SEMINAR: MAPPING CLASS GROUPS, HOMOLOGICAL STABLITY AND MADSEN-WEISS THEORY

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The goal of this seminar is to understand the Madsen-Weiss theorem:

Theorem 1 (Madsen-Weiss [19])**.** *Let $S_{g,r}$ be a connected compact surface of genus g with r boundary components. There is a map*

$$B\mathrm{Diff}_\partial(F) \to \Omega^\infty{}_0\mathrm{MTSO}(2)$$

which induces an isomorphism on integral homology groups in degrees $\leq \frac{2}{3}g$.

Talk 1 (Classifying spaces of fibre bundles and groups, JE, 18.10.)**.** ...

Topology of surfaces.

Talk 2 (Surfaces, mapping class groups and surface bundles, Georg Frenck, 25.10.)**.** Sketch of the classification of surfaces up to diffeomorphism using Morse Theory [9], including the case with boundary. Diffeomorphism group $\mathrm{Diff}_\partial(F)$ of a surface as a topological group. Mapping class groups $\Gamma_{g,r} := \pi_0(\mathrm{Diff}_\partial(S_{g,r})$. Statement of the Dehn-Nielsen-Baer theorem [8, §8]. Also consult [10], [2].

Surface bundles, vertical tangent bundle [4, §I.1,I.2,I.3]. At this point, you can define the Miller-Morita-Mumford classes κ_i as in [17, §I.3] or [2, §4.2] and state the Mumford conjecture.

There is a deep connection to complex algebraic geometry via Teichmüller theory, as explained in [2, §4.2.2] or [4]. If time permits, you could mention it, but be aware that this might overwhelm listeners not familiar with complex algebraic geometry.

Talk 3 (The Earle-Eells theorem, Kevin Poljsak, 8.11.)**.** The theorem states that the components of $\mathrm{Diff}_\partial(F)$ are contractible if F is a surface except S^2 and T^2. The original proof by Earle and Eells [3] is based on geometric analysis, the talk should follow the topological proof by Gramain [16] as explained by Hatcher [11, Appendix B].

Homological stability.

Talk 4 (Homological stability I, Christian Rösner, 15.11.)**.** Recapitulation: homology of groups. Statement of the Harer stability theorem [17, §1]. The go through [17, §2] and prove as much as your time permits (the connectivity of the arc complex is treated in a later talk).

Talk 5 (Homological stability II, Prasanth Sivabalasingam, 22.11.)**.** Spectral sequence argument [17, §3]. A simpler and instructive special case is that of symmetric groups [12, §5.2]. For the connectivity argument [13].

Talk 6 (Homological stability III, Jannes Bantje, 29.11.)**.** Connectivity of the curve complexes [17, §4]. For time reasons, you cannot discuss everything in detail, and I suggest to discuss Theorems 4.1 and 4.9 and omit the other proofs. The proofs of those two results contain all ideas involved in the other proofs of [17, §4].

The 0-dimensional Case: symmetric groups and stable homotopy.

Talk 7 (J.E., 13.12.)**.** Configuration spaces and relation to $B\Sigma_n$. Pontrjagin-Thom map to QS^0. Spaces of infinite configurations and homotopy type of the space of all configurations.

Talk 8 (Lukas Buggisch, 20.12.)**.** Semisimplicial spaces. Spectrum of configuration spaces. Delooping argument. Group completion theorem and Quillen plus construction.

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The Madsen-Weiss theorem.

Talk 9 (Pontrjagin-Thom-Theory, Divya Sharma, 10.1.)**.** Classifying spaces of diffeomorphism groups [11, Appendix A]. The spectra MTSO(n). Construction of the map $B\mathrm{Diff}^+(M^n) \to \Omega^\infty \mathrm{MTSO}(n)$. Statement of the Madsen-Weiss theorem.

