arXiv:2203.03414v1 [math.AT] 7 Mar 2022

SOME RATIONAL HOMOLOGY COMPUTATIONS FOR DIFFEOMORPHISMS OF ODD-DIMENSIONAL MANIFOLDS

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Abstract. We calculate the rational cohomology of the classifying space of the diffeomorphism group of the manifolds $U^n_{g,1} := \#^g(S^n \times S^{n+1}) \setminus \mathrm{int}(D^{2n+1})$, for large g and n, up to approximately degree n. The answer is that it is a free graded commutative algebra on an appropriate set of Miller–Morita–Mumford classes.

Our proof goes through the classical three-step procedure: (a) compute the cohomology of the homotopy automorphisms, (b) use surgery to compare this to block diffeomorphisms, (c) use pseudoisotopy theory and algebraic K-theory to get at actual diffeomorphism groups.

Contents

Date: March 8, 2022.

The authors were supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 427320536 – SFB 1442, as well as under Germany's Excellence Strategy EXC 2044 – 390685587, Mathematics Münster: Dynamics–Geometry–Structure.

1. Introduction

1.1. **Context: Madsen–Weiss type theorems.** For a smooth compact manifold with boundary M, let $\mathrm{Diff}_\partial(M)$ denote the group of diffeomorphisms of a smooth compact manifold M which are equal to the identity near ∂M. One of the success stories of differential topology in the 21st century was a (partial) computation of the cohomology of the classifying space $B\mathrm{Diff}_\partial(M)$ for some even–dimensional manifolds, by Madsen–Weiss [48] (for surfaces) and by Galatius–Randal-Williams [24] [26] [25] (in the higher dimensional case). The simplest case of these results concern the manifolds

$$W_{g,1}^n := \#^g(S^n \times S^n) \setminus \mathrm{int}(D^{2n}),$$

the connected sum of g copies of $S^n \times S^n$, minus the interior of a disc, and are formulated in terms of the Madsen–Tillmann spectra[1] $\mathrm{MT}\theta_{2n}^n$, the Thom spectrum of the additive inverse of the universal $2n$-dimensional vector bundle over the n-connected cover $BO(2n)\langle n\rangle \to BO(2n)$. There is a natural map $\alpha_g : B\mathrm{Diff}_\partial(W_{g,1}^n) \to \Omega_0^\infty \mathrm{MT}\theta_{2n}^n$ to the unit component of the infinite loop space. These maps are compatible for varying value of g, and induce a map

$$\alpha_\infty : \mathrm{hocolim}_{g\to\infty} B\mathrm{Diff}_\partial(W_{g,1}^n) \to \Omega_0^\infty \mathrm{MT}\theta_{2n}^n$$

in the limiting case, which is an integral homology equivalence (for $n = 1$ by [48], for $n \geq 3$ by [24] and for $n = 2$ by [25]). This is complemented by homological stability theorems (unless $n = 2$) due to [33] and [26], so that α_g induces an isomorphism in homology in a range of degrees increasing with g.

The rational cohomology of $\Omega_0^\infty \mathrm{MT}\theta_{2n}^n$ (and more general Madsen–Tillmann spectra) is easily calculated using the standard tools from algebraic topology. The answer is that it is the free graded-commutative algebra generated by the vector space $(s^{-2n}H^*(BO(2n)\langle n\rangle;\mathbb{Q}))_{>0}$, the positive degree part of the desuspension of $H^*(BO(2n)\langle n\rangle;\mathbb{Q})$. Let $\mu_c \in H^{k-2n}(\Omega_0^\infty \mathrm{MT}\theta_n;\mathbb{Q})$ be the element corresponding to $c \in H^k(BO(2n)\langle n\rangle;\mathbb{Q})$. The pullback $\alpha_g^*\mu_c \in H^{k-2n}(B\mathrm{Diff}_\partial(W_{g,1}^n);\mathbb{Q})$ is the tautological class κ_c of the universal bundle over $B\mathrm{Diff}_\partial(W_{g,1}^n)$. Finally $H^*(BO(2n)\langle n\rangle;\mathbb{Q})$ is the polynomial algebra generated by the Pontrjagin classes p_m with $\frac{n+1}{4} \leq m \leq n-1$ and the Euler class. So altogether, in a range of degrees, $H^*(B\mathrm{Diff}_\partial(W_{g,1}^n);\mathbb{Q})$ is a polynomial algebra in certain tautological classes.

All these results are for *even-dimensional* manifolds. The construction of the map α_g can be generalized to any manifold, and yields for oriented M of dimension

[1]We use the notation from [37] instead of that from [24].

d a map $\alpha_M : B\mathrm{Diff}^+_\partial(M) \to \Omega^\infty_0 \mathrm{MT}SO(d)$ (the maps α_g above are a refinement of this construction due to the fact that $W^n_{g,1}$ is $(n-1)$-connected and n-parallelizable). It has been observed by the first named author [13] that the classes $\alpha_M^* \mu_{L_m}$ associated to the components of the Hirzebruch L-class vanish, for each odd-dimensional d, though μ_{L_m} is nonzero. Hence any naive generalization of say [48] or [24] will fail in odd dimensions.

1.2. **Main result.** Even though some substantial inroads into the odd-dimensional situation have been made recently [57], [7], [56], [37], it remains a mystery and there does not seem to be a convincing conjectural odd-dimensional analogue of the main result of [24]. Our modest hope in this work is that our main result, Theorem A below, might eventually serve as a piece of evidence which helps to formulate an odd-dimensional version of these results. Let us consider the manifolds

$$U^n_{g,1} := \sharp^g (S^n \times S^{n+1}) \setminus \mathrm{int}(D^{2n+1})$$

which we consider as an odd-dimensional variant of the manifolds $W^n_{g,1}$. Being $(n-1)$-connected and n-parallelizable, one obtains a map

$$\beta^n_g : B\mathrm{Diff}_\partial(U^n_{g,1}) \to \Omega^\infty_0 \mathrm{MT}\theta^n_{2n+1}, \tag{1.1}$$

where the target is the Madsen–Tillmann spectrum of $BO(2n+1)\langle n\rangle \to BO(2n+1)$. Note that

$$H^*(BO(2n+1)\langle n\rangle; \mathbb{Q}) \cong \mathbb{Q}[L_m | \frac{n+1}{4} \leq m \leq n]$$

(there is no Euler class; and it is more convenient to use the components of the Hirzebruch L-class instead of the Pontrjagin classes as generators). Hence $H^*(\Omega^\infty_0 \mathrm{MT}\theta^n_{2n+1}; \mathbb{Q})$ is the exterior algebra generated by the elements $\{\mu_{L_{m_1} \cdots L_{m_r}}\}$, where $\frac{n+1}{4} \leq m \leq n$.

Theorem A. *Assume that $n \geq 5$. Then the map*

$$(\beta^n_g)^* : H^*(\Omega^\infty_0 \mathrm{MT}\theta^n_{2n+1}; \mathbb{Q}) \to H^*(B\mathrm{Diff}_\partial(U^n_{g,1}); \mathbb{Q})$$

is surjective in degrees $* \leq \min(\frac{g-2}{2}, n-4)$, and in that range of degrees, the kernel is the ideal generated by the classes μ_{L_m} (all m) and by the linear subspace $H^1(\Omega^\infty_0 \mathrm{MT}\theta^n_{2n+1}; \mathbb{Q})$.

Remark 1.2.
(1) That μ_{L_m} lies in the kernel of $(\beta^n_g)^*$ is the main result of [13].
(2) The space $H^1(\Omega^\infty_0 \mathrm{MT}\theta^n_{2n+1}; \mathbb{Q})$ is zero unless $n \equiv 3 \pmod 4$, say $n = 4k-1$. In that case, one checks that

$$H^1(\Omega^\infty_0 \mathrm{MT}\theta^{4k-1}_{8k+7}; \mathbb{Q}) = \mathbb{Q}\{\mu_{L_{2k}}, \mu_{L_k^2}\}.$$

Hence the only new relation is $\kappa_{L_k^2} = 0$, which holds more generally for all stably parallelizable manifolds of those dimensions. We give the fairly elementary proof in Proposition 2.10 below; for the special manifolds $U^n_{g,1}$ the relation comes out of the proof of Theorem A.
(3) The bound in g stems from a homological stability result due to Perlmutter [56, Corollary 1.3.2]: the stabilization map $B\mathrm{Diff}_\partial(U^n_{g,1}) \to B\mathrm{Diff}_\partial(U^n_{g+1,1})$ is homologically $\frac{g-2}{2}$-connected (with integral coefficients).
(4) The bound in n comes from our method of proof which we describe informally in §1.4 below.

1.3. **Relation to Hebestreit–Perlmutter's work.** Let us comment on the relationship of the present work with [37]. The disjoint union

$$B\mathcal{D} := \coprod_g B\mathrm{Diff}_\partial(U^n_{g,1})$$

carries a natural structure of an algebra over the operad of little $(2n+1)$-discs. Hence we can form its group completion $\Omega B(B\mathcal{D})$ which is a $(2n+1)$-fold loop space. We clearly have $\pi_0(\Omega B(B\mathcal{D})) \cong \mathbb{Z}$, and an application of the group completion theorem shows that the homology of each of the components is $H_*(\Omega_0 B(B\mathcal{D})) \cong H_*(\mathrm{hocolim}_g B\mathrm{Diff}_\partial(U^n_{g,1}))$; hence Theorem A also evaluates the rational cohomology of $\Omega_0 B(B\mathcal{D})$ in a range of degrees. In [37], a larger E_{2n+1}-algebra is considered, namely

$$\mathcal{M}_{2n+1} := \coprod_{[W]} B\mathrm{Diff}_\partial(W),$$

where $[W]$ ranges through all diffeomorphism classes of $(n-1)$-connected $(2n+1)$-manifolds W with boundary S^{2n} which are moreover n-parallelizable, i.e. the restriction of TW to the n-skeleton is trivial. The main result of [37] is that the group completion of $\mathcal{M}_{2n+1}$ has the homotopy type of an infinite loop space if $n \geq 4$ and $n \neq 7$ (this is an odd-dimensional version of a theorem by Tillmann [69] for surfaces). They do this by showing that $\Omega B\mathcal{M}_{2n+1}$ is homotopy equivalent to the infinite loop space of a spectrum denoted $\mathrm{MT}\mathcal{L}_{2n+1}$. The latter is not a Madsen–Tillmann spectrum despite the notation, but rather obtained from a certain cobordism category of manifolds equipped with certain subspaces of their homology by using infinite loop space machinery. That cobordism category does not fit into the general theory of cobordism categories as in [27]; there is however a map $\mathrm{MT}\mathcal{L}_{2n+1} \to \mathrm{MT}\theta^n_{2n+1}$ of spectra.

Clearly $B\mathcal{D} \subset \mathcal{M}_{2n+1}$ is a union of path components. However, while $\pi_0(B\mathcal{D}) \cong \mathbb{N}_0$, $\pi_0(\mathcal{M}_{2n+1})$ is much larger; [37, Proposition 3.2.5] deduces a description of $\pi_0(\mathcal{M}_{2n+1})$ from [74]. It is therefore not clear how to relate the group completions of $B\mathcal{D}$ and of $\mathcal{M}_{2n+1}$. As Fabian Hebestreit and Manuel Krannich pointed out to us, it seems conceivable that the map $\Omega_0 B(B\mathcal{D}) \to \Omega_0 B\mathcal{M}_{2n+1}$ is a rational homology equivalence. If that turns out to be true, Theorem A computes the rational homology of $\Omega_0^\infty \mathrm{MT}\mathcal{L}_{2n+1}$ in a range of degrees.

1.4. **Method of proof.** Having said that the methods of [24] must fail in the odd-dimensional case we need to say how we approach Theorem A. There is an established three-stage procedure to describe the topology of $B\mathrm{Diff}_\partial(M)$ for a high-dimensional manifold ($d = \dim(M) \geq 5$) in a range depending on d. The *first step* is to get a hold on $B\mathrm{hAut}_\partial(M)$, the classifying space of the homotopy automorphisms of M, relative to the boundary. The *second step* uses Quinn's space-level version of the surgery exact sequence [61] to compare $B\mathrm{hAut}_\partial(M)$ with the classifying space $B\widetilde{\mathrm{Diff}}_\partial(M)$ of the *block diffeomorphism* group; the difference is in terms of the L-theory of the group ring of $\pi_1(M)$. The *third step* compares block diffeomorphisms to diffeomorphisms in a range of degrees, in terms of algebraic K-theory.

Our strategy in this paper is to first compute $H^*(B\widetilde{\mathrm{Diff}}_\partial(U^n_{g,1});\mathbb{Q})$ in the same range of degrees as in Theorem A (the first and second step can largely be merged), and then to use the comparison from the third step (which holds in a larger range of degrees) to arrive at $H^*(B\mathrm{Diff}_\partial(U^n_{g,1});\mathbb{Q})$. Let us first note that the map (1.1) does not extend to block diffeomorphisms, hence the spectrum $\mathrm{MT}\theta^n_{2n+1}$ does not play an important role in the calculation. However, the tautological classes can be extended by [15]. To state our results property, let us introduce some notation. Define a graded vector space $Z(d)$ as follows:

$$Z(d)_i = \begin{cases} 0 & i \leq 0 \\ H^{d+i}(BO;\mathbb{Q}) & i > 0. \end{cases}$$

The element of $Z(d)_i$ corresponding to $c \in H^{d+i}(BO;\mathbb{Q})$ is denoted k_c.

For each d-manifold M, we therefore obtain maps

$$\Phi_M : \mathbb{F}(Z(d)) \to H^*(B\mathrm{Diff}^+_\partial(M);\mathbb{Q})$$

and

$$\tilde{\Phi}_M : \mathbb{F}(Z(d)) \to H^*(B\widetilde{\mathrm{Diff}}^+_\partial(M);\mathbb{Q})$$

from the free graded-commutative algebra generated by $Z(d)$. Sending k_c to $\mu_c \in H^*(\Omega^\infty_0 \mathrm{MT}\theta^n_{2n+1};\mathbb{Q})$ also gives an algebra map

$$\Psi : \mathbb{F}(Z(2n+1)) \to H^*(\Omega^\infty_0 \mathrm{MT}\theta^n_{2n+1};\mathbb{Q})$$

which is surjective and whose kernel is the ideal generated by all $k_{L_m c}$ with $4m \leq n$ and $c \in H^*(BO;\mathbb{Q})$, and by all k_c with $c \in \ker(H^*(BO;\mathbb{Q}) \to H^*(BO(2n+1);\mathbb{Q})$. Moreover $(\beta^n_g)^* \circ \Psi = \Phi_{U^n_{g,1}}$. Therefore Theorem A is equivalent to the following result.

