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On the divisibility of characteristic classes of non-oriented surface bundles

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ABSTRACT

In this note we introduce a construction which assigns to an arbitrary manifold bundle its fiberwise orientation covering. This is used to show that the zeta classes of non-oriented surface bundles are not divisible in the stable range.

1. Introduction

The mapping class group $\mathcal{N}_g$ of a non-orientable surface S_g of genus g (that is, the connected sum of g copies of $\mathbb{RP}^2$) is defined to be

$$\mathcal{N}_g := \pi_0(\mathrm{Diff}(S_g)),$$

the group of components of the diffeomorphism group of that surface. If $g \geqslant 3$, the components of $\mathrm{Diff}(S_g)$ are contractible [3], hence $B\mathcal{N}_g \simeq B\,\mathrm{Diff}(S_g)$, and so the cohomology of $B\mathcal{N}_g$ (or the group cohomology of $\mathcal{N}_g$) can be interpreted as the ring of characteristic classes for S_g-bundles.

Wahl [10] has proved a homological stability theorem for these groups, which says that in degrees $* \leqslant (g-3)/4$ the cohomology groups $H^*(\mathcal{N}_g)$ are independent of the genus g. We call this range of degrees the *stable range*. Combining Wahl's result with that of Galatius, Madsen, Tillmann and Weiss [7], the stable rational cohomology of these groups can be identified: there are certain integrally defined characteristic classes ζ_i in degrees $4i$ (defined in Section 3) and the map

$$\mathbb{Q}[\zeta_1, \zeta_2, \zeta_3, \ldots] \to H^*(\mathcal{N}_g; \mathbb{Q})$$

is an isomorphism in the stable range. In [9] the second author calculates these stable groups with coefficients in a finite field, and tabulates some low-dimensional integral groups.

The classes ζ_i are analogues of the even Miller–Morita–Mumford classes, for non-oriented surface bundles. Galatius, Madsen and Tillmann [6] have studied the divisibility of the Miller–Morita–Mumford classes $\kappa_i \in H^*(\Gamma_\infty; \mathbb{Z})$ in the free

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quotient of the integral cohomology of the stable mapping class group Γ_∞. They find that the even classes are divisible by 2 and the odd classes are divisible by a denominator of a Bernoulli number. In [5] the first author studied the divisibility of the Miller–Morita–Mumford classes for surface bundles with spin structures, and it was shown that the divisibility increases by a certain power of 2 relative to the non-spin case. Continuing the study of divisibility of characteristic classes of surface bundles, we prove

Theorem A. *The universal zeta classes, $\zeta_n \in H^{4n}(\mathcal{N}_g;\mathbb{Z})$, are not divisible in the stable range. Indeed, they are not divisible in the free quotient $H^{4n}_{free}(\mathcal{N}_g;\mathbb{Z})$ of $H^{4n}(\mathcal{N}_g;\mathbb{Z})$ in this range.*

This gives the trend that extra structure on the vertical tangent bundle, such as an orientation or a spin structure, gives extra divisibility of characteristic classes of surface bundles.

2. Lifting diffeomorphisms to orientation coverings

In this section, we will construct a natural homomorphism from the diffeomorphism group $\mathrm{Diff}(M)$ of a smooth d-manifold to the group $\mathrm{Diff}^+(\tilde{M})$ of orientation-preserving diffeomorphisms of the orientation covering of M. This implies that any smooth fiber bundle $p: E \to B$ admits a two-fold covering $\pi: \tilde{E} \to E$, such that $p \circ \pi: \tilde{E} \to B$ is an oriented smooth fiber bundle and that the restriction of $\pi: \tilde{E} \to E$ to a fiber of p is the orientation covering.

Let M be a smooth d-manifold, $d > 0$, and let $\Lambda^d TM$ be the highest exterior power of the tangent bundle, which is a real line bundle. The total space of the orientation covering of M can be defined as the quotient

$$\tilde{M} := \left(\Lambda^d TM \setminus 0\right)/\mathbb{R}_{>0}. \tag{2.1}$$

The canonical map $\pi: \tilde{M} \to M$ is a two-sheeted covering. The space $\tilde{M}$ is a smooth oriented manifold with a preferred orientation. To see this, recall that an orientation of a d-dimensional real vector space V is a component of $\Lambda^d V \setminus 0$, or in other words, one of the two points of $(\Lambda^d V \setminus 0)/\mathbb{R}_{>0}$. Thus a point in $x \in \tilde{M}$ is by definition an orientation of the tangent space $T_{\pi(x)}M$. The differential $T_x\pi$ at $x \in \tilde{M}$ is a linear isomorphism $T_x\tilde{M} \to T_{\pi(x)}M$ so the orientation of $T_{\pi(x)}M$ given by x gives us a preferred orientation of $T_x\tilde{M}$. Using local coordinates on M, it is easy to see that these orientations of the tangent spaces $T_x\tilde{M}$ fit together continuously and define an orientation of $\tilde{M}$, the *preferred orientation*.

