully stated. The main point of the proof of 4.5 is Proposition 6.18 and this is completely parallel to the classical "surgery below the middle dimension". This parallel should become clear!

Talk 8. (Surgery in the middle dimension I, Federico Cantero) This talk should cover §5 [2]. State Theorem 5.3 carefully. The argument for this is largely parallel to that for Theorem 4.1. As we saw the general pattern, the focus in this talk should be on the differences. Give an informal discussion along the lines of §5, introduction. Then introduce spherical tangential structures §5.1. Once the standard family is introduced, the construction of the surgery data and the proof of the theorem is completely parallel to talk 6.

Talk 9. (Surgery in the middle dimension II, William Gollinger) The argument will be completed by proving that the space of surgery data is contractible, [2] §6.5. Here the Whitney trick and the even-dimensionality plays an important role. Hints: again, many arguments are similar to those of talk 7 and the emphasis should be on the differences. The key result of the whole story is Lemma 6.21. The result is similar to a result of Kreck [5], p. 722, and this should also be discussed. Not too much is lost if you restrict to the case $\pi_1(B) = 1$, and this simplifies the argument considerably. In the proof of Lemma 6.12, a reference to the "proof of the h-cobordism theorem" is made. The relevant result is Theorem VIII.4.1 in [4].

4. The group completion argument

The last two talks are optional. In case we find enough speakers; I might change the seminar program.

Talk 10. (Group completion in the simplest case, Martin Palmer) This talk gives the proof of the main result, [2], §7.1. Before, recall the group completion theorem [7]. It is a good idea to give the proof of the $n = 0$ case (this is an exercise for the speaker) as a motivation, and also to discuss the 2-dimensional case [3], §7. Hint: in the present context, the most useful expression of the group completion theorem is *not* the statement of Proposition 1 [7]. Consult [3], §7 to see how the results by Mc Duff and Segal are used.

Talk 11. (Group completion argument - general case I, optional) §7.2, §7.3. The main result is the characterization and construction of "universal ends".

Talk 12. (Group completion argument - general case II, optional) Here the proof of the main results is completed. This involves the universal ends from the previous talk, the group completion theorem and Moore-Postnikov towers. §7.4, 7.5. To conclude the whole seminar, you might want to discuss some examples.

References

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