Theorem 1.3. *Assume that $n \geq 5$. Then $\Phi_{U^n_{g,1}}$ is surjective in degrees $* \leq \min(\frac{g-2}{2}, n-4)$, and in that range of degrees, the kernel is the ideal generated by the following list of elements:*

$$k_{L_m} \quad \textit{all } m, \tag{1.4}$$

$$k_{L_m c} \quad 4m \leq n,\ c \in H^*(BO;\mathbb{Q}), \tag{1.5}$$

$$k_{L_{m_0} L_{m_1}} \quad 4(m_0+m_1) = 2n+2. \tag{1.6}$$

To state our result concerning $B\widetilde{\mathrm{Diff}}_\partial(U^n_{g,1})$, let B be the graded vector space

$$B := \bigoplus_{k \geq 1} \mathbb{Q}[4k+1]$$

and recall that $H^*(B\mathrm{GL}_\infty(\mathbb{Z});\mathbb{Q}) \cong \mathbb{F}(B)$ by Borel's famous calculation [5]. The action of the block diffeomorphism group on $H_n(U^n_{g,1};\mathbb{Z}) \cong \mathbb{Z}^g$ gives a map

$$\mathbb{F}(B) \cong H^*(B\mathrm{GL}_\infty(\mathbb{Z});\mathbb{Q}) \to H^*(B\mathrm{GL}_g(\mathbb{Z});\mathbb{Q}) \to H^*(B\widetilde{\mathrm{Diff}}_\partial(U^n_{g,1});\mathbb{Q}).$$

Combining this with $\tilde{\Phi}_{U^n_{g,1}}$, we obtain an algebra map

$$\Gamma : \mathbb{F}(B \oplus Z(2n+1)) \to H^*(B\widetilde{\mathrm{Diff}}_\partial(U^n_{g,1});\mathbb{Q}) \tag{1.7}$$

and we will prove the following result.

Theorem 1.8. *Let $n \geq 5$. The map Γ is surjective in degree $* \leq n-4$ and for $g \gg n$, and the kernel is spanned by the same elements as given in Theorem 1.3.*

Grey [29, Theorem B] has shown a homological stability result for block diffeomorphism groups which applies to the manifolds $U^n_{g,1}$ and can be used to give an explicit lower bound for g for which Theorem 1.8 holds (a worse bound can also be deduced from our proofs).

As already said, the proof of Theorem 1.8 uses Quinn's theory, which roughly expresses the homotopy fibre of the forgetful map $B\widetilde{\mathrm{Diff}}_\partial(M) \to B\mathrm{hAut}_\partial(M)$ in terms of L-theory which is quite manageable for simply connected M. We refer to [2] for a more informative survey; more importantly, that paper contains a consequence of Quinn's theory [2, Theorem 1.1] which allows us to use surgery theory completely as a black box. The result is that the calculation of $H^*(B\widetilde{\mathrm{Diff}}_\partial(U^n_{g,1});\mathbb{Q})$ is equivalent to the calculation of

$$H^*(\mathrm{map}_*(U^n_{g,1};BO_\mathbb{Q})^0 /\!\!/ \mathrm{hAut}_\partial(U^n_{g,1});\mathbb{Q}), \tag{1.9}$$

up to some smallprint that we shall ignore for the moment ($\mathrm{map}_*(U^n_{g,1};BO_\mathbb{Q})^0$ is the component of the pointed mapping space containing the constant map). The computation of (1.9) fills the largest portion of this paper. The component group $\pi_0(\mathrm{hAut}_\partial(U^n_{g,1}))$ is very close to $\mathrm{GL}_g(\mathbb{Z})$ so that naturally Borel's work on the cohomology of arithmetic group enters. This is one of the reasons why g needs to be large in Theorem 1.8. The cohomology of mapping spaces such as $\mathrm{map}_*(U^n_{g,1};BO_\mathbb{Q})^0$ is fairly easy to compute, but rather large. To get from there to Theorem 1.8, we employ Borel's vanishing theorem in a similar way to its use in [16] or [45], and a calculation in classical invariant theory.

While in principle the general theory allows us to compute the cohomology of block diffeomorphisms in arbitrary degrees, we ran into several difficulties which we could only resolve in small degrees (the main results of [2] about $H^*(B\widetilde{\mathrm{Diff}}_\partial(W^n_{g,1});\mathbb{Q})$ show that $H^*(B\widetilde{\mathrm{Diff}}_\partial(W^n_{g,1});\mathbb{Q})$ behaves very differently from $H^*(B\mathrm{Diff}_\partial(W^n_{g,1});\mathbb{Q})$ in high degrees, so that this is certainly to be expected). We invite the curious and capable reader to figure out $H^*(B\widetilde{\mathrm{Diff}}_\partial(U^n_{g,1});\mathbb{Q})$ in the homological stability range given by [29], or at least up to degree $2n-6$, which is the range of degrees in which the third step, the comparison of diffeomorphisms and block diffeomorphisms, is valid.

Let us now turn to this step. One feature is that the Borel classes coming from the action on homology vanish on $H^*(B\mathrm{Diff}_\partial(U^n_{g,1});\mathbb{Q})$. We deduce this from the Dwyer–Weiss–Williams index theorem [12] in Proposition 2.14. In general, the comparison of diffeomorphisms and block diffeomorphisms is in terms of pseudoisotopy theory and algebraic K-theory, with the stable h-cobordism theorem [72] and Igusa's stability theorem [40] as the main points; the last one enforces a bound depending on the dimension of the manifold. An elaborate formulation of this step was given by Weiss–Williams in [75], but a simpler variant suffices for us. Let us describe briefly how the comparison is done. Recall the classical result by Farrell–Hsiang [20] stating that

$$\pi_k(B\mathrm{Diff}_\partial(D^{2n+1}))\otimes\mathbb{Q} \cong \begin{cases}\mathbb{Q} & k\equiv 0 \pmod 4\\ 0 & k\not\equiv 0 \pmod 4\end{cases}$$

holds in a range of degrees. This range which is nowadays known to be roughly $2n$, by recent work of Krannich [43] and Krannich–Randal-Williams [42]. An instance of Morlet's lemma of disjunction states that the homotopy fibre of

$$B\mathrm{Diff}_\partial(U^n_{g,1}) \to B\widetilde{\mathrm{Diff}}_\partial(U^n_{g,1}) \tag{1.10}$$

is rationally equivalent to $B\mathrm{Diff}_\partial(D^{2n+1})$ up to degree approximately $2n$, and the main result of [14] (which is a consequence of [7] and [57]) says that the inclusion of the homotopy fibre into the total space of (1.10) induces the trivial map on rational homology in a range of degrees. To combine those facts, we use an elementary argument borrowed from [43] to show that the fibration (1.10) is "plus-constructible".

Our treatment of this step is analogous to an argument in Krannich's paper [43]. He considers

$$B\mathrm{Diff}_D(V_g^n) \to B\widetilde{\mathrm{Diff}}_D(V_g^n) \tag{1.11}$$

instead, where V_g^n is the boundary connected sum of g copies of $S^n \times D^{n+1}$, and D is a fixed disc in ∂V_g^n. The homotopy fibre of (1.11) is also rationally equivalent to $B\mathrm{Diff}_\partial(D^{2n+1})$. Krannich used knowledge about $B\mathrm{Diff}_D(V_g^n)$ from [7] and $B\widetilde{\mathrm{Diff}}_D(V_g^n)$ from surgery theory to deduce knowledge about the homotopy fibre of (1.11) (and thereby extends the range in Farrell-Hsiang's theorem to roughly $2n$). In the present paper, the logic is reversed: we use knowledge about the base of (1.10) from surgery theory and the fibre from Farrell-Hsiang's theorem (or Krannich's improvement thereof) to deduce knowledge about the total space.

1.5. **Overview of the chapters.** To navigate the reader through this rather long paper, let us briefly describe the content of the chapters. §2: after setting up notation, we introduce the tautological classes and show in §2.3 the three vanishing theorems implied by Theorems A and 1.8 (one of the vanishing results is needed for the proof, the other two fall out as byproducts). In §2.4, we prove another vanishing result, namely Proposition 2.14 which says that the Borel classes on $B\widetilde{\mathrm{Diff}}_\partial(U_{g,1}^n)$ vanish when pulled back to $B\mathrm{Diff}_\partial(U_{g,1}^n)$; this is a fairly straightfoward consequence of the Dwyer–Weiss–Williams theorem.

§3 is about the general theory behind the proof of Theorem 1.8. The goal is Proposition 3.27 which essentially gives a formula for the rational cohomology of $B\mathrm{Diff}_\partial(M)$ under some hypotheses on M, which are satisfied by $U_{g,1}^n$. The consitutents for this formula are $\mathrm{hAut}_\partial(M_\mathbb{Q})$, the homotopy automorphisms (relative to the boundary) of the rationalization of M, and the mapping space $\mathrm{map}_\partial(M; BO_\mathbb{Q})$. This is derived from surgery theory, through a result of Berglund–Madsen [2]. §3.1 reviews some generalities on rational homotopy theory, and §3.3 describes the cohomology of mapping spaces of the above sort.

In §4, we begin to apply this theory to the manifolds $U_{g,1}^n$. Many of the results are exercises in standard homotopy theory (and already contained in [29]). The goal is Proposition 4.30, which computes the E_2-page of the spectral sequence of the fibration $\mathrm{map}_\partial(U_{g,1}^n; BO_\mathbb{Q})^0 /\!\!/ \mathrm{hAut}_\partial(U_{g,1}^n)^{\mathrm{id}} \to B\mathrm{hAut}_\partial(U_{g,1}^n)^{\mathrm{id}}$ (in a range of degrees, and including the d_2-differential).

Having determined the E_2-page of the spectral sequence, our tactical goal is to calculate the $\pi_0(\mathrm{hAut}_\partial(U_{g,1}^n))$-*invariant* part of the E_∞-page. For this, we need a calculation in classical invariant theory which we carry out in the purely algebraic §5. In §6, we eventually complete the proof of Theorem 1.8, with a use of Borel's vanishing theorem [6]. The rather short section §7 derives Theorem A from Theorem 1.8 along the lines sketched above.

Notations. We use the following standard notations in this paper. For two $\mathbb{N}_0$-graded algebras A and B, $A \otimes B$ always denotes the *graded* tensor product, to conform the conventions of homological algebra. The free graded-commutative algebra generated by a graded vector space V is denoted $\mathbb{F}(V)$. If V is a vector space and $n \in \mathbb{N}_0$, we let $V[n]$ be the graded vector space which is V in degree n and 0 otherwise (and not the degree shift of a graded vector space).

1.6. **Acknowledgements.** The authors would like to thank Alexander Berglund, Fabian Hebestreit, Lutz Hille, Manuel Krannich, Sander Kupers, Thomas Nikolaus and Oscar Randal-Williams for helpful conversations about various aspects of this work. Last but not least, it is a pleasure to thank Jerzy Weyman. Without his help, we would not have been able to carry out the crucial invariant-theoretic calculation in Proposition 5.7, and would not have been able to get this project to a conclusion.

2. Characteristic classes of smooth and block bundles

2.1. **Automorphism groups.** Let us first establish some notation. Let M^d be a compact oriented smooth manifold with boundary. We write $\mathrm{Diff}(M)$ for the diffeomorphism group and $\mathrm{Diff}_\partial(M) \subset \mathrm{Diff}(M)$ for the subgroup of diffeomorphisms which are the identity near ∂M. We furthermore let $\mathrm{Diff}^+(M) \subset \mathrm{Diff}(M)$ and $\mathrm{Diff}^+_\partial(M) \subset \mathrm{Diff}_\partial(M)$ the subgroups of orientation-preserving diffeomorphisms; note that $\mathrm{Diff}^+_\partial(M) = \mathrm{Diff}_\partial(M)$ if the inclusion $\partial M \to M$ is 0-connected.

In the present paper, we make heavy use of the *block diffeomorphism* group $\widetilde{\mathrm{Diff}}(M)$. We won't repeat the definition here and refer instead to §1 and 2 of [43] for an up-to-date exposition. There are block analogues $\widetilde{\mathrm{Diff}}_\partial(M)$, $\widetilde{\mathrm{Diff}}^+(M)$, $\widetilde{\mathrm{Diff}}^+_\partial(M)$ of respective diffeomorphism groups. Let us also note that the natural map

$$I : \mathrm{Diff}(M) \to \widetilde{\mathrm{Diff}}(M)$$

is *by definition* 0-connected; the same holds for the decorated versions.

We shall need various flavours of homotopy automorphism groups. When forming mapping spaces, we secretly replace all spaces that occur by their singular simplicial set, and view the mapping space as a simplicial set. For a CW-pair (X, A) and a pointed space Y, we let $\mathrm{map}_A(X; Y)$ be the space of maps $X \to Y$ whose restriction to A is the constant map to the basepoint in Y (or more formally the fibre of the restriction map $\mathrm{map}(X; Y) \to \mathrm{map}(A; Y)$ over the constant map). For a CW-pair (X, A), we let $\mathrm{hAut}_A(X)$ be the monoid of all homotopy self-equivalences of X which are the identity on A. For an oriented compact manifold, we let also $\mathrm{hAut}_\partial(M)^+ \subset \mathrm{hAut}_\partial(M)$ the submonoid of those self-equivalences which preserve the fundamental homology class. There are natural maps

$$\widetilde{\mathrm{Diff}}_\partial(M) \to \mathrm{hAut}_\partial(M) \text{ and } \widetilde{\mathrm{Diff}}^+_\partial(M) \to \mathrm{hAut}^+_\partial(M)$$

(or rather a zig-zag, see the discussion in [2, p. 98 f] for more details).

Assume that $V \to X$ is a vector bundle and that $C \subset A$ is a subcomplex. We let $\mathrm{hAut}^C_A(V)$ be the monoid of all pairs $(f, \hat{f})$ where $f \in \mathrm{hAut}_A(X)$ and $\hat{f} : V \to V$ is a bundle map covering f which is fibrewise an isomorphism, and such that $f|_{V|_C}$ is the identity, see [2, p. 107f] for more details. There is a stable version of that construction given on p. 110 loc.cit.; we define

$$\mathrm{hAut}^C_A(V)^{\mathrm{st}} := \mathrm{colim}_k \mathrm{hAut}^C_A(V \oplus \mathbb{R}^k).$$

All the monoids we just introduced are grouplike E_1-spaces and therefore admit classifying spaces. The classifying space $B\mathrm{hAut}^C_A(V)^{\mathrm{st}}$ has a convenient description as follows. If $V|_C$ is stably trivial and a stable trivialization is chosen, Proposition 4.13 of [2] provides a weak equivalence

$$B\mathrm{hAut}^C_A(V)^{\mathrm{st}} \simeq (\mathrm{map}_C(X; BO) /\!\!/ \mathrm{hAut}_A(X))_V, \tag{2.1}$$

where

$$(\mathrm{map}_C(X;BO) /\!\!/ \mathrm{hAut}_A(X))_V \subset \mathrm{map}_C(X;BO) /\!\!/ \mathrm{hAut}_A(X)$$

denotes the connected component determined by a fixed classifying map $\lambda : X \to BO$ of V which extends the given trivialization of $V|_C$, viewed as a point in $\mathrm{map}_C(X;BO)$. The map in (2.1) arises as follows: the total space of the universal fibration over $B\mathrm{hAut}_A^C(V)^{\mathrm{st}}$ with fibre X carries a stable vector bundle which is built from V. For a detailed construction on the point-set level, we refer to [2].