Moreover, this construction is natural: a diffeomorphism $f: M \to N$ of smooth manifolds induces a diffeomorphism $\tilde{f}: \tilde{M} \to \tilde{N}$ which covers f. It is easy to see that $\tilde{f}$ is orientation-preserving provided $\tilde{M}$ and $\tilde{N}$ are endowed with the preferred orientations. If $g: N \to P$ is another diffeomorphism, then $\widetilde{g \circ f} = \tilde{g} \circ \tilde{f}$. Also, $\widetilde{\mathrm{id}_M} = \mathrm{id}_{\tilde{M}}$. Finally, we did not use that f is a diffeomorphism, but only that the differential of f was nonsingular. It follows that the assignments $M \mapsto \tilde{M}$ and $f \mapsto \tilde{f}$ define a functor $\mathcal{L}$ from the category $\mathcal{X}_d$ of smooth d-manifolds and local diffeomorphisms to the category $\mathcal{X}_d^+$ of oriented d-manifolds and orientation-preserving local diffeomorphisms. In particular, we defined a group homomorphism $\mathcal{L}_M: \mathrm{Diff}(M) \to \mathrm{Diff}^+(\tilde{M})$.

For a manifold M, we denote by π_M the covering map $\tilde{M} \to M$ and by $\iota_M: \tilde{M} \to \tilde{M}$ the unique nontrivial deck transformation. If $f: M \to N$ is a (local) diffeomorphism, the following relations hold

$$\pi_N \circ \tilde{f} = f \circ \pi_M; \qquad \tilde{f} \circ \iota_M = \iota_N \circ \tilde{f}. \tag{2.2}$$

The morphism spaces of the categories $\mathcal{X}_d$ and $\mathcal{X}_d^+$ have a natural topology, the weak C^∞-topology, with respect to which the composition maps are continuous. Thus $\mathcal{X}_d$ and $\mathcal{X}_d^+$ are topological categories. Using local coordinates, it is easy to see that the functor $\mathcal{L}$ is continuous. In particular, the homomorphism $\mathcal{L}_M: \mathrm{Diff}(M) \to \mathrm{Diff}^+(\tilde{M})$ is continuous.

Let us now discuss smooth fiber bundles. Let $p: E \to B$ be a smooth fiber bundle with fiber a d-dimensional smooth manifold M and structural group $\mathrm{Diff}(M)$ (with the weak C^∞-topology). Consider the associated $\mathrm{Diff}(M)$-principal bundle $Q \to B$, which has the property that $Q \times_{\mathrm{Diff}(M)} M \cong E$. Via the homomorphism $\mathcal{L}_M$, the manifold $\tilde{M}$ has a $\mathrm{Diff}(M)$-action by orientation-preserving diffeomorphisms. Hence the fiber bundle

$$q: \tilde{E} := Q \times_{\mathrm{Diff}(M)} \tilde{M} \to B$$

is an oriented smooth fiber bundle with fiber $\tilde{M}$. Because of (2.2), there is a twofold covering $\pi_E: \tilde{E} \to E$, such that $q = p \circ \pi_E$. Furthermore, there is a fiber-preserving and orientation-reversing involution ι_E on $\tilde{E}$. We call $\tilde{E}$ the *fiberwise orientation cover* of E. We summarize the results of this section.

Theorem 2.1. *The fiberwise orientation covering $\pi_E: \tilde{E} \to E$ of a smooth fiber bundle $p: E \to B$ is a two-sheeted covering whose restriction to any fiber E_b of p is the orientation covering of E_b. The composition $q = p \circ \pi_E$ is an oriented fiber bundle. Furthermore, $\tilde{E}$ and π_E are uniquely determined by these properties (up to orientation-preserving isomorphism).*

We conclude with a simple remark. All the constructions in this section make sense when the manifold M (or the fiber bundle E) is orientable. If this is the case, then $\tilde{M}$ is the disjoint sum of two copies of M. The choice of an orientation of M singles out a component of $\tilde{M}$.