Talk 10 (Cohomology of $\Omega^\infty\mathrm{MTSO}(n)$, Danial Sanusi, 17.1.)**.** In this talk, the proof is given that the Madsen-Weiss theorem implies the Mumford conjecture. This involves the computation of $H^*(\Omega^\infty \mathrm{MT}SO(2);\mathbb{Q})$.

Methods of proof.

Talk 11 (Spaces of manifolds, Gordan Fröhlich, 24.1.)**.** Here, the space $\psi_d(U)$ of d-dimensional submanifolds of a manifold U should be introduced [14, §2]. The main player in the following talks is the space $\psi_{n,k}$ of submanifolds of $\mathbb{R}^n$ which are "noncompact in k directions", defined in [14, Definition 3.5]. Instead of focussing on the technicalities of the proofs in [14, §2], you should try to follow an axiomatic approach and explain how to work with these topologies. A good exercise in that direction is the computation of $\pi_0(\psi_{n,k})$ [14, §3.1] and [13]. Remark: in [14], a version with tangential structures θ is developed. This is absolutely essential for the high-dimensional theory [15], but for the 2-dimensional case, orientations are enough. This offers some room for simplifications.

Talk 12 (Cobordism categories and their classifying spaces, Raphael Reinauer, 31.1.)**.** Introduce the embedded cobordism category [14, §3.2] and mention the connection to classifying spaces of diffeomorphism groups as in the introduction to [15]. The main results of this talk are Theorems 3.9 and 3.10 of [14], and you should cover the proofs of these results. Then mention Theorems 3.12, 3.13 and 3.22 and explain how these results together determine the homotopy type of the cobordism category category (the proofs of these results are given in the following two talks). Please resist the temptation to discuss anything related to higher category theory in this talk.

Talk 13 (Homotopy type of the space of all manifolds, william Gollinger, 7.2.)**.** Discuss the proof of [14, Theorem 3.22].

Talk 14 (The scanning principle, Robin Loose, 14.2.)**.** Discuss the proof of [14, Theorem 3.13].

***-Talk 15** (The positive boundary subcategory I)**.** Introduce the positive boundary cobordism category as in the beginning of [18, §6] and state Theorem 6.1. Then show how this is used in [18, §7], together with the group compltion theorem to finish the proof of the Madsen-Weiss theorem.

***-Talk 16** (The positive boundary subcategory II)**.** Present the surgery argument [18, §6].

References

[1] Galatius. *Lectures on the Madsen-Weiss theorem* https://math.stanford.edu/ galatius/notes3.pdf
[2] Morita: *Geometry of characteristic classes*
[3] Earle, Eells: *A fibre bundle description of Teichmüller theory*
[4] Wahl: *The Mumford conjecture, Madsen-Weiss and homological stability* www.math.ku.dk/ wahl/PCMIlec.pdf
[5] Rosenberg: *Introduction to algebraic K-Theory*
[6] Tom Dieck: *Algebraic topology*
[7] Stong: *Notes on cobordism theory*
[8] Farb, Margalit: *A primer on mapping class groups*
[9] Hirsch: *Differential topology*
[10] Ivanov: *Mapping class groups*
[11] Hatcher: *A short exposition of the Madsen-Weiss theorem*
[12] Randal-Williams: *Homological stability for unordered configuration spaces*
[13] J. Ebert: *Personal communication*
[14] Galatius, Randal-Williams: *Monoids of moduli spaces of manifolds*
[15] Galatius, Randal-Williams: *Stable moduli spaces of high-dimensional manifolds*
[16] Gramain: *Le type d'homotopie du groupe des diffómorphismes d'une surface compacte*
[17] Wahl: *Homological stability for mapping class groups of surfaces*
[18] Galatius, Madsen, Tillmann, Weiss: *The homotopy type of the cobordism category*
[19] Madsen, Weiss: *The stable moduli space of Riemann surfaces: Mumford's conjecture*
[20] Madsen, Tillmann: *The stable mapping class group and* $\mathbb{CP}^\infty_+$

[21] Tillmann: *The homotopy of the stable mappings class group*

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