One important feature of block diffeomorphisms is the existence of the *derivative map*

$$D : \widetilde{\mathrm{Diff}}_\partial(M) \to \mathrm{hAut}_\partial^\partial(TM)^{\mathrm{st}}, \tag{2.2}$$

which is a map of E_1-monoids and can therefore be delooped to a map

$$BD : B\widetilde{\mathrm{Diff}}_\partial(M) \to B\mathrm{hAut}_\partial^\partial(TM)^{\mathrm{st}}. \tag{2.3}$$

The derivative map is constructed in [2, §4.3], see also [43, §1.9], and is an expression of the fact, first proven in [15] and expanded on in [36, §2.4], that block bundles have a stable vertical tangent bundle. By virtue of its definition, the derivative map is a map over $\mathrm{hAut}_\partial(M)$.

2.2. **Tautological classes.** Tautological classes (aka Miller–Morita–Mumford classes or κ-classes) for block bundles have been constructed in [15] and more systematically in [36]. The most streamlined construction can be given using the derivative map (2.3), and we sketch the definition briefly, in a level of generality that will prove to be useful for us later on.

Construction 2.4. Assume that M^d is a compact oriented smooth manifold with boundary. Consider the universal fibration pair $(E, \partial E)$ with fibre $(M, \partial M)$ over the space $B := \mathrm{map}_\partial(M;BO) /\!\!/ \mathrm{hAut}_\partial^+(M)$. The evaluation map

$$\mathrm{ev} : \mathrm{map}_\partial(M;BO) \times M \to BO$$

is $\mathrm{hAut}_\partial(M)$-equivariant and maps $\mathrm{map}_\partial(M;BO) \times \partial M$ to the basepoint. As $E = (\mathrm{map}_\partial(M;BO) \times M) /\!\!/ \mathrm{hAut}_\partial^+(M)$ and $\partial E = (\mathrm{map}_\partial(M;BO) \times \partial M) /\!\!/ \mathrm{hAut}_\partial^+(M)$, ev induces a map $\epsilon : E \to BO$ sending ∂E to the basepoint. We may think of ϵ as a stable vector bundle on E, trivialized on ∂E. The Leray–Serre spectral sequence for the fibration pair $(E, \partial E) \to B$ yields maps

$$\pi_! : H^k(E;\partial E) \to E_\infty^{k-d,d} \subset E_2^{k-d,d} \cong H^{k-d}(B;H^d(M;\partial M)) \to H^{k-d}(B)$$

(with coefficients in an arbitrary ring). Given a class $c \in H^k(BO)$, we can therefore form

$$\kappa_c := \pi_!(\epsilon^* c) \in H^{k-d}(\mathrm{map}_\partial(M;BO) /\!\!/ \mathrm{hAut}_\partial^+(M)).$$

Construction 2.5. Let M^d be an oriented compact smooth manifold with boundary and assume that $TM|_{\partial M}$ is stably trivial. Combining the derivative map BD with (2.1) yields a map

$$B\widetilde{\mathrm{Diff}}_\partial^+(M) \to B\mathrm{hAut}_\partial^\partial(TM)^{\mathrm{st}} \to \mathrm{map}_\partial(M;BO) /\!\!/ \mathrm{hAut}_\partial^+(M),$$

and we can pull back κ_c to a class, also denoted $\kappa_c \in H^{k-d}(B\widetilde{\mathrm{Diff}}_\partial^+(M))$. Pulling this further back along $BI : B\mathrm{Diff}_\partial(M) \to B\widetilde{\mathrm{Diff}}_\partial(M)$, we obtain the usual κ-classes on classifying spaces of diffeomorphism groups.

A map γ from a space X to one of the classifying spaces $B\mathrm{Diff}^+_\partial(M)$, $B\widetilde{\mathrm{Diff}}^+_\partial(M)$ or $\mathrm{map}_\partial(M;BO) /\!\!/ \mathrm{hAut}^+_\partial(M)$ classifies $E \to X$, which is a smooth fibre bundle / block bundle / fibration with a stable vector bundle on its total space. In such situations, we use the suggestive notation $\kappa_c(E) := \gamma^*\kappa_c$.

We now specialize to rational coefficients. Recall that $H^*(BO;\mathbb{Q})$ is the polynomial algebra in the Pontrjagin classes. For our purposes, it is useful to observe that one can also write

$$H^*(BO;\mathbb{Q}) \cong \mathbb{Q}[L_1, L_2, \ldots],$$

where $L_m \in H^{4m}(BO;\mathbb{Q})$ is the mth component of the Hirzebruch L-class. The above is true by the well-known fact [39, p. 14] that the coefficient a_m of p_m in L_m is nonzero.

2.3. **Some vanishing theorems for tautological classes.**

In this subsection, we review the three vanishing theorems that are entailed by Theorem A. Only one of them (Theorem 2.8) is actually used in the proof of our main theorem; the other two (Proposition 2.7 and Proposition 2.10) are only stated for sake of completeness; the fact that they are valid for the manifolds $U^n_{g,1}$ is a byproduct of our computations below.

An additivity property. Let us begin with a fact which is well-known for diffeomorphism groups; the argument we give is essentially contained in [52], [54].

Lemma 2.6. *Let M and N be compact oriented d-manifolds with boundary with a common (closed) part $\partial_0 \subset \partial M, \partial N$ of their boundary. Let*

$$\mu : \mathrm{map}_\partial(M;BO) /\!\!/ \mathrm{hAut}^+_\partial(M) \times \mathrm{map}_\partial(N;BO) /\!\!/ \mathrm{hAut}^+_\partial(N) \to$$
$$\to \mathrm{map}_\partial(M \cup_{\partial_0} N;BO) /\!\!/ \mathrm{hAut}^+_\partial(M \cup_{\partial_0} N)$$

be the obvious gluing map, let p_M be the projection to $\mathrm{map}_\partial(M;BO) /\!\!/ \mathrm{hAut}^+_\partial(M)$, and define p_N similarly. Then for each $c \in H^k(BO)$ with $k > d$, we have

$$\mu^*\kappa_c = p_M^*\kappa_c + p_N^*\kappa_c.$$

Proof. We must show the following: assume that $\pi^E : E \to X$ and $\pi^F : F \to X$ are two oriented fibrations with fibres M and N and trivialized boundaries, and that E and F contain a common part $A \subset \partial E, \partial F$ of their boundary. Then for two vector bundles $V \to E$ and $W \to F$ of the same rank n which are trivialized over the respective boundaries and each $c \in H^k(BO)$, we have

$$\pi^{E\cup_A F}_!(c(V \cup W)) = \pi^E_!(c(V)) + \pi^F(c(W)) \in H^{k-d}(X),$$

provided that $k - d > 0$. We may suppose that X is a finite CW complex. It is an exercise in linear algebra to prove an isomorphism

$$(V \cup_A \underline{\mathbb{R}^n}_F) \oplus (\underline{\mathbb{R}^n}_E \cup_A W) \cong (V \cup_A W) \oplus (\underline{\mathbb{R}^n}_E \cup_A \underline{\mathbb{R}^n}_F)$$

of clutched bundles on $E \cup_A F$ (hint: picking bundle maps $f : \underline{\mathbb{R}^n} \to V$ and $g : \underline{\mathbb{R}^n} \to W$ which are the identity on the boundary is a good start). Hence the classifying map $\gamma_{E\cup_A F} : E \cup_A F \to BO$ of $V \cup_A W$ can be factored as

$$E \cup_A F \to E/\partial \vee F/\partial \stackrel{\gamma_E \vee \gamma_F}{\to} BO \vee BO \stackrel{\mathrm{fold}}{\to} BO.$$

The claim follows from the observation that

$$H^*(E,\partial E) \cong H^*(E \cup_A F, \partial E \cup F) \to H^*(E \cup_A F, \partial(E \cup_A F)) \stackrel{\pi^{E\cup_A F}_!}{\to} H^{*-d}(X)$$

is nothing else than $\pi_!^E$ (and the similar fact for F in place of E). □

The next result is fairly simple-minded; compare [24, Lemma 7.16] for a more elaborate, but closely related result about diffeomorphism groups.

Proposition 2.7. *Let M be a compact oriented d-manifold, assume that $TM|_{\partial M}$ is stably trivial, that $\partial M \to M$ is $(k-1)$-connected and assume that all rational Pontrjagin classes of TM up to degree k are zero. Then*

$$\kappa_c = 0 \in H^{|c|-d}(B\widetilde{\mathrm{Diff}}_\partial^+(M);\mathbb{Q})$$

whenever $c \in H^*(BO;\mathbb{Q})$ with $d < |c| \leq k$.

Proof. We use an easy cohomological argument as follows. The cohomological Leray–Serre spectral sequence for the universal block bundle pair $(E,\partial E)$ over $(B\widetilde{\mathrm{Diff}}_\partial^+(M),*)$ reads as follows:

$$E_2^{p,q} = H^p(B\widetilde{\mathrm{Diff}}_\partial^+(M),*;H^q(M,\partial M;\mathbb{Q})) \Rightarrow H^{p+q}(E, M\cup\partial E).$$

The connectivity assumption on M prove that $H^*(E,M\cup\partial E)=0$ whenever $*\leq k$. Hence $H^*(E)\to H^*(M)\oplus H^*(\partial E)$ is injective for $*\leq k$. The Pontrjagin classes of T_vE go to $(p(TM),\mathrm{pr}_{\partial M}^*p(TM))=0$ and therefore vanish in degrees $\leq k$. Hence $c(T_vE)=0$, and a fortiori $\kappa_c=0$. □

The family signature theorem. The classical family signature theorem for smooth fibre bundles (which uses elliptic operators in its proof) holds more generally for block bundles, as shown by Randal–Williams in [62]. It has two cases, the odd-dimensional and the even-dimensional case. The odd case reads as follows.

Theorem 2.8. *Let M^d be an odd-dimensional oriented manifold, and assume for simplicity that $TM|_{\partial M}$ is stably trivial. Then*

$$\kappa_{L_m} = 0 \in H^{4k-d}(B\widetilde{\mathrm{Diff}}_\partial^+ M;\mathbb{Q})$$

for each $m\in\mathbb{N}$ (and hence the same is true for diffeomorphisms).

References. By Lemma 2.6, it is enough to show the theorem for closed M, and this is done in [62, Theorem 3.1] for block diffeomorphisms, and in [13] for diffeomorphisms. The proof in the latter paper uses families of elliptic operators and cannot be generalized to block diffeomorphisms. □

Let us state the even-dimensional case for sake of completeness, and only for the case where M is closed or ∂M is a sphere. The point is that in the case $\dim(M)=2n$, the action of $\mathrm{hAut}_\partial^+(M)$ on $H_n(M;\mathbb{Q})$ preserves the (nondegenerate) intersection form I_M. This fact produces a map $f:\mathrm{hAut}_\partial^+(M)\to B\mathrm{Aut}(H_n(M;\mathbb{Q});I_M)$; the latter is the classifying space of a symplectic or an orthogonal group, depending on the parity of n. Randal–Williams defines classes $\sigma_i\in H^i(B\mathrm{Aut}(H_n(M;\mathbb{Q});I_M);\mathbb{Q})$, which live in degrees $i\equiv 2 \pmod 4$ if n is odd and $i\equiv 0 \pmod 4$ if n is even, and shows in [62, Theorem 3.1] that

$$\kappa_{L_m} = f^*\sigma_{4m-2n} \in H^{4m-2n}(B\widetilde{\mathrm{Diff}}_\partial^+(M);\mathbb{Q}); \tag{2.9}$$

he uses index theory to identify these classes in terms of ordinary characteristic classes, see [62, Theorem 4.1].

Consequence of the family signature theorem.

Proposition 2.10. *Let M be a compact oriented d-manifold, assume that $TM|_{\partial M}$ is stably trivial, and suppose that all rational Pontrjagin classes of M are trivial. Then for each $c \in H^k(BO;\mathbb{Q})$, $k > \dim(M)$, the homomorphism*

$$\pi_{k-d}(B\widetilde{\mathrm{Diff}}^+_\partial(M)) \to \mathbb{Q}$$

given by

$$[f] \mapsto \langle f^*\kappa_c; [S^{k-d}]\rangle$$

is the zero map. In particular, if $k = d+1$, then $\kappa_c = 0 \in H^1(B\widetilde{\mathrm{Diff}}^+_\partial(M);\mathbb{Q})$.

Proof. For diffeomorphisms, this is a well-known fact, see e.g. [44, Proposition 13] or [32, Proposition 1.9]. To see that the proof also applies to block diffeomorphisms, we review the argument.

Firstly, the double $M \cup_{\partial M} M$ has trivial rational Pontrjagin classes, by an argument given in the proof of [17, Theorem F]; hence by Lemma 2.6 it suffices to give the argument for closed M.

Let $k = 4m$ and let $\pi : E \to S^{4m-d}$ be an oriented block bundle with fibre M, classified by $f : S^{4m-d} \to B\widetilde{\mathrm{Diff}}^+(M)$. The restriction of the stable vertical tangent bundle T_vE to the fibre M over $*$ is stably isomorphic to TM. A brief inspection of the Leray–Serre spectral sequence of π proves that $c(T_vE) = 0$ if c can be written as a product of classes in positive degrees. Hence we only have to prove that $\kappa_{L_m}(E) = 0$. This follows from [62, Theorem 3.1] (in the even-dimensional case, we only need to consider bundles over spheres of even dimension, which are simply-connected and (2.9) shows that κ_{L_m} is pulled back from an asperical space).

The last sentence follows by the Hurewicz theorem. □

2.4. **Borel classes.** Let $K(\mathbb{Z})$ denote the algebraic K-theory spectrum of the integers; recall that $\pi_0(K(\mathbb{Z})) \cong \mathbb{Z}$ and that $\Omega^\infty_0 K(\mathbb{Z}) \simeq B\mathrm{GL}_\infty(\mathbb{Z})^+$. A celebrated result of Borel describes the rational cohomology of the latter space.