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3. Characteristic classes of surface bundles

In this section, we give a brief review of the theory of characteristic classes of surface bundles, both oriented and non-oriented. First we discuss the oriented case. Let $\pi : E \to B$ be an oriented surface bundle and let $T_v E$ be the vertical tangent bundle. It is an oriented 2-dimensional real vector bundle on E and thus it has an Euler class $e(T_v(E)) \in H^2(E;\mathbb{Z})$. We can consider $T_v E$ also as a complex line bundle (there is a complex structure on it, which is unique up to isomorphism) and the Euler class agrees with the first Chern class. The Miller–Morita–Mumford classes are defined to be

$$\kappa_n(E) := \pi_!\big(e(T_v E)^{n+1}\big) \in H^{2n}(B;\mathbb{Z}),$$

where $\pi_! : H^*(E;\mathbb{Z}) \to H^{*-2}(B;\mathbb{Z})$ is the umkehr, or cohomological fiber-integration, map. This definition cannot be generalized to the non-oriented case without further effort, because both the Euler class and the umkehr map only exist for oriented surface bundles.

The concept needed for a generalization is the Becker–Gottlieb transfer [1]. Let $p : E \to B$ be a smooth fiber bundle with compact fibers diffeomorphic to F (not necessarily of dimension 2). The transfer is a stable map in the converse direction, more precisely, it is a map of suspension spectra

$$\mathrm{trf}_p : \Sigma^\infty B_+ \to \Sigma^\infty E_+.$$

Recall that the spectrum cohomology of the suspension spectrum of a space $\Sigma^\infty X_+$ agrees with the usual cohomology of the space X. Thus we can form the map $\mathrm{trf}_p^* \circ p^* : H^*(B;\mathbb{Z}) \to H^*(B;\mathbb{Z})$, and for all $x \in H^*(B;\mathbb{Z})$ we have

$$\mathrm{trf}_p^* \circ p^*(x) = \chi(F)\cdot x, \tag{3.1}$$

where $\chi(F)$ denotes the Euler number of the fiber [1, Theorem 5.5]. Furthermore, if $q : \tilde{E} \to E$ is another smooth fiber bundle with compact fibers, then $p \circ q$ is also such a fiber bundle. In this situation the composition of the transfers is homotopic to the transfer of the composition (see [2, Eq. (2.3), p. 137]):

$$\mathrm{trf}_{p\circ q} \simeq \mathrm{trf}_q \circ \mathrm{trf}_p. \tag{3.2}$$

A diffeomorphism $f : M \to N$ of manifolds can be considered as a fiber bundle whose fiber is a point. By (3.1),

$$\mathrm{trf}_f^* \circ f^* = \mathrm{id}_{H^*(N;\mathbb{Z})}, \qquad f^* \circ \mathrm{trf}_f^* = \mathrm{id}_{H^*(M;\mathbb{Z})}. \tag{3.3}$$

In fact, trf_f and $\Sigma^\infty(f^{-1})$ are homotopic, but we do not need this fact. The transfer of an oriented fiber bundle $p : E \to B$ is closely related to the umkehr map. For all $x \in H^*(E;\mathbb{Z})$, one has (see [1, Theorem 4.3])

$$\mathrm{trf}_p^*(x) = p_!\big(x \cup e(T_v E)\big). \tag{3.4}$$

The identity (3.4) implies that

$$\kappa_n(E) = \mathrm{trf}_p^*\big(e(T_v E)^n\big) \tag{3.5}$$

for the Miller–Morita–Mumford classes of an oriented surface bundle $p : E \to B$. Because of the identity $p_1(L) = e(L)^2$ for the Pontrjagin class of a 2-dimensional oriented real vector bundle L, we see that

$$\kappa_{2n}(E) = \mathrm{trf}_p^*\big(p_1(T_v E)^n\big). \tag{3.6}$$

This can be generalized to the non-oriented case. Wahl defines [10, p. 3]

$$\zeta_i(E) := \mathrm{trf}_p^*\big(p_1(T_v E)^i\big) \in H^{4i}(B;\mathbb{Z}), \tag{3.7}$$

for a non-oriented surface bundle $p : E \to B$, where $p_1(T_v E) \in H^4(E;\mathbb{Z})$ is the first Pontrjagin class of the vertical tangent bundle.

The spaces $B\,\mathrm{Diff}^+(F_g)$ and $B\,\mathrm{Diff}(S_g)$ carry universal oriented and non-orientable surface bundles of a fixed genus, so the above constructions define classes $\kappa_i \in H^{2i}(B\,\mathrm{Diff}^+(F_g);\mathbb{Z})$ and $\zeta_i \in H^{4i}(B\,\mathrm{Diff}(S_g);\mathbb{Z})$ which we call the *universal classes*, and omit the universal bundle from the notation.

