SEMINAR ON THE BAUM–CONNES CONJECTURE

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The seminar takes place on Mondays, 4:15 – 6:00, M6.

Goal of the seminar. In the first part of the seminar, we wish to understand the formulation of the Baum–Connes conjecture: if G is a discrete group, the Baum–Connes assembly map

$$\mu_G : K_*^G(\underline{E}G) \to K_*(C_r^*(G))$$

is an isomorphism. The second part is devoted to the following result by Higson and Kasparov [8], following the exposition of [7]: if G has the Haagerup property, then μ_G is an isomorphism.

Prerequisites for the seminar. We have to assume that the participants are more or less familiar with the basics of C^*-algebra theory which is typically covered in the lecture course "Operator algebras I". We also have to assume familiarity with K-theory of C^*-algebras, even though talks 3–5 constitute a crash course into that subject.

An overview.

Talk 1 (Overview, Siegfried Echterhoff, 7.10. and 14.10.)**.** This talk and the next should give an introduction into the ideas around the Baum–Connes conjecture. The topics that have to be touched upon are K-Theory of $C*$-algebras, group C^*-algebras and crossed products, equivariant K-homology, the classifying space of proper actions, construction of the assembly map (in one way or the other) and statement of the conjecture. Perhaps one should define the assembly map first without referring to bivariant theory, as in [1] which is nicely explained in [14, §6.1]. The idea of bivariant K-theory should also be explained (but note that the source for the technical part of the seminar [7] is formulated in the framework of E-theory).

Ordinary K-Theory and asymptotic morphisms.

Talk 2 (The spectral picture of K-theory I, Timo Siebenand, 21.10.)**.** In this talk, the spectral picture of K-theory should be described. Main reference: [7], §1.5–1.7, but this also involves material from §1.2–1.4. The original source [13] might be useful, as well as the relevant chapter in the book project [16].

Talk 3 (The spectral picture of K-theory II, André Schemaitat, 28.10.)**.** Introduce asymptotic morphisms [7, §1.8] and describe how to associate maps between K-theory groups from them (this will motivate the introduction of E-theory later on). Then discuss [7, §1.9] and end with the statement of the Bott periodicity theorem and the formal part of its proof [7, §1.10].

Talk 4 (The spectral picture of K-theory III, Jannes Bantje, 4.11.)**.** In this talk, the Bott periodicity theorem is proven using asymptotic morphisms, following [7, §1.11–1.13]. The harmonic oscillator appearing in [7, §1.13] will play an important rome later on.

Bivariant K-theory.

Talk 5 (E-Theory, Michael Joachim, 11.11.)**.** This and the next talk discusses the bivariant E-Theory groups. The relevant material is contained in [7, §2.1–2.6] , and a more detailed approach is in [6]. It might be useful for the audience to compare this to KK-theory.

Talk 6 (Equivariant E-theory, Thomas Nikolaus, 2.12/9.12.)**.** In this section, the theory from the preceeding talks will be made equivariant, following [7, §2.7–2.9].

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Talk 7 (Statement of the Baum–Connes conjecture in the context of E-theory and the proof for finite groups, Julian Kranz, 9.12./16.12.)**.** [7, §2.10–2.16]. Discuss the proof in the case of finite groups with some care, as this will give some insight.

Talk 8 (Proper algebras, a general criterion for the BCC and the case of free abelian groups, Johannes Ebert, 16.12./6.1.)**.** [7, §2.17–2.19]. The general criterion [7, Theorem 2.20] will be used later on and must be carefully stated. The proof for $G = \mathbb{Z}^n$ is a role model for what comes next...

The Baum-Connes conjecture for a(T)-menable groups.

Talk 9 (The Haagerup property and amenability I, Markus Schmetkamp, Timo Gessner, 13.1.)**.** This talk does not involve K-theory at all! It introduces the class of groups for which the proof of the Baum–Connes conjecture will be presented in the sequel: groups with the Haagerup property ("a(T)-menability" is just another name for that). This is done in [7, §3.1–3.3]. The talk should also recall the more familiar notion of amenability and sketch the proof that amenable groups have the Haagerup property. Either follow [7, Theorem 3.1] (the characterization of amenability that is used in the proof is discussed in [3, §2.6]) or [15]. A further interesting and large class with the Haagerup property are groups that act on cubical $CAT(0)$-complexes. See also [2] or [4].

Talk 10 (The Haagerup property II, Markus Schmetkamp, Timo Gessner, 20.1.)**.** See description of previous talk.

Talk 11 (Proof of the main theorem I, Arthur Bartels, Omar Mohsen and Rudolf Zeidler, 27.1.)**.** The main goal of the seminar is [7, Theorem 3.3 and Corollary 3.3], which states that a(T)-menable groups satisfy the Baum–Connes conjecture. This and the next talk will discuss the proof, which covers [7, §3.5–§3.7], and the precise organization of the material is left to the speakers. Remark: from p.203 on (i.e. exactly from the start of the technical construction on), [7] contains a systematic LaTeX error which makes the text hard to decipher. On Guentner's webpage http://www.math.hawaii.edu/ erik/research.html, there is a version without this error, but with different numbering.

Talk 12 (Proof of the main theorem II, Arthur Bartels, Omar Mohsen and Rudolf Zeidler, 3.2.)**.** See description of the previous talk.

Talk 13 (Proof of the main theorem III, Arthur Bartels, Omar Mohsen and Rudolf Zeidler, 10.2.)**.** See description of the previous talk.

Bonus material

Talk 14 (Assembly and higher index theory, Johannes Ebert, 17.2.)**.** This talk must be aligned with the next one. For content and further references, see [5].

Talk 15 (Injectivity of the Baum–Connes map implies the Novikov conjecture, Jens Reinhold, 24.2.)**.** If the assembly map $K_*^G(\underline{E}G) \to K_*(C_r^*(G))$ is rationally injective, then the Novikov conjecture on the homotopy invariance of higher signatures is true for G. In other words, if M, N are closed oriented n-manifolds, $h : M \to N$ is an orientation-preserving homotopy equivalence, $f : N \to BG$ a map and $x \in H^*(BG; \mathbb{Q})$, then $\langle h^* f^* x \cup L(TM); [M] \rangle = \langle f^* x \cup L(TN); [N] \rangle$. This is of course an old result [10], [12], [9], but it is not straightforward to extract a geodesic proof of that. This talk must be prepared jointly with the previous one. Consult [5].

Talk 16 (The Kadison–Kaplansky conjecture, Giles Gardam, 3.3.)**.** Surjectivity of the Baum–Connes assembly map for a torsionfree group G implies that $C_r^*(G)$ has no nontrivial idempotents and a fortiori the purely algebraic result that $\mathbb{C}[G]$ has no nontrivial idempotents. There are two proofs of this in the literature: one that uses Atiyah's L^2-index theorem in [14, §1 and §6.3], and another one which uses a wonderful group homology trick in [11, Lemma 7.1].

References

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