Theorem 2.11 (Borel)**.** *The rational cohomology $H^*(\Omega^\infty_0 K(\mathbb{Z});\mathbb{Q}) = H^*(B\mathrm{GL}_\infty(\mathbb{Z});\mathbb{Q})$ is the exterior algebra with generators $\beta_{4k+1} \in H^{4k+1}(\Omega^\infty_0 K(\mathbb{Z});\mathbb{Q})$, $k \geq 1$. The classes β_{4k+1} are primitive. The restriction maps*

$$H^p(B\mathrm{GL}_\infty(\mathbb{Z});\mathbb{Q}) \to H^p(B\mathrm{GL}_g(\mathbb{Z});\mathbb{Q}) \to H^p(B\mathrm{SL}_g(\mathbb{Z});\mathbb{Q})$$

are isomorphisms provided that $g \geq 2p+2$. The group homomorphism $\kappa : \mathrm{GL}_g(\mathbb{Z}) \to \mathrm{GL}_g(\mathbb{Z})$ given by $\kappa(x) := (x^\top)^{-1}$ has the following effect on these classes:

$$(B\kappa)^*\beta_{4k+1} = -\beta_{4k+1}.$$

References. The first part is of course a famous theorem of Borel [5] (he treats real cohomology which makes little difference as the cohomology spaces are all finite-dimensional by [60]). A range in which the map from the stable cohomology to the unstable cohomology is an isomorphism is also determined in Borel's paper; the range stated above follows from Van der Kallen's homological stability theorem [71, Theorem 4.11], using that the Bass stable rank of the integers is 2; see [31, §4.1.11]. The last statement can also easily be deduced from Borel's work. Since we do not know a reference, we shall indicate the proof here. It suffices to prove the statement for $\mathrm{SL}_g(\mathbb{Z})$ instead of $\mathrm{GL}_g(\mathbb{Z})$, since the covering map $B\mathrm{SL}_g(\mathbb{Z}) \to B\mathrm{GL}_g(\mathbb{Z})$ induces an injection in rational cohomology. We need to recall how the Borel classes are constructed. Let X be the symmetric space $\mathrm{SL}_g(\mathbb{R})/SO(g)$ and let $\mathcal{A}^*(X)^{\mathrm{SL}_g(\mathbb{R})}$ be the chain complex of invariant differential forms, which has trivial differential as each $\mathrm{SL}_g(\mathbb{R})$-invariant differential form on X is closed, by a general fact about symmetric spaces. On the other hand, X is contractible and the $\mathrm{SL}_g(\mathbb{Z})$-action is proper, so that there is a natural isomorphism

$$H^*(B\mathrm{SL}_g(\mathbb{Z});\mathbb{R}) \cong H^*(\mathcal{A}^*(X)^{\mathrm{SL}_g(\mathbb{Z})}).$$

On X, there is the Cartan involution $\tau : X \to X$, $\tau(xSO(g)) := (x^\top)^{-1}SO(g)$. It is easily verified that τ induces an involution on $\mathcal{A}^*(X)^{\mathrm{SL}_g(\mathbb{R})}$, and that the diagram

$$\begin{array}{ccc} \mathcal{A}^*(X)^{\mathrm{SL}_g(\mathbb{R})} & \longrightarrow & H^*(B\mathrm{SL}_g(\mathbb{Z});\mathbb{R}) \\ \big\downarrow{\scriptstyle \tau^*} & & \big\downarrow{\scriptstyle B\iota^*} \\ \mathcal{A}^*(X)^{\mathrm{SL}_g(\mathbb{R})} & \longrightarrow & H^*(B\mathrm{SL}_g(\mathbb{Z});\mathbb{R}) \end{array}$$

commutes. By definition, the Borel classes come from certain invariant forms on X. It is therefore enough to show that $\tau^* : \mathcal{A}^p(X)^{\mathrm{SL}_g(\mathbb{R})} \to \mathcal{A}^p(X)^{\mathrm{SL}_g(\mathbb{R})}$ is multiplication by $(-1)^p$. The involution τ fixes the basepoint $o := SO(g) \in X$, and since invariant forms on X are determined by their values at x, it is enough to check that $T_o\tau = -1$. But this is easily verified by a direct calculation. $\square$

Now let X be a finite CW-complex. For each p, the action on $H_p(X;\mathbb{Z})$ provides a map

$$B\rho_p : B\mathrm{hAut}(X) \to B\mathrm{GL}(H_p(X;\mathbb{Z})).$$

For each finitely generated abelian group A with torsion subgroup TA, there is a map

$$\iota : B\mathrm{GL}(A) \to B\mathrm{GL}(A/TA) \to \Omega^\infty K(\mathbb{Z})$$

well defined up to homotopy; it hits the component of $\mathrm{rank}(A) \in \mathbb{Z} = \pi_0(K(\mathbb{Z}))$. Composing ι and $B\rho_p$ gives classes

$$\beta^p_{4k+1} := (\iota \circ B\rho_p)^*\beta_{4k+1} \in H^{4k+1}(B\mathrm{hAut}(X);\mathbb{Q}).$$

Using the infinite loop space structure on $\Omega^\infty K(\mathbb{Z})$, we can form the alternating sum

$$\chi := \sum_{p\geq 0} (-1)^p \iota \circ B\rho_p : B\mathrm{hAut}(X) \to \Omega^\infty K(\mathbb{Z}),$$

the *algebraic K-theory Euler characteristic*. Because the Borel classes are primitive, the relation

$$\chi^*\beta_{4k+1} = \sum_{p\geq 0} (-1)^p \beta^p_{4k+1} \in H^{4k+1}(B\mathrm{hAut}(X);\mathbb{Q})$$

holds.

Theorem 2.12 (Dwyer–Weiss–Williams)**.** *Let M be a smooth compact manifold, possibly with boundary. Then*

$$\chi^*\beta_{4k+1} = 0 \in H^{4k+1}(B\mathrm{Diff}(M);\mathbb{Q}).$$

Let us remark that the analogue of Theorem 2.12 for block diffeomorphism groups is *false*; in fact our computation of $H^*(B\widetilde{\mathrm{Diff}}_\partial(U^n_{g,1});\mathbb{Q})$ certifies its failure.

References. The Dwyer–Weiss–Williams index theorem [12, Corollary 8.12] shows that the map χ factors through the Becker–Gottlieb transfer $B\mathrm{Diff}(M) \to QS^0$, so that the result simply follows from Serre's finiteness theorem. See also §2 of [14] for a more detailed summary. □

We now use Poincaré duality to deduce a sharper vanishing theorem from Theorem 2.12.

Lemma 2.13. *Let M^d be a connected smooth oriented manifold with boundary, and suppose that $\partial M = S^{d-1}$. Then*

$$\beta^p_{4k+1} = -\beta^{d-p}_{4k+1} \in H^{4k+1}(B\mathrm{Diff}^+_\partial(M); \mathbb{Q})$$

for all p.

Proof. We can consider $\hat{M} := M \cup_{\partial M} D^d$ instead, without changing the Borel classes. Poincaré duality, the universal coefficient theorem and the last statement of Theorem 2.11 proves the claim. □

Proposition 2.14. *Let M be as in Lemma 2.13. If $d = 2n$, then*

$$\beta^n_{4k+1} = 0 \in H^{4k+1}(B\mathrm{Diff}^+_\partial(M); \mathbb{Q}).$$

If $d = 2n+1$, then

$$\sum_{p=0}^{n} (-1)^p \beta^p_{4k+1} = 0 \in H^{4k+1}(B\mathrm{Diff}^+_\partial(M); \mathbb{Q}).$$

Proof. For $d = 2n$, compute

$$0 \overset{(2.12)}{=} \sum_{p=0}^{n-1} \Big((-1)^p \beta^p_{4k+1} + (-1)^{2n-p} \beta^{2n-p}_{4k+1} \Big) + (-1)^n \beta^n_{4k+1} =$$

$$\overset{(2.13)}{=} \sum_{p=0}^{n-1} (-1)^p \Big(\beta^p_{4k+1} - \beta^p_{4k+1} \Big) + (-1)^n \beta^n_{4k+1} = (-1)^n \beta^n_{4k+1}.$$

For $d = 2n+1$, compute

$$0 \overset{(2.12)}{=} \sum_{p=0}^{n} \Big((-1)^p \beta^p_{4k+1} + (-1)^{2n+1-p} \beta^{2n+1-p}_{4k+1} \Big) =$$

$$\overset{(2.13)}{=} \sum_{p=0}^{n} \Big((-1)^p \beta^p_{4k+1} + (-1)^{2n+1-p+1} \beta^p_{4k+1} \Big) =$$

$$2 \sum_{p=0}^{n} (-1)^p \beta^p_{4k+1}.$$

□

Remark 2.15. The even-dimensional case of Proposition 2.14 can be shown directly from Borel's work, without recourse to [12]. The point is that by Poincaré duality, $B\rho_n$ factors through the symplectic group or through an orthogonal group of some signature, depending on the parity of n, and Borel also computed the stable rational cohomologies of such groups: they are concentrated in even degrees.

3. Rational cohomology of block diffeomorphism spaces: general theory

3.1. **Some words about rational homotopy theory.** Let us recall some notions and results from rational homotopy theory. For us, a *space* will be a Kan complex. The category of spaces is denoted $\mathcal{S}$, and the category of pointed spaces by $\mathcal{S}_*$. Recall that the category $\mathcal{S}$ is enriched over itself.

We say that a space X is *finite* if the geometric realization $|X|$ is homotopy equivalent to a finite CW complex.

A map $f: X \to Y$ is a $H\mathbb{Q}$*-equivalence* if the induced map $f_*: H_*(X, \mathbb{Q}) \to H_*(Y, \mathbb{Q})$ is an isomorphism. When all path components of X and Y are nilpotent (e.g. simply connected or simple), this requirement is equivalent to saying that $f_*: \pi_0(X) \to \pi_0(Y)$ is bijective and that $f_*: \pi_k(X, x) \otimes \mathbb{Q} \to \pi_k(Y, f(x)) \otimes \mathbb{Q}$ is an isomorphism for all $k \geq 1$ and all $x \in X$ (for an arbitrary nilpotent group G, we use the notation $G \otimes \mathbb{Q}$ for the $\mathbb{Q}$-localization of G, see [38, §I]). If source and target are nilpotent, we call an $H\mathbb{Q}$-equivalence also a *rational homotopy equivalence*.

A space Z is $\mathbb{Q}$*-local* if for each $H\mathbb{Q}$-equivalence $f: X \to Y$ and all choices of basepoints, the map

$$\_ \circ f : \mathrm{map}_*(Y; Z) \to \mathrm{map}_*(X; Z)$$

is a weak equivalence (equivalently the map induced by $\_ \circ f$ on π_0 is bijective for all such f). If Z is nilpotent, this is equivalent to saying that all homotopy groups $\pi_k(Z, z)$ for $k \geq 2$ are $\mathbb{Q}$-vector spaces and that the fundamental groups $\pi_1(Z, z)$ are $\mathbb{Q}$-local nilpotent groups in the sense of [38, p.4].

A map $f: X \to Y$ is a $\mathbb{Q}$*-localization* if f is an $H\mathbb{Q}$-equivalence and Y is $\mathbb{Q}$-local. Such a map, if it exists, is unique up to weak equivalence. It was proven by Sullivan [67] that each nilpotent space admits a $\mathbb{Q}$-localization. We make use of the following fact, which follows from Theorem 2.5 on p.66 and Theorem 3.11 on p. 77 of [38].

Theorem 3.1. *Let X and Y be connected pointed spaces and assume that X is finite. Then each component $\mathrm{map}_*(X; Y)_g \subset \mathrm{map}_*(X; Y)$ of the pointed mapping space is nilpotent. Moreover, if Y is nilpotent and $f: Y \to Z$ is a $\mathbb{Q}$-localization, the induced map*

$$\mathrm{map}_*(X; Y)_g \to \mathrm{map}_*(X; Z)_{f \circ g}$$

is a rational homotopy equivalence, for each choice of g.

For our purposes, it will be convenient to have a strictly functorial $\mathbb{Q}$-localization[2].

Theorem 3.2. *There is an enriched functor $(\_)_\mathbb{Q} : \mathcal{S} \to \mathcal{S}$, together with an enriched natural transformation $\eta : \mathrm{id} \to (\_)_\mathbb{Q}$, such that for each $X \in \mathcal{S}$, the map $\eta_\mathbb{Q} : X \to X_\mathbb{Q}$ is a $\mathbb{Q}$-localization.*

Being enriched means that the functor comes along with natural maps

$$i_{X,Y} : \mathrm{map}(X; Y) \to \mathrm{map}(X_\mathbb{Q}; Y_\mathbb{Q}),$$

and the enriched natural transformation η is given by maps $\eta_X : X \to X_\mathbb{Q}$ such that the composition

$$\mathrm{map}(X; Y) \overset{i_{X,Y}}{\to} \mathrm{map}(X_\mathbb{Q}; Y_\mathbb{Q}) \overset{\_ \circ \eta_X}{\to} \mathrm{map}(X; Y_\mathbb{Q})$$

[2]The only place in the paper where this enters essentially is in Observation 4.24.

agrees with the map $\eta_Y \circ \_$. See [65, §3] for more details on the vocabulary of enriched category theory. There is an induced enriched functor $\mathcal{S}_* \to \mathcal{S}_*$ of pointed spaces.

References for Theorem 3.2. This is due to Bousfield; first in [8] without the word "enriched", the enrichment is constructed in [9, §5]. □

Lemma 3.3. *Let X and Z be connected pointed spaces and let $g : X \to Z$ be a pointed map. Then if X is finite, the natural map*

$$i^g_{X,Z} : \mathrm{map}_*(X; Z)^g \to \mathrm{map}_*(X_\mathbb{Q}; Z_\mathbb{Q})^{g_\mathbb{Q}}$$

is a rational homotopy equivalence.

Proof. The composition

$$\mathrm{map}_*(X; Z)^g \overset{i^g_{X,Z}}{\to} \mathrm{map}_*(X_\mathbb{Q}; Z_\mathbb{Q})^{g_\mathbb{Q}} \overset{\_\circ\eta_X}{\to} \mathrm{map}_*(X; Z_\mathbb{Q})^{g_\mathbb{Q}\circ\eta_X}$$

is equal to $\eta_Z \circ \_$. The three spaces are nilpotent by Theorem 3.1, and the second map is a weak equivalence since $Z_\mathbb{Q}$ is $\mathbb{Q}$-local and η_X is an $H\mathbb{Q}$-equivalence. The composition is a rational homotopy equivalence by Theorem 3.1, and so $i^g_{X,Z}$ is a rational homotopy equivalence as well. □

A similar fact is true for homotopy automorphisms. Rationalization gives maps

$$j_X : \mathrm{hAut}_*(X) \to \mathrm{hAut}_*(X_\mathbb{Q})$$

and

$$j_{X,A} : \mathrm{hAut}_A(X) \to \mathrm{hAut}_{A_\mathbb{Q}}(X_\mathbb{Q})$$

when X is a pointed space or (X, A) is a space pair. We write $\mathrm{hAut}_A(X)^{\mathrm{id}}$ for the unit component of $\mathrm{hAut}_A(X)$, and let moreover

$$\mathrm{hAut}_{A_\mathbb{Q}}(X_\mathbb{Q})_\mathbb{Z} \subset \mathrm{hAut}_{A_\mathbb{Q}}(X_\mathbb{Q})$$

be the union of all path components which are hit by $j_{X,A}$; this are clearly grouplike submonoids.

Lemma 3.4. *Suppose that (X, A) is a pair of finite spaces. Then the natural map*

$$j_{X,A} : \mathrm{hAut}_A(X)^{\mathrm{id}} \to \mathrm{hAut}_{A_\mathbb{Q}}(X_\mathbb{Q})^{\mathrm{id}}$$

is a rational homotopy equivalence.

Proof. The case $A = *$ is a special case of Lemma 3.3. In the general case, consider the commutative diagram

$$\begin{array}{ccc} \mathrm{hAut}_*(X)^{\mathrm{id}} & \xrightarrow{j_X} & \mathrm{hAut}_*(X_\mathbb{Q})^{\mathrm{id}} \\ \downarrow & & \downarrow \\ \mathrm{map}_*(A; X)^{\mathrm{inc}} & \xrightarrow{i^{\mathrm{inc}}_{A,X}} & \mathrm{map}_*(A_\mathbb{Q}; X_\mathbb{Q})^{\mathrm{inc}}. \end{array}$$

The horizontal maps are rational homotopy equivalences by Lemma 3.3. It follows that the induced map on vertical homotopy fibres induces an isomorphism on all rational homotopy groups in degrees ≥ 2. The same is true on fundamental groups, using [38, Proposition 1.10]. On the other hand, restricting the map on vertical fibres to the path component of the identity gives the map $j_{A,X}$. □

Having understood these matters, we usually abuse notation and write

$$\mathrm{hAut}_A(X_{\mathbb{Q}}) := \mathrm{hAut}_{A_{\mathbb{Q}}}(X_{\mathbb{Q}})$$

and use the notation $\mathrm{hAut}_A(X_{\mathbb{Q}})_{\mathbb{Z}}$, $\mathrm{hAut}_A(X)^{\mathrm{id}}$ similarly.