Now we can state and prove the main result of this section.

Theorem 3.1. *Let $p : E \to B$ be a non-oriented surface bundle with compact fibers and let $c : \tilde{E} \to E$ be its fiberwise orientation covering. Denote $q := p \circ c : \tilde{E} \to B$. Then the following relations hold for all $n \geqslant 0$:*

(1) $\kappa_{2n}(\tilde{E}) = 2 \cdot \zeta_n(E)$.
(2) $2 \cdot \kappa_{2n+1}(\tilde{E}) = 0$.

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Proof. For the identity (1), we compute

$$\begin{aligned}\kappa_{2n}(\tilde{E}) &= \mathrm{trf}_q^*\big(p_1(T_v\tilde{E})^n\big)\\ &= \mathrm{trf}_p^*\big(\mathrm{trf}_c^*\big(c^*\big(p_1(T_vE)^n\big)\big)\big)\\ &= \mathrm{trf}_p^*\big(2\cdot p_1(T_vE)^n\big)\\ &= 2\cdot\zeta_n(E).\end{aligned}$$

The first equality is (3.6). Because $c:\tilde{E}\to E$ is a smooth covering in every fiber, $c^*(T_vE)\cong T_v(\tilde{E})$, whence $p_1(T_v\tilde{E}) = c^*(p_1(T_vE))$. Together with (3.2), this fact implies the second equality. Because c is a double covering, the Euler number of its fiber is 2. Thus $\mathrm{trf}_c^*\circ c^* = 2$, which gives the third equality. The fourth equality is the definition.

For the proof of identity (2), we use the orientation-reversing involution ι on $\tilde{E}$. By (3.3), $\mathrm{trf}_\iota^* = (\iota^*)^{-1} = \iota^*$. Because $c\circ\iota = c$, it follows that $\mathrm{trf}_c^* = \mathrm{trf}_c^*\circ\mathrm{trf}_\iota^* = \mathrm{trf}_c^*\circ\iota^*$. Because ι is an orientation-reversing fiberwise diffeomorphism, it induces an orientation-reversing vector bundle isomorphism $d\iota: T_v\tilde{E}\to\iota^*T_v\tilde{E}$. Thus $e(T_v\tilde{E}) = -\iota^*e(T_v\tilde{E})$. Thus

$$\begin{aligned}\kappa_{2n+1}(\tilde{E}) &= \mathrm{trf}_p^*\big(\mathrm{trf}_c^*\big(e(T_v\tilde{E})^{2n+1}\big)\big)\\ &= \mathrm{trf}_p^*\big(\mathrm{trf}_c^*\big(\iota^*\big(e(T_v\tilde{E})^{2n+1}\big)\big)\big)\\ &= (-1)^{2n+1}\mathrm{trf}_p^*\big(\mathrm{trf}_c^*\big(e(T_v\tilde{E})^{2n+1}\big)\big)\\ &= -\kappa_{2n+1}(\tilde{E}). \qquad\square\end{aligned}$$

Remark 3.2. An implication of this theorem is that for an oriented surface bundle $E'\to B$, the characteristic classes $2\cdot\kappa_{2n+1}(E')$ are obstructions to E' admitting an orientation-reversing, fixed-point free, fiberwise involution. Furthermore, for bundles which do admit such an involution, it gives an interpretation of $\frac{1}{2}\kappa_{2n}(E')$ as the zeta classes of the associated quotient bundle of non-orientable surfaces.

4. An example

In this section, we consider the example of a genus zero surface bundle. Let $\gamma_3\to BSO(3)$ be the universal 3-dimensional oriented Riemannian real vector bundle and let $\mathbb{S}(\gamma_3)\to BSO(3)$ be its unit sphere bundle. It is known that this is the universal smooth oriented bundle with fiber S^2, but we do not need this fact. In [4, Proposition 5.2.4] the first author has computed that $\kappa_{2n}(\mathbb{S}(\gamma_3)) = 2p_1^n$. The bundle $\mathbb{S}(\gamma_3)$ admits an orientation-reversing, fixed-point free involution on its fibers, namely the antipodal map $-\mathrm{id}$. The quotient is $\mathbb{P}(\gamma_3)$, the $\mathbb{RP}^2$-bundle associated to γ_3. By Theorem 3.1, we have

$$2\zeta_n\big(\mathbb{P}(\gamma_3)\big) = 2p_1^n. \tag{4.1}$$

It is well known that the free quotient $H^*_{free}(BSO(3);\mathbb{Z})$ is the polynomial algebra $\mathbb{Z}[p_1]$. In particular, the powers p_1^n are not divisible in the free quotient of $H^*(BSO(3);\mathbb{Z})$. We have shown:

Proposition 4.1. *The class $\zeta_n(\mathbb{P}(\gamma_3))$ is not divisible in $H^*_{free}(BSO(3);\mathbb{Z})$.*

5. A review of the stable homotopy theory of surfaces and proof of Theorem A

In this section, we give a brief introduction to the modern homotopy theory of surface bundles developed by Galatius, Madsen, Tillmann and Weiss. A good survey can be found in [6, Sections 2 and 3],[3] and full proofs can be found in [7]. Let us first discuss the oriented case.