3.2. **Block diffeomorphisms versus tangential homotopy automorphisms.** We shall use the surgery-theoretic approach to the topology of diffeomorphism groups which is due to Quinn [61]; a detailed exposition is available in [55]. For our purposes, work of Berglund and Madsen [2, §4] enables us to treat all the surgery theory as a black box. Let us describe the result we need, starting with the introduction of some more notation.

Notation 3.5.

(1) We write $\mathrm{Diff}_\partial(M)^{\sim\mathrm{id}} \subset \mathrm{Diff}_\partial(M)$ for the subgroup of diffeomorphisms which are homotopic to the identity (relative boundary), and define $\widetilde{\mathrm{Diff}}_\partial(M)^{\sim\mathrm{id}} \subset \widetilde{\mathrm{Diff}}_\partial(M)$ analogously.

(2) Assume that $C \subset A \subset X$ are subcomplexes and that $V \to X$ is a vector bundle. In that situation, we denote by

$$\mathrm{hAut}_A^C(V)^{\sim\mathrm{id}} \subset \mathrm{hAut}_A^C(V)$$

the preimage of $\mathrm{hAut}_A(X)^{\mathrm{id}}$ under the forgetful map $\mathrm{hAut}_A^C(V) \to \mathrm{hAut}_A(X)$; in other words the space of pairs $(f, \hat{f})$ with $f \sim \mathrm{id}$ (relative A). We define $\mathrm{hAut}_A^C(V)^{\mathrm{st},\sim\mathrm{id}} \subset \mathrm{hAut}_A^C(V)^{\mathrm{st}}$ similarly.

The derivative map (2.2) is, by virtue of its definition, a map over $\mathrm{hAut}_\partial(M)$, and hence it restricts to a map

$$D : \widetilde{\mathrm{Diff}}_\partial(M)^{\sim\mathrm{id}} \to \mathrm{hAut}_\partial^\partial(TM)^{\mathrm{st},\sim\mathrm{id}}.$$

If $* \in \partial M$ is a basepoint, we can furthermore compose the derivative map with the forgetful map $\mathrm{hAut}_\partial^\partial(TM)^{\mathrm{st}} \to \mathrm{hAut}_\partial^*(TM)^{\mathrm{st}}$. Hence by restriction and taking classifying spaces, we obtain a map

$$BD : B\widetilde{\mathrm{Diff}}_\partial(M)^{\sim\mathrm{id}} \to B\mathrm{hAut}_\partial^*(TM)^{\mathrm{st},\sim\mathrm{id}}. \tag{3.6}$$

All the surgery theory we need enters the proof of the following result.

Theorem 3.7. [2, Theorem 1.1] *Assume that M^d is 1-connected, $\partial M = S^{d-1}$ and that $d \geq 5$. Then the spaces $B\widetilde{\mathrm{Diff}}_\partial(M)^{\sim\mathrm{id}}$ and $B\mathrm{hAut}_\partial^*(TM)^{\mathrm{st},\sim\mathrm{id}}$ are nilpotent, and the map BD from (3.6) is a rational homotopy equivalence.*

In the rest of this subsection, we derive a version of Theorem 3.7 which involves the full block diffeomorphism group and not just $\mathrm{Diff}_\partial(M)^{\sim\mathrm{id}}$ and which is directly applicable to the manifolds $U_{g,1}^n$. Our goal is Theorem 3.12 below.

We begin with the introduction of some more pieces of notation. We let

$$\mathrm{hAut}_\partial(M)^{\mathrm{id}} \subset \mathrm{hAut}_\partial(M)^{\cong} \subset \mathrm{hAut}_\partial(M)$$

be the union of the components which are hit by the forgetful map $\widetilde{\mathrm{Diff}}_\partial(M) \to \mathrm{hAut}_\partial(M)$, and we define for a subcomplex $C \subset \partial M$

$$\mathrm{hAut}_\partial^C(TM)^{\mathrm{st},\cong} \subset \mathrm{hAut}_\partial^C(TM)^{\mathrm{st}}$$

as the preimage of $\mathrm{hAut}_\partial(M)^{\cong}$ under the forgetful map. Using this notation, the derivative map yields

$$BD : B\widetilde{\mathrm{Diff}}_\partial(M) \to B\mathrm{hAut}_\partial^\partial(TM)^{\mathrm{st},\cong} \to B\mathrm{hAut}_\partial^*(TM)^{\mathrm{st},\cong}. \tag{3.8}$$

Corollary 3.9. *Let M be as in Theorem 3.7. Then the composition 3.8 is an $H\mathbb{Q}$-equivalence.*

Proof. There is a diagram

$$\begin{array}{ccc}
B\widetilde{\mathrm{Diff}}_\partial(M)^{\sim \mathrm{id}} & \xrightarrow{BD} & B\mathrm{hAut}_\partial(TM)^{\mathrm{st},\sim\mathrm{id}} \\
\downarrow & & \downarrow \\
B\widetilde{\mathrm{Diff}}_\partial(M) & \xrightarrow{BD} & B\mathrm{hAut}_\partial(TM)^{\mathrm{st},\cong} \\
\downarrow & & \downarrow \\
B\pi_0(\mathrm{hAut}_\partial(M)^{\cong}) & = & B\pi_0(\mathrm{hAut}_\partial(M)^{\cong})
\end{array}$$

whose columns are fibre sequences, so the claim follows immediately from Theorem 3.7 and an application of the Leray–Serre spectral sequence. □

For stably parallelizable manifolds, such as $U^n_{g,1}$, the spaces $B\mathrm{hAut}^\partial_\partial(TM)^{\mathrm{st},\cong}$ and $B\mathrm{hAut}^*_\partial(TM)^{\mathrm{st},\cong}$ have a simpler description.

Lemma 3.10. *Assume that $V \to X$ is stably trivial, and that a stable trivialization of $V|_C$ is fixed. Then there is a weak equivalence*

$$B\mathrm{hAut}^C_A(V)^{\mathrm{st}} \simeq \mathrm{map}_C(X;BO)^0 /\!\!/ \mathrm{hAut}_A(X),$$

where $\mathrm{map}_C(X;BO)$ is the space of maps which are constant on C, and $\mathrm{map}_C(X;BO)^0$ is the component of the constant map.

Proof. This is an almost immediate consequence of [2, Proposition 4.13] which was stated above as (2.1). The component

$$(\mathrm{map}_C(X;BO) /\!\!/ \mathrm{hAut}_A(X))_V \subset \mathrm{map}_C(X;BO) /\!\!/ \mathrm{hAut}_A(X)$$

agrees with

$$\mathrm{map}_C(X;BO)_V /\!\!/ \mathrm{hAut}_A(X)$$

where $(\mathrm{map}_C(X;BO) /\!\!/ \mathrm{hAut}_A(X))_V \subset \mathrm{map}_C(X;BO) /\!\!/ \mathrm{hAut}_A(X)$ is the union of components which belong to the orbit of a classifying map $\lambda : X \to BO$. If V is stably trivial, we can pick λ to be the constant map. On the other hand $\pi_0(\mathrm{map}_C(X;BO)) = KO^0(X,C)$ is an abelian group and $\mathrm{hAut}_A(X)$ acts by group automorphism and therefore fixes the neutral element. It follows that for stably trivial V, $\mathrm{map}_C(X;BO)_V = \mathrm{map}_C(X;BO)^0$ is the component of the constant map. □

With the help of Lemma 3.10, we can replace the spaces in (3.8) and reformulate Corollary 3.9 as follows.

Corollary 3.11. *Assume that M is a simply connected and stably parallelizable manifold of dimension $d \geq 5$, and that $\partial M = S^{d-1}$. Then the derivative map induces maps*

$$B\widetilde{\mathrm{Diff}}_\partial(M) \to \mathrm{map}_\partial(M;BO)^0 /\!\!/ \mathrm{hAut}_\partial(M)^{\cong} \to \mathrm{map}_*(M;BO)^0 /\!\!/ \mathrm{hAut}_\partial(M)^{\cong}$$

whose composition is an $H\mathbb{Q}$-equivalence. □

Using the rationalization functor, we now give the variant of Theorem 3.7 which we shall eventually use. We let

$$\mathrm{hAut}_\partial(M_\mathbb{Q})^\cong \subset \mathrm{hAut}_\partial(M_\mathbb{Q})_\mathbb{Z}$$

the union of components which are hit by the rationalization map

$$\mathrm{hAut}_\partial(M)^\cong \to \mathrm{hAut}_\partial(M_\mathbb{Q})_\mathbb{Z}.$$

Theorem 3.12. *Let M be a simply connected manifold of dimension $d \geq 5$, assume that M is stably parallelizable and that $\partial M = S^{d-1}$. Then the composition*

$$B\widetilde{\mathrm{Diff}}_\partial(M) \to \mathrm{map}_*(M; BO)^0 \,/\!\!/\, \mathrm{hAut}_\partial(M)^\cong \to \mathrm{map}_*(M_\mathbb{Q}; BO_\mathbb{Q})^0 \,/\!\!/\, \mathrm{hAut}_\partial(M_\mathbb{Q})^\cong$$

is an $H\mathbb{Q}$-equivalence.

Proof. Corollary 3.11 leaves us with the task of proving that the second map is an $H\mathbb{Q}$-equivalence. This second map can clearly be factored as

$$\mathrm{map}_*(M; BO)^0 /\!\!/ \mathrm{hAut}_\partial(M)^\cong \to \mathrm{map}_*(M_\mathbb{Q}; BO_\mathbb{Q})^0 /\!\!/ \mathrm{hAut}_\partial(M)^\cong \to \mathrm{map}_*(M_\mathbb{Q}; BO_\mathbb{Q})^0 /\!\!/ \mathrm{hAut}_\partial(M_\mathbb{Q})^\cong.$$

The first of those maps is a $H\mathbb{Q}$-equivalence, by Lemma 3.3 and a straightfoward spectral sequence argument. To prove that the second map is also an $H\mathbb{Q}$-equivalence, observe that there is a homotopy cartesian diagram

$$\begin{array}{ccc} \mathrm{map}_*(M_\mathbb{Q}; BO_\mathbb{Q})^0 \,/\!\!/\, \mathrm{hAut}_\partial(M)^\cong & \longrightarrow & B\mathrm{hAut}_\partial(M)^\cong \\ \downarrow & & \downarrow \\ \mathrm{map}_*(M_\mathbb{Q}; BO_\mathbb{Q})^0 \,/\!\!/\, \mathrm{hAut}_\partial(M_\mathbb{Q})^\cong & \longrightarrow & B\mathrm{hAut}_\partial(M_\mathbb{Q})^\cong. \end{array}$$

Therefore, it is sufficient to prove that the homotopy fibre F of the right vertical map is rationally acyclic.

The definition of $B\mathrm{hAut}_\partial(M_\mathbb{Q})^\cong$ shows that F is connected, and Lemma 3.4 shows that $\pi_k(F) \otimes \mathbb{Q} = 0$ for all $k \geq 2$. Hence the universal cover $\tilde{F}$ is rationally acyclic, and the Leray–Serre spectral sequence of the fibre sequence $\tilde{F} \to F \to B\pi_1(F)$ shows that

$$H_k(F; \mathbb{Q}) \cong H_k(B\pi_1(F); \mathbb{Q}). \tag{3.13}$$

The exact sequence

$$\pi_1(\mathrm{hAut}_\partial(M)^\cong) \to \pi_1(\mathrm{hAut}_\partial(M_\mathbb{Q})^\cong) \to \pi_1(F) \to \pi_0(\mathrm{hAut}_\partial(M)^\cong) \to \pi_0(\mathrm{hAut}_\partial(M_\mathbb{Q})^\cong)$$

yields a short exact sequence

$$0 \to T \to \pi_1(F) \to Q \to 1 \tag{3.14}$$

of groups, where

$$T := \mathrm{coker}\Big(\pi_1(\mathrm{hAut}_\partial(M)^\cong) \to \pi_1(\mathrm{hAut}_\partial(M_\mathbb{Q})^\cong)\Big)$$

is an abelian torsion group by Lemma 3.4 and

$$Q := \ker\Big(\pi_0(\mathrm{hAut}_\partial(M)^\cong) \to \pi_0(\mathrm{hAut}_\partial(M_\mathbb{Q})^\cong)\Big)$$

is finite by [18, Theorem 1.1]. The latter is a relative version of a Theorem of Sullivan [68, Theorem 10.2] which asserts that $\pi_0(\mathrm{hAut}_*(X)) \to \pi_0(\mathrm{hAut}_*(X_\mathbb{Q}))$ has finite kernel whenever X is a finite and simply connected CW complex. As Q is finite, we get

$$H_k(\pi_1(F); \mathbb{Q}) \cong H_k(T; \mathbb{Q})_Q$$

from the Lyndon–Hochschild–Serre spectral sequence of the group extension (3.14). Since $H_k(T;\mathbb{Q}) = 0$ for $k \geq 1$, being the colimit of $H_k(H;\mathbb{Q})$, where H runs through the finitely generated (and hence finite, as T is torsion) subgroups of T, $B\pi_1(F)$ is rationally acyclic. By (3.13), F is rationally acyclic as claimed. □

3.3. **Cohomology of mapping spaces.** Theorem 3.12 shows that we need to understand the cohomology of mapping spaces $\mathrm{map}_*(M; BO_{\mathbb{Q}})^0$ and $\mathrm{map}_\partial(M; BO_{\mathbb{Q}})^0$. Since $\mathrm{map}_\partial(M; BO_{\mathbb{Q}})^0 = \mathrm{map}_*(M/\partial M; BO_{\mathbb{Q}})^0$, it suffices to consider pointed mapping spaces.

In what follows, all homology and cohomology groups are taken with coefficients in $\mathbb{Q}$. Though this is not needed for large parts of the section, it is all we shall need later on. To ease notation, we often write $\alpha\beta := \alpha \cup \beta$ for the cup product of two cohomology classes.

The slant product. Let us first recall from [66, Chapter 6.1] the slant product

$$H^n(Y \times X) \otimes H_k(X) \to H^{n-k}(Y),\ (\xi, x) \mapsto \xi/x,$$

which is related to the cohomology cross product and the Kronecker product by the formula

$$(\eta \times \zeta)/x = \eta\langle \zeta, x\rangle$$

for $\eta \in H^*(Y)$, $\zeta \in H^*(X)$ and $x \in H_*(X)$. Assume that X has finite type over $\mathbb{Q}$ (i.e. each $H_k(X)$ is finite-dimensional) and pick a homogeneous basis $(b_i)_i$ of $H_*(X)$ and let $(\beta_i)_i$ be the dual basis of $H^*(X)$. An arbitrary $\gamma \in H^*(Y \times X)$ can be written in the form $\gamma = \sum_i \gamma_i \times \beta_i$ by the Künneth formula, and we get

$$\gamma/b_j = \sum_i (\gamma_i \times \beta_i)/b_j = \sum_i \gamma_i \langle \beta_i, b_j\rangle = \gamma_j.$$

or

$$\gamma = \sum_i (\gamma/b_i) \times \beta_i \tag{3.15}$$

The λ-classes.

Definition 3.16. Let X and Z be pointed spaces, with X of finite type over $\mathbb{Q}$. The evaluation map $\mathrm{ev} : \mathrm{map}_*(X; Z) \times X \to Z$ and the slant product yields

$$\lambda : H^n(Z) \otimes H_k(X) \to H^{n-k}(\mathrm{map}_*(X; Z)),\ \xi \otimes x \mapsto \lambda_{x,\xi} := (\mathrm{ev}^*\xi)/x.$$

This construction enjoys a naturality property, which is most concisely expressed by saying that λ is a natural transformation of functors

$$\mathcal{S}_* \times \mathcal{S}_*^{\mathrm{op}} \to \mathbb{Q} - \mathrm{Mod}$$

from $(X, Z) \mapsto H^n(Z) \otimes H_k(X)$ to $(X, Z) \mapsto H^{n-k}(\mathrm{map}_*(X; Z))$. The formula (3.15) leads to the equation

$$\mathrm{ev}^* z = \sum_i (\mathrm{ev}^* z/b_i) \times \beta_i = \sum_i \lambda_{b_i, z} \times \beta_i \in H^*(\mathrm{map}_*(X; Z) \times X) \tag{3.17}$$

for each $z \in H^*(Z)$.

Lemma 3.18. *Let $c^k_{i,j}$ be the structure constants of the algebra $H^*(X)$ with respect to the basis (β_i), i.e.*

$$\beta_i \beta_j = \sum_k c^k_{i,j} \beta_k.$$

Then

$$\lambda_{b_k,zy} = \sum_{i,j} (-1)^{|\beta_i|(|y|-|b_j|)} c^k_{i,j} \lambda_{b_i,z}\lambda_{b_j,y}$$

for all $z, y \in H^*(Z)$.

Proof. Using (3.17), one checks that firstly

$$\mathrm{ev}^*(z)\mathrm{ev}^*(y) = \mathrm{ev}^*(zy) = \sum_k \lambda_{b_k,zy} \times \beta_k,$$

and secondly

$$\begin{aligned}\mathrm{ev}^*(z)\mathrm{ev}^*(y) = (\sum_i \lambda_{b_i,z} \times \beta_i)(\sum_j \lambda_{b_j,y} \times \beta_j) = \\ = \sum_{i,j} (-1)^{|\beta_i|(|y|-|b_j|)} (\lambda_{b_i,z}\lambda_{b_j,y}) \times (\beta_i\beta_j) = \\ = \sum_{i,j,k} (-1)^{|\beta_i|(|y|-|b_j|)} c^k_{i,j} (\lambda_{b_i,z}\lambda_{b_j,y}) \times \beta_k.\end{aligned}$$

Comparing coefficients yields the result. □

Lemma 3.19. *Assume that* X *is connected and that the cup length of* X *is* $\leq r-1$ *(i.e. cup products of* r *elements of* $\tilde{H}^*(X)$ *vanish). Then for* $z_1, \dots, z_r \in H^*(Z)$*, we have*

$$\lambda_{b_k,z_1\cdots z_r} = 0.$$

Proof. Using (3.17), we compute

$$\begin{aligned}\sum_k \lambda_{b_k,z_1\cdots z_r} \times \beta_k = \mathrm{ev}^*(z_1 \cdots z_r) = \mathrm{ev}^*(z_1)\cdots\mathrm{ev}^*(z_r) = \\ = \sum_{j_1,\dots,j_r} \epsilon_{j_1,\dots,j_r} (\lambda_{b_{j_1},z_1} \cdots \lambda_{b_{j_r},z_r}) \times (\beta_{j_1}\cdots\beta_{j_r}) = 0\end{aligned}$$

for some signs $\epsilon_{j_1,\dots,j_r} \in \{\pm 1\}$. The claim follows immediately. □

The case of an Eilenberg–Mac-Lane space. Now we consider the case $Z = K(\mathbb{Q}, m)$ and let $u_m \in H^m(K(\mathbb{Q}, m); \mathbb{Q})$ be the fundamental class. We assume that X is connected, and we wish to calculate $H^*(\mathrm{map}_*(X; K(\mathbb{Q}, m))^0)$ in terms of the classes λ_{a_i,u^r}, where (a_i) is a homogeneous basis of $H_*(X)$ and (α_i) is dual basis.

There are a couple of relations between the classes λ_{a_i,u^r} which we state first.

Lemma 3.20. *Assume that* X *is connected. Then*

$$\lambda_{a_i,u_m^r} = 0 \in H^*(\mathrm{map}_*(X; K(\mathbb{Q}, m))^0)$$

unless $0 < |a_i| < rm$.

Proof. For degree reasons, we get that $\lambda_{a_i,u_m^r} = 0$ if $|a_i| > rm$. For the remaining cases, note that the evaluation map $\mathrm{map}_*(X; K(\mathbb{Q}, m))^0 \times X \to K(\mathbb{Q}, m)$ factors through the smash product $\mathrm{map}_*(X; K(\mathbb{Q}, m))^0 \wedge X$. It follows that in the sum

$$\mathrm{ev}^* u_m^r = \sum_i \lambda_{a_i,u_m^r} \times \alpha_i,$$

all terms in which one factor has degree 0 must vanish. This happens if $|\alpha_i| = |a_i| = 0$ and $rm - |a_i| = 0$. □

Proposition 3.21. *Assume that X is of finite type over $\mathbb{Q}$. Then the natural map*

$$\mathbb{F}(\bigoplus_{k=1}^{m-1} H_k(M)[m-k]) \to H^*(\mathrm{map}_*(X; K(\mathbb{Q}, m)^0)$$

induced by the maps

$$H_k(M) \to H^{m-k}(\mathrm{map}_*(X; K(\mathbb{Q}, m)^0), \ a \mapsto \lambda_{a,u_m}$$

is an isomorphism. Hence $H^*(\mathrm{map}_*(X; K(\mathbb{Q}, m)^0)$ is the free graded commutative algebra generated by the elements λ_{a_i,u_m}, where a_i runs through a homogeneous basis for $\tilde{H}_{*<m}(X)$.

Proof. The space $\mathrm{map}_*(X; K(\mathbb{Q}, m))^0$ is a connected infinite loop space; in fact it is a generalized Eilenberg–Mac Lane space with homotopy groups

$$\pi_k(\mathrm{map}_*(X; K(\mathbb{Q}, m))) \otimes \mathbb{Q} = [X, \Omega^k K(\mathbb{Q}, m)]_* = [X, K(\mathbb{Q}, m-k)]_* = \tilde{H}^{m-k}(X) \tag{3.22}$$

in positive degrees. It follows from the Milnor–Moore theorem [53] and the assumption that X is of finite type that $H^*(\mathrm{map}_*(X; K(\mathbb{Q}, m)^0)$ is isomorphic (as a graded algebra) to $\mathbb{F}(\bigoplus_{k=1}^{m-1} H_k(M)[m-k])$. The only issue is to verify that the map in question is indeed an isomorphism.

To achieve this, we use the following general principle. Assume that Y is a connected infinite loop space[3] of finite type, let V_* be a $\mathbb{N}_0$-graded, degreewise finite-dimensional $\mathbb{Q}$-vector space with $V_0 = 0$ and let $\sigma : V_* \to H^*(Y; \mathbb{Q})$ be a graded linear map. Then the induced algebra map

$$\mathbb{F}(\sigma) : \mathbb{F}(V_*) \to H^*(Y; \mathbb{Q})$$

from the free graded-commutative algebra on V to the cohomology of Y is an isomorphism if and only if the bilinear form

$$B_k : \pi_k(Y) \otimes \mathbb{Q} \times V_k \to \mathbb{Q}, \ (f, v) \mapsto \langle \sigma(v), \mathrm{hur}(f) \rangle$$

is nondegenerate for each k. This principle is easily shown for $Y = K(\mathbb{Q}, n)$; the general case follows from that and the fact that Y splits rationally as a product of Eilenberg–Mac-Lane spaces [21, §16].

We must therefore show that the bilinear form

$$B_{X,k} : \pi_{m-k}(\mathrm{map}_*(X; K(\mathbb{Q}, m)^0)) \times \tilde{H}_k(X) \to \mathbb{Q}$$

given by

$$([f], a) \mapsto \langle \lambda_{a,u}, \mathrm{hur}([f]) \rangle$$

is nondegenerate. If $F : X \to Y$ is a map of pointed spaces, we denote by $F^\sharp : \mathrm{map}_*(Y; K(\mathbb{Q}, m))^0 \to \mathrm{map}_*(X; K(\mathbb{Q}, m))^0$ the induced map. By the naturality of the slant product and the Kronecker product, we have

$$B_{Y,k}([f], F_* a) = B_{X,k}((F^\sharp)_*[f], a).$$

If $F_* : H_k(X) \to H_k(Y)$ is an isomorphism then so is the induced map $(F^\sharp)_* : \pi_{m-k}(\mathrm{map}_*(Y; K(\mathbb{Q}, m))^0) \to \pi_{m-k}(\mathrm{map}_*(X; K(\mathbb{Q}, m))^0)$ by (3.22), and it follows that nondegeneracy of $B_{Y,k}$ is equivalent to nondegeneracy of $B_{X,k}$ (in degree k). Now there are maps $X \to K(H_k(X; \mathbb{Q}), k)$ and $\bigvee^g S^k \to K(H_k(X; \mathbb{Q}), k)$ inducing isomorphisms on $H_k(\_; \mathbb{Q})$. This argument proves that it suffices to consider the case where X is a wedge of finitely many k-spheres. Using the naturality again, the

[3] It would be enough to assume that Y is a connected homotopy-commutative H-space.

form $B_{\bigvee^g S^k,k}$ decomposes as the direct sum of k copies of the form $B_{S^k,k}$, and so we are left with the case $X = S^k$.

Both, $\pi_{m-k}(\mathrm{map}_*(S^k; K(\mathbb{Q}, m)^0)) \otimes \mathbb{Q}$ and $\tilde{H}_k(S^k;\mathbb{Q})$ are 1-dimensional and $\mathrm{map}_*(S^k; K(\mathbb{Q}, m))^0 \simeq K(\mathbb{Q}, m-k)$ is $(m-k-1)$-connected. Hence (by the Hurewicz theorem) it is left to be proven that $\lambda_{[S^k],u_m} \in H^{m-k}(\mathrm{map}_*(S^k; K(\mathbb{Q}, m)))$ is nonzero. On the other hand, the map

$$H^m(K(\mathbb{Q}, m);\mathbb{Q}) \to H^{m-k}(\mathrm{map}_*(S^k; K(\mathbb{Q}, m))) = H^{m-k}(\Omega^k K(\mathbb{Q}, m)),\ u \mapsto \mathrm{ev}^*(u)/[S^k]$$

can be identified with the "transgression" map

$$H^m(K(\mathbb{Q}, m);\mathbb{Q}) = [K(\mathbb{Q}, m); K(\mathbb{Q}, m)] \overset{\Omega^k}{\to} [\Omega^k K(\mathbb{Q}, m); \Omega^k K(\mathbb{Q}, m)] = H^{m-k}(\Omega^k K(\mathbb{Q}, n))$$

which is well-known to be an isomorphism. □

Corollary 3.23. *Let X be a connected space of finite type over $\mathbb{Q}$. Then the map*

$$\mathbb{F}\Big(\bigoplus_{m\geq 1, 0<k<4m} H_k(M)[4m-k]\Big) \to H^*(\mathrm{map}_*(X; BO_{\mathbb{Q}})^0;\mathbb{Q})$$

which on the generators is the direct sum of the maps

$$H_k(M)[4m-k] \to H^{4m-k}(H^*(\mathrm{map}_*(X; BO_{\mathbb{Q}})^0;\mathbb{Q}));\ a \mapsto \lambda_{a,L_m}$$

is an isomorphism. Thus $H^*(\mathrm{map}_*(X; BO_{\mathbb{Q}})^0;\mathbb{Q})$ is the free graded-commutative algebra with generators

$$\lambda_{a_i,L_m} \in H^{4m-|a_i|}(\mathrm{map}_*(X; BO_{\mathbb{Q}})^0;\mathbb{Q})$$

where a_i runs through a homogeneous basis for $\tilde{H}_*(X)$ and m through the natural numbers and $4m-|a_i| > 0$. The classes $\lambda_{a_i,L_{m_1}\cdots L_{m_r}}$ are determined by the relation stated in Lemma 3.18. □

The naturality of the construction of the λ-classes, together with Lemma 3.3, has the following consequence which turns out to be helpful for us.

Corollary 3.24. *Let X be a connected space of finite type over $\mathbb{Q}$. Then the $\pi_0(\mathrm{hAut}_*(X))$-action on $H^*(\mathrm{map}_*(X; BO_{\mathbb{Q}})^0;\mathbb{Q})$ is, under the isomorphism of Corollary 3.23, induced by the usual action of $\pi_0(\mathrm{hAut}_*(X))$ on the rational homology of X. Furthermore, the action extends to an action of $\pi_0(\mathrm{hAut}_*(X_{\mathbb{Q}}))$ on $H^*(\mathrm{map}_*(X; BO_{\mathbb{Q}})^0;\mathbb{Q})$ which is induced by the usual action of $\pi_0(\mathrm{hAut}_*(X_{\mathbb{Q}}))$ on $H_*(X_{\mathbb{Q}};\mathbb{Q}) \cong H_*(X;\mathbb{Q})$.*

3.4. **Consequences for block diffeomorphism spaces.** We now collect some consequences of the above calculations.

Lemma 3.25. *For M be as in Construction 2.4, the fibre inclusion*

$$q : \mathrm{map}_\partial(M; BO) \to \mathrm{map}_\partial(M; BO) /\!\!/ \mathrm{hAut}^+_\partial(M)$$

pulls back the κ-classes to the λ-classes, i.e.

$$q^*\kappa_c = \lambda_c$$

for each $c \in H^*(BO;\mathbb{Q})$.

Proof. This is a consequence of the definitions and the fact that for each space Y, the Gysin map of the trivial bundle $\pi : Y \times (M/\partial M) \to Y$ is given by $\pi_!(\xi) = \xi/[M]$. □

Corollary 3.26. *Let M^d be an oriented compact manifold. Then the algebra homomorphism*

$$\mathbb{F}(\kappa_{L_m} | 4m-d>0) \to H^*(\mathrm{map}_\partial(M;BO)^0 /\!\!/ \mathrm{hAut}^+_\partial(M))$$

is injective. The same conclusion is true for the homotopy quotient

$$\mathrm{map}_\partial(M;BO)^0 /\!\!/ G,$$

where G is any group complete E_1-monoid with an E_1-map $\varphi : G \to \mathrm{hAut}^+_\partial(M)$. Moreover, these statements continue to hold if $\mathrm{map}_\partial(M;BO)^0$ gets replaced by $\mathrm{map}_\partial(M_\mathbb{Q};BO_\mathbb{Q})^0$ and φ by an E_1-morphism $G \to \mathrm{hAut}^+_\partial(M_\mathbb{Q})$.