Consider the universal complex line bundle $L\to BSO(2)$. There does not exist a vector bundle V such that $V\oplus L$ is trivial, but we can define an additive inverse $L^\perp$ of L as a *stable vector bundle*. The *Madsen–Tillmann spectrum* $\mathbf{MTSO}(2)$ is defined to be the Thom spectrum of $L^\perp$. For any oriented surface bundle $E\to B$, there exists a natural map

$$\alpha_E: B\to\Omega_0^\infty\mathbf{MTSO}(2)$$

into the unit component of the infinite loop space of the Madsen–Tillmann spectrum. In particular, it can be defined for the universal oriented surface bundle with fibers a surface F_g of genus g, to obtain a universal map

$$\alpha_g: B\,\mathrm{Diff}^+(F_g)\to\Omega_0^\infty\mathbf{MTSO}(2).$$

For each $n>0$, there is a cohomology class $y_n\in H^{2n}(\Omega_0^\infty\mathbf{MTSO}(2);\mathbb{Z})$ such that for any surface bundle as above

$$\alpha_E^*(y_n) = \kappa_n(E). \tag{5.1}$$

[3] Note that this paper uses a different notation: they denote $\mathbf{MTSO}(2)$ by $\mathbb{CP}^\infty_{-1}$.

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The rational cohomology of $\Omega_0^\infty \mathbf{MTSO}(2)$ is isomorphic to the polynomial ring $\mathbb{Q}[y_1, y_2, \ldots]$. The main result of [7], originally due to Madsen and Weiss [8], implies that the map α_g induces an isomorphism on homology groups in the stable range, that is,

$$H_k(\alpha_g) : H_k\big(B\,\mathrm{Diff}^+(F_g); \mathbb{Z}\big) \to H_k\big(\Omega_0^\infty \mathbf{MTSO}(2); \mathbb{Z}\big)$$

is an isomorphism as long as $g \geqslant 2k + 2$.

Similar results hold in the non-oriented case, and are detailed in [10, Section 6]. The Madsen–Tillmann spectrum is replaced by $\mathbf{MTO}(2)$, which is the Thom spectrum of the stable inverse of the universal 2-dimensional real vector bundle over $BO(2)$. There is an analogue of the map α_E for any non-oriented surface bundle $E \to B$, and there are classes $x_n \in H^{4n}(\Omega_0^\infty \mathbf{MTO}(2); \mathbb{Z})$ for $n > 0$, such that $\alpha_E^*(x_n) = \zeta_n(E)$. The rational cohomology ring of $\Omega_0^\infty \mathbf{MTO}(2)$ is isomorphic to the polynomial ring $\mathbb{Q}[x_1, x_2, \ldots]$, in complete analogy to the oriented case.

The analogue of the Madsen–Weiss theorem is also true in the non-oriented case, by [10] and [7]. More precisely

$$H_k(\alpha_g; \mathbb{Z}) : H_k\big(B\,\mathrm{Diff}(S_g); \mathbb{Z}\big) \to H_k\big(\Omega_0^\infty \mathbf{MTO}(2); \mathbb{Z}\big) \tag{5.2}$$

is an isomorphism as long as $4k + 3 \leqslant g$, and similarly in cohomology.

Proof of Theorem A. This is now straightforward. We assume that the universal class ζ_n is divisible in the stable range. Under the isomorphism (5.2), ζ_n corresponds to the class $x_n \in H^{4n}(\Omega^\infty \mathbf{MTO}(2); \mathbb{Z})$, which must also be divisible. We have seen in Proposition 4.1 that the image of $x_n \in H^{4n}(\Omega^\infty \mathbf{MTO}(2); \mathbb{Z})$ under the map $\alpha_{\mathbb{P}(\gamma_3)} : BSO(3) \to \Omega^\infty \mathbf{MTO}(2)$ is p_1^n in the free quotient and so not divisible. This is a contradiction. □

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