Proof. Corollary 3.23, together with Lemma 3.25, proves both statements for $G=1$, and the general follows immediately. □

Proposition 3.27. *Assume that M^d is a smooth manifold which satisfies the hypotheses of Theorem 3.12. Then the algebra map*

$$H^*(\mathrm{map}_\partial(M_\mathbb{Q};BO_\mathbb{Q})^0 /\!\!/ \mathrm{hAut}_\partial(M_\mathbb{Q})^{\cong}) \to H^*(B\widetilde{\mathrm{Diff}}_\partial(M))$$

is surjective. Moreover

1. *If d is odd, the kernel is the ideal generated by the classes κ_{L_m}, $4m-d>0$.*
2. *If $d=2n$ is even, the kernel is the ideal generated by the classes $\kappa_{L_m} - \sigma_{4m-d}$, $4m-d>0$.*

Proof. The composition

$$B\widetilde{\mathrm{Diff}}_\partial(M) \to \mathrm{map}_\partial(M_\mathbb{Q};BO_\mathbb{Q})^0/\!\!/\mathrm{hAut}_\partial(M_\mathbb{Q})^{\cong} \to \mathrm{map}_*(M_\mathbb{Q};BO_\mathbb{Q})^0/\!\!/\mathrm{hAut}_\partial(M_\mathbb{Q})^{\cong}$$

is an $H\mathbb{Q}$-equivalence by Theorem 3.12, and so surjectivity is immediate. Theorem 2.8 and (2.9) prove that the classes listed in (1) and (2) lie in the kernel, and so it remains for us to show that the kernel is not larger. Consider the map

$$\Pi : \mathrm{map}_\partial(M_\mathbb{Q};BO_\mathbb{Q})^0/\!\!/\mathrm{hAut}^+_\partial(M_\mathbb{Q})^{\cong} \to \mathrm{map}_*(M_\mathbb{Q};BO_\mathbb{Q})^0/\!\!/\mathrm{hAut}^+_\partial(M_\mathbb{Q})^{\cong} \times \prod_{4m-d>0} K(\mathbb{Q},4m-d) \tag{3.28}$$

made out of the forgetful map and the classes κ_{L_m} or $\kappa_{L_m} - \sigma_{4m-d}$ (for odd d or even d). If we can prove that Π is a weak equivalence, the claim follows.

To prove this, note first that Π is a map over $B\mathrm{hAut}^+_\partial(M_\mathbb{Q})_\mathbb{Z}$. The induced map on the homotopy fibres of the respective maps to $B\mathrm{hAut}^+_\partial(M_\mathbb{Q})_\mathbb{Z}$ is the map

$$\mathrm{map}_\partial(M_\mathbb{Q};BO_\mathbb{Q})^0 \to \mathrm{map}_*(M_\mathbb{Q};BO_\mathbb{Q})^0 \times \prod_{4m-d>0} K(\mathbb{Q},4m-d); \tag{3.29}$$

made out of the forgetful map and the classes λ_{L_m}. This follows from Lemma 3.25 and the fact that the classes σ_{4m-d} are pulled back from $B\mathrm{hAut}^+_\partial(M_\mathbb{Q})_\mathbb{Z}$.

To see that (3.29) is a weak equivalence, one uses Corollary 3.23 for both mapping spaces and the naturality of the λ-classes. □

Remark 3.30. The proof suggests that one might extract a proof of Theorem 2.8 out of the arguments in [2, §4]. While surely true, it seems simpler to us to use the independent proof of Theorem 2.8, as it allows us to treat [2, §4] as a black box.

4. Homotopy calculations for the manifolds $U^n_{g,1}$

4.1. **Low dimensional homotopy groups.** Let us now focus our attention to the manifolds we are actually interested in, i.e.

$$U^n_{g,1} := \sharp^g (S^n \times S^{n+1}) \setminus \mathrm{int}(D^{2n+1}). \tag{4.1}$$

Most of the following facts are also proven in [29], but as the setup in [29] is more general, we prefer to indicate the proofs here. Note that

$$U^n_{g,1} \simeq \bigvee^g S^n \vee \bigvee^g S^{n+1}.$$

Let us denote

$$N(g)_{\mathbb{Z}} := H_n(U^n_{g,1}; \mathbb{Z}) \cong \mathbb{Z}^g$$

and

$$N(g) := H_n(U^n_{g,1}; \mathbb{Q}) = N(g)_{\mathbb{Z}} \otimes \mathbb{Q}.$$

By Poincaré duality and the universal coefficient theorem, we have

$$H_{n+1}(U^n_{g,1}; \mathbb{Z}) \cong H^n(U^n_{g,1}; \mathbb{Z}) \cong N(g)^\vee_{\mathbb{Z}}.$$

These isomorphisms are natural with respect to the action of $\mathrm{hAut}_\partial(U^n_{g,1})$, and hold similarly for rational coefficients. The group $\mathrm{hAut}_\partial(U^n_{g,1})$ acts trivially on $H_{2n+1}(U^n_{g,1}, \partial; \mathbb{Z})$, $H_{2n+1}(U^n_{g,1}, \partial; \mathbb{Q})$, $H^{2n+1}(U^n_{g,1}, \partial; \mathbb{Z})$ and $H^{2n+1}(U^n_{g,1}, \partial; \mathbb{Q})$.

Let $x_1, \ldots, x_g \in \pi_n(U^n_{g,1})$ be the elements represented by the inclusion of the g different S^n's, and similarly let $y_1, \ldots, y_g \in \pi_{n+1}(U^n_{g,1})$ be represented by the g copies of S^{n+1}. The inclusion of the boundary $S^{2n} = \partial U^n_{g,1} \to U^n_{g,1}$ represents an element $\omega \in \pi_{2n}(U^n_{g,1})$ which agrees with the sum

$$\omega = \sum_{j=1}^g [x_j, y_j] \tag{4.2}$$

of Whitehead products of the generators, if the numbering and the signs of the generators are chosen appropriately. This is obvious from the definition of the Whitehead product when $g = 1$, and follows for higher g. We denote by

$$a_j := \mathrm{hur}(x_j) \in H_n(U^n_{g,1}; \mathbb{Z}) \text{ and } b_j := \mathrm{hur}(y_j) \in H_{n+1}(U^n_{g,1}; \mathbb{Z})$$

the images under the Hurewicz homomorphism, and use the same symbol for the image in rational homology.

We let $(\alpha_1, \ldots, \alpha_g)$ the basis of $H^n(U^n_{g,1}; \mathbb{Z}) = H^n(U^n_{g,1}, \partial; \mathbb{Z})$ dual to $(a_1, \ldots, a_g)$ and let $(\beta_1, \ldots, \beta_g)$ be the basis of $H^{n+1}(U^n_{g,1}; \mathbb{Z}) = H^{n+1}(U^n_{g,1}, \partial; \mathbb{Z})$ dual to $(b_1, \ldots, b_g)$. Let $\nu \in H^{2n+1}(U^n_{g,1}, \partial; \mathbb{Z})$ be dual to the fundamental class $[U] = [U^n_{g,1}]$. The cup product structure of $H^*(U^n_{g,1}; \mathbb{Z})$ is then given by

$$\alpha_i \beta_j = \delta_{ij} \nu; \tag{4.3}$$

all other cup products are zero for degree reasons. The cup length of $U^n_{g,1}/\partial$ is therefore 2, and it follows from Lemma 3.18 that

$$\lambda_{[U], L_m L_k} = \sum_{j=1}^g \lambda_{a_j, L_m} \lambda_{b_j, L_k} + \sum_{j=1}^g \lambda_{b_j, L_m} \lambda_{a_j, L_k}. \tag{4.4}$$

Having understood the homological structure, let us turn to homotopy groups. The Hurewicz homomorphism

$$\mathrm{hur} : \pi_n(U^n_{g,1}) \to H_n(U^n_{g,1}; \mathbb{Z})$$

is an isomorphism. An element $\rho \in \pi_{n+m}(S^n)$ yields a natural map

$$\theta_\rho : \pi_n(X) \to \pi_{n+m}(X)$$

given by composition with a representative for ρ. For $\eta \in \pi_{n+1}(S^n)$ the suspension of the Hopf map, we obtain

$$\theta_\eta : \pi_n(X) \to \pi_{n+1}(X),$$

which is a group homomorphism when $n \geq 3$, by [76, Corollary XI.1.12]. Moreover, [76, XI.1(1.1)] and $2\eta = 0$ proves that θ_η descends to a map $\pi_n(U^n_{g,1}) \otimes \mathbb{Z}/2 \to \pi_{n+1}(U^n_{g,1})$.

Lemma 4.5. *For $n \geq 3$, the sequence*

$$0 \to \pi_n(U^n_{g,1}) \otimes \mathbb{Z}/2 \xrightarrow{\theta_\eta} \pi_{n+1}(U^n_{g,1}) \xrightarrow{\text{hur}} H_{n+1}(U^n_{g,1}; \mathbb{Z}) \to 0$$

is exact.

Proof. The inclusion

$$U^n_{g,1} \simeq \bigvee^g S^n \vee \bigvee^g S^{n+1} \to (S^n)^g \times (S^{n+1})^g$$

is the inclusion of the $(2n-1)$-skeleton and hence $(2n-1) \geq (n+2)$-connected. It is therefore enough to prove the lemma for $(S^n)^g \times (S^{n+1})^g = (S^n \times S^{n+1})^g$ in place of $U^n_{g,1}$. The Künneth formula reduces to the case $g = 1$, where the claim is clear because $\pi_{n+1}(S^n) \cong \mathbb{Z}/2$ as long as $n \geq 3$. □

4.2. **Homotopy automorphisms.**

Lemma 4.6. *The map*

$$\pi_0(\mathrm{hAut}_*(U^n_{g,1})) \to \mathrm{GL}(N(g)_\mathbb{Z}) \times \mathrm{GL}(N(g)^\vee_\mathbb{Z})$$

given by the action on integral homology is surjective when $n \geq 1$, and has finite kernel, provided that $n \geq 3$.

The analogous map

$$\pi_0(\mathrm{hAut}_*((U^n_{g,1})_\mathbb{Q})) \to \mathrm{GL}(N(g)) \times \mathrm{GL}(N(g)^\vee)$$

is an isomorphism, which maps the subgroup $\pi_0(\mathrm{hAut}_*((U^n_{g,1})_\mathbb{Q})_\mathbb{Z}) \subset \pi_0(\mathrm{hAut}_*((U^n_{g,1})_\mathbb{Q}))$ onto the subgroup $\mathrm{GL}(N(g)_\mathbb{Z}) \times \mathrm{GL}(N(g)^\vee_\mathbb{Z})$.

Proof. Since $\mathrm{hAut}_*(\bigvee^g S^n) \to \mathrm{GL}(H_n(\bigvee^g S^n; \mathbb{Z}))$ is surjective, and similarly for a wedge of $(n+1)$-spheres, surjectivity follows. To see that the kernel is finite, we factor the map in question as

$$\pi_0(\mathrm{hAut}(U^n_{g,1})) \to \mathrm{GL}(\pi_n(U^n_{g,1})) \times \mathrm{GL}(\pi_{n+1}(U^n_{g,1})) \to \mathrm{GL}(N(g)_\mathbb{Z}) \times \mathrm{GL}(N(g)^\vee_\mathbb{Z}). \tag{4.7}$$

The first map is injective: a pointed map $f : U^n_{g,1} \to U^n_{g,1}$ which induces the identity on both, π_n and π_{n+1}, must be homotopic to the identity, as $U^n_{g,1}$ is a wedge of spheres. The second map is the product of two maps, the first induced by the isomorphism $\pi_n(U^n_{g,1}) \cong N(g)_\mathbb{Z}$, and the second coming from the fact that $N(g)^\vee_\mathbb{Z}$ is the torsionfree quotient of $\pi_{n+1}(U^n_{g,1})$, which stems from Lemma 4.5. By that Lemma, the kernel of the second map in (4.7) can be identified with the finite group

$$\mathrm{Hom}(H_{n+1}(U^n_{g,1}; \mathbb{Z}); \pi_n(U^n_{g,1}) \otimes \mathbb{Z}/2) \cong \mathrm{Hom}(\mathbb{Z}^g; \mathbb{Z}/2^g).$$

The rational case is analogous. The proof of surjectivity is similar (using elementary matrices to generate $\mathrm{GL}_g(\mathbb{Q})$), and in the factorization analogous to (4.7), the second map is an isomorphism by Lemma 4.5. The last statement follows easily from the others. □

Let us turn to a description of some of the higher homotopy groups of $\mathrm{hAut}_*(U^n_{g,1})$. Lemma 3.4 shows that the map

$$\pi_k(\mathrm{hAut}_*(U^n_{g,1}))\otimes\mathbb{Q}\to\pi_k(\mathrm{hAut}_*((U^n_{g,1})_{\mathbb{Q}}))\otimes\mathbb{Q}=\pi_k(\mathrm{hAut}_*((U^n_{g,1})_{\mathbb{Q}}))$$

is an isomorphism when $k\geq 1$. Let us moreover note that these homotopy groups carry representations of the group $\pi_0(\mathrm{hAut}_*((U^n_{g,1})_{\mathbb{Q}}))\cong\mathrm{GL}(N(g))\times\mathrm{GL}(N(g)^\vee)$.

Lemma 4.8. *We have*

$$\dim_{\mathbb{Q}}(\pi_k(B\mathrm{hAut}_*(U^n_{g,1}))\otimes\mathbb{Q})=\begin{cases}g^2 & k=2 \text{ and } n\geq 3,\\ 0 & 3\leq k\leq n-2.\end{cases}$$

The same is true for $(U^n_{g,1})_{\mathbb{Q}}$ *in place of* $U^n_{g,1}$*. Moreover*

$$\dim_{\mathbb{Q}}(\pi_{n-1}(B\mathrm{hAut}_*(U^n_{g,1}))\otimes\mathbb{Q})\neq 0.$$

Proof. For $k\geq 1$, we have $\pi_k(\mathrm{hAut}_*(U^n_{g,1}))\cong\pi_k(\mathrm{map}_*(U^n_{g,1};U^n_{g,1});\mathrm{id})$. Since the inclusion $U^n_{g,1}\to(S^n\times S^{n+1})^g$ is $(2n-1)$-connected, the induced map

$$\mathrm{map}_*(U^n_{g,1};U^n_{g,1})\to\mathrm{map}_*(U^n_{g,1};(S^n\times S^{n+1})^g)$$

is $(2n-1)-(n+1)=(n-2)$-connected. Since

$$\mathrm{map}_*(U^n_{g,1};(S^n\times S^{n+1})^g)\simeq(\Omega^n(S^n\times S^{n+1})^g))^g\times(\Omega^{n+1}(S^n\times S^{n+1})^g))^g,$$

we obtain an isomorphism

$$\pi_k(\mathrm{map}_*(U^n_{g,1};U^n_{g,1}))\cong$$
$$(\pi_{n+k}(S^n))^{\oplus g^2}\oplus(\pi_{n+k}(S^{n+1}))^{\oplus g^2}\oplus(\pi_{n+1+k}(S^n))^{\oplus g^2}\oplus(\pi_{n+k+1}(S^{n+1}))^{\oplus g^2}$$

for $1\leq k\leq n-3$, and an epimorphism if $k=n-2$. The claim follows by using the known rational homotopy groups of the spheres.

The rational case is an immediate consequence of the integral one and Lemma 3.4. □

We need to know a precise description of $\pi_1(\mathrm{hAut}_*(U^n_{g,1}))\otimes\mathbb{Q}\cong\pi_1(\mathrm{hAut}_*((U^n_{g,1})_{\mathbb{Q}}))$, not merely its dimension which we just computed. We identify

$$\pi_1(\mathrm{hAut}_*(U^n_{g,1}))\cong\pi_2(B\mathrm{hAut}_*(U^n_{g,1}))=H_2((B\mathrm{hAut}_*(U^n_{g,1}))^{\mathrm{id}};\mathbb{Z}).$$

An element $\gamma\in\pi_2(B\mathrm{hAut}_*(U^n_{g,1}))$ classifies a fibration $E_\gamma\to S^2$, together with a cross-section and a homotopy equivalence of the fibre over the basepoint with $U^n_{g,1}$.

The only potentially nonzero differential of the homological Leray–Serre spectral sequence of $E_\gamma\to S^2$ is the map

$$d^2_{2,n}:E^2_{2,n}\to E^2_{0,n+1},$$

which can be rewritten as

$$d(\gamma):N(g)_{\mathbb{Z}}\cong H_2(S^2;N(g)_{\mathbb{Z}})\to H_0(S^2;N(g)^\vee_{\mathbb{Z}})\cong N(g)^\vee_{\mathbb{Z}}.$$

Therefore, assigning $\gamma\mapsto d(\gamma)$ gives a map

$$d:\pi_2(B\mathrm{hAut}_*(U^n_{g,1}))\to\mathrm{Hom}(N(g)_{\mathbb{Z}};N(g)^\vee_{\mathbb{Z}}). \tag{4.9}$$

Similarly, we obtain

$$d_{\mathbb{Q}}:\pi_2(B\mathrm{hAut}_*((U^n_{g,1})_{\mathbb{Q}})\to\mathrm{Hom}(N(g);N(g)^\vee). \tag{4.10}$$

Source and target of (4.9) are $\mathbb{Z}[\pi_0(\mathrm{hAut}_*(U^n_{g,1}))]$-modules; and source and target of (4.10) are $\mathbb{Q}[\pi_0(\mathrm{hAut}_*(U^n_{g,1})_{\mathbb{Q}})]$-modules.

Lemma 4.11. *The map d is a homomorphism of $\mathbb{Z}[\pi_0(\mathrm{hAut}_*(U^n_{g,1}))]$-modules, and similarly $d_{\mathbb{Q}}$ is a homomorphism of $\mathbb{Q}[\pi_0(\mathrm{hAut}_*(U^n_{g,1})_{\mathbb{Q}})]$-modules. When $n \geq 4$, $d_{\mathbb{Q}}$ is an isomorphism.*

Proof. The group $\pi_0(\mathrm{hAut}_*(U^n_{g,1}))$ acts on $\pi_2(B\mathrm{hAut}_*(U^n_{g,1}))$ by changing the identification of the fibres. Therefore, it is clear that d is $\pi_0(\mathrm{hAut}_*(U^n_{g,1}))$-equivariant; and the same argument applies to $d_{\mathbb{Q}}$.

That d and $d_{\mathbb{Q}}$ are additive requires a more detailed argument. The argument is the same in both cases, so we concentrate on d. Let γ_0, γ_1 be two elements of $\pi_2(B\mathrm{hAut}_*(U^n_{g,1}))$. Identification of the fibres over the basepoint gives a fibration over $S^2 \vee S^2$ which we denote by $E_{\gamma_0} \vee E_{\gamma_1} \to S^2 \vee S^2$, abusing notation for simplicity. The fold map $S^2 \vee S^2 \to S^2$ is covered by a fibrewise homotopy equivalence $E_{\gamma_0} \vee E_{\gamma_1} \to E_{\gamma_0+\gamma_1}$, and the inclusions $S^2 \to S^2 \vee S^2$ are covered by fibrewise homotopy equivalences $E_{\gamma_j} \to E_{\gamma_0} \vee E_{\gamma_1}$. We obtain a commutative diagram

$$\begin{array}{ccccc} H_2(S^2; N(g)_{\mathbb{Z}}) \oplus H_2(S^2; N(g)_{\mathbb{Z}}) & \longrightarrow & H_2(S^2 \vee S^2; N(g)_{\mathbb{Z}}) & \longrightarrow & H_2(S^2; N(g)_{\mathbb{Z}}) \\ \downarrow{\scriptstyle d(\gamma_0)\oplus d(\gamma_1)} & & \downarrow & & \downarrow{\scriptstyle d(\gamma_0+\gamma_1)} \\ H_0(S^2; N(g)^{\vee}_{\mathbb{Z}}) \oplus H_0(S^2; N(g)^{\vee}_{\mathbb{Z}}) & \longrightarrow & H_0(S^2 \vee S^2; N(g)^{\vee}_{\mathbb{Z}}) & \longrightarrow & H_0(S^2; N(g)^{\vee}_{\mathbb{Z}}) \end{array}$$

and the two horizontal composition maps are just the addition. Additivity of d follows. In a similar way, one proves that $d_{\mathbb{Q}}$ is additive and hence $\mathbb{Q}$-linear.

For the second part of the proof, consider the diagram

$$\begin{array}{ccc} \pi_2(B\mathrm{hAut}_*(U^n_{g,1})) & \xrightarrow{d} & \mathrm{Hom}(N(g)_{\mathbb{Z}}; N(g)^{\vee}_{\mathbb{Z}}) \\ \downarrow & & \downarrow \\ \pi_2(B\mathrm{hAut}_*((U^n_{g,1})_{\mathbb{Q}}) & \xrightarrow{d_{\mathbb{Q}}} & \mathrm{Hom}(N(g); N(g)^{\vee}) \end{array} \tag{4.12}$$

which obviously commutes. We will show that

$$\ker(d) \subset \ker\Big(\pi_2(B\mathrm{hAut}_*(U^n_{g,1})) \to \pi_2(B\mathrm{hAut}_*((U^n_{g,1})_{\mathbb{Q}}))\Big). \tag{4.13}$$

Both vertical maps in (4.12) are rationalizations (the left one by Lemma 3.4), and so (4.13) shows that $\ker(d)$ is finite. Hence $d_{\mathbb{Q}}$ is injective, and Lemma 4.8, together with a dimension count, proves that $d_{\mathbb{Q}}$ is an isomorphism.

To show (4.13), we must show that a fibration $\pi : E \to S^2$ with fibre $U^n_{g,1}$ and a cross-section whose Leray–Serre spectral sequence collapses at the E^2-stage is rationally fibre-homotopy equivalent to $S^2 \times (U^n_{g,1})_{\mathbb{Q}}$. Since the spectral sequence collapses and since $n \geq 3$, the inclusion of the fibre $U^n_{g,1} \to E$ induces isomorphisms on H_n and H_{n+1} and so gives isomorphisms $N(g)_{\mathbb{Z}} \to H_n(E)$ and $N(g)^{\vee}_{\mathbb{Z}} \to H_{n+1}(E)$. From the inverses of those isomorphisms, we obtain a map

$$f : E \to K(N(g)_{\mathbb{Z}}, n) \times K(N(g)^{\vee}_{\mathbb{Z}}, n+1) \to K(N(g), n) \times K(N(g)^{\vee}, n+1)$$

inducing an isomorphism on rational homology in degrees n and $n+1$. The natural map $(U^n_{g,1})_{\mathbb{Q}} \to K(N(g), n) \times K(N(g)^{\vee}, n+1)$, which induces the identity on homology in degrees n and $n+1$, is $(2n-1) \geq (n+3)$-connected, and since the homotopy dimension of E is $n+3$, we can deform f to a map $g : E \to (U^n_{g,1})_{\mathbb{Q}}$, such that the composition with the inclusion of the fibre is the rationalization map $U^n_{g,1} \to (U^n_{g,1})_{\mathbb{Q}}$. The map

$$h := (\pi, g) : E \to S^2 \times (U^n_{g,1})_{\mathbb{Q}}$$

over S^2 induces an isomorphism in rational homology up to degree $n+1$. Using a spectral sequence comparison argument, we obtain that h is a rational homology equivalence, and hence also a rational homology equivalence on the fibres. ☐

4.3. **Homotopy automorphisms relative to the boundary.** At this point, we understand the rational homotopy groups of $\mathrm{hAut}_*(U^n_{g,1})$ in a range of degrees. We are interested in those of $\mathrm{hAut}_\partial(U^n_{g,1})$, however. These two spaces are related by a fibre sequence

$$\mathrm{hAut}_\partial(U^n_{g,1}) \to \mathrm{hAut}_*(U^n_{g,1}) \to \Omega^{2n} U^n_{g,1}. \tag{4.14}$$

Lemma 4.15. *For $n \geq 3$, the map*

$$\pi_0(\mathrm{hAut}_\partial(U^n_{g,1})) \to \mathrm{GL}(N(g)_{\mathbb{Z}})$$

induced by the action on nth homology is surjective and has finite kernel. The map

$$\pi_0(\mathrm{hAut}_\partial((U^n_{g,1})_{\mathbb{Q}})) \to \mathrm{GL}(N(g))$$

is an isomorphism for such n, and maps the subgroup $\pi_0(\mathrm{hAut}_\partial((U^n_{g,1})_{\mathbb{Q}})_{\mathbb{Z}})$ onto $\mathrm{GL}(N(g)_{\mathbb{Z}})$.

Proof. Consider the diagram

$$\begin{array}{ccc} \pi_0(\mathrm{hAut}_\partial(U^n_{g,1})) & \longrightarrow & \mathrm{GL}(N(g)_{\mathbb{Z}}) \\ \downarrow & & \downarrow \Delta \\ \pi_0(\mathrm{hAut}_*(U^n_{g,1})) & \longrightarrow & \mathrm{GL}(N(g)_{\mathbb{Z}}) \times \mathrm{GL}(N(g)^\vee_{\mathbb{Z}}); \end{array} \tag{4.16}$$

the left vertical map is the obvious one, and the map Δ sends $h \in \mathrm{GL}(N(g)_{\mathbb{Z}})$ to $(h, (h^{-1})^\vee)$. The horizontal maps are induced by the action on homology. To verify that (4.16) commutes, pick $\varphi \in \mathrm{hAut}_\partial(U^n_{g,1})$. Let $f = (f_{ij})$ and $g = (g_{ij})$ be the matrices which describe the effect of φ on $H_n(U^n_{g,1})$ and $H_{n+1}(U^n_{g,1})$, respectively, in terms of the bases $(a_1, \ldots, a_g)$ and $(b_1, \ldots, b_g)$. Since φ fixes $\partial U^n_{g,1}$ pointwise, we have $\varphi_*(\omega) = \omega$. Using (4.2), this condition translates into the relation

$$f^\top g = 1 \in \mathrm{GL}_g(\mathbb{Z}),$$

which is equivalent to the commutativity of (4.16).

The top horizontal map of (4.16) is surjective: start with $h \in \mathrm{GL}(N(g))$ and use Lemma 4.6 to find $\varphi \in \mathrm{hAut}_*(U^n_{g,1})$ which realizes $\Delta(h)$ on $H_n \oplus H_{n+1}$. Reading the above computation backwards, this means $\varphi_*(\omega) = \omega$, and so φ is homotopic relative to the basepoint to a map which is the identity on $\partial U^n_{g,1}$.

The left vertical map in (4.16) fits into an exact sequence

$$\pi_{2n+1}(U^n_{g,1}) \xrightarrow{\delta} \pi_0(\mathrm{hAut}_\partial(U^n_{g,1})) \to \pi_0(\mathrm{hAut}_*(U^n_{g,1}))$$

coming from the fibre sequence (4.14). It is proven in [29] that the connecting homomorphism δ has finite image (see Proof of Proposition 6.6 and Remark 6.7 of the quoted paper). Together with Lemma 4.6, this finishes the proof of the integral statement. The proof of the rational statement is analogous. ☐

Proposition 4.17. *We have*

$$\pi_k(B\mathrm{hAut}_\partial(U^n_{g,1})) \otimes \mathbb{Q} = 0$$

for $3 \leq k \leq n-3$. *If* $n \geq 5$, *the map*

$$\pi_2(B\mathrm{hAut}_\partial(U^n_{g,1})) \otimes \mathbb{Q} \to \pi_2(B\mathrm{hAut}_*(U^n_{g,1})) \otimes \mathbb{Q}$$

is injective and can be identified, $\pi_0(\mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})$*-equivariantly, with the inclusion of*

$$\begin{cases} S^2(N(g)^\vee) & n \text{ even} \\ \Lambda^2(N(g)^\vee) & n \text{ odd} \end{cases}$$

into $N(g)^\vee \otimes N(g)^\vee = \mathrm{Hom}(N(g); N(g)^\vee)$.

Proof. For a graded vector space V, we denote the desuspension by $(s^{-1}V)_k := V_{k+1}$. For a (simply connected) space X, $s^{-1}\pi_*(X_\mathbb{Q})$ is a graded Lie algebra under the Whitehead product, and since $U^n_{g,1}$ is a wedge of spheres, we have

$$s^{-1}\pi_*((U^n_{g,1})_\mathbb{Q}) \cong \mathbb{L}(\pi_n((U^n_{g,1})_\mathbb{Q}) \oplus \pi_{n+1}((U^n_{g,1})_\mathbb{Q})),$$

the free Lie algebra on the graded vector space which is $\pi_n((U^n_{g,1})_\mathbb{Q})$ in degree $n-1$ and $\pi_{n+1}((U^n_{g,1})_\mathbb{Q})$ in degree n, by [21, Theorem 24.5]. It follows that

$$\pi_{2n+k}(U^n_{g,1}) \otimes \mathbb{Q} = 0 \text{ if } 2 \leq k \leq n-3.$$

Hence $\pi_k(B\mathrm{hAut}_\partial(U^n_{g,1}))\otimes\mathbb{Q} \to \pi_k(B\mathrm{hAut}_*(U^n_{g,1}))\otimes\mathbb{Q}$ is injective if $2 \leq k \leq n-3$, by the fibre sequence (4.14). Together with Lemma 4.8, the first claim follows, and also the injectivity on π_2.

The fibre sequence (4.14), together with $\pi_{2n+2}((U^n_{g,1})_\mathbb{Q}) = 0$ (which holds when $n \geq 5$), gives us an exact sequence

$$\begin{aligned} 0 &\to \pi_2(B\mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})) \to \pi_2(B\mathrm{hAut}_*((U^n_{g,1})_\mathbb{Q})) \to \pi_{2n+1}((U^n_{g,1})_\mathbb{Q}) \to \\ &\to \pi_1(B\mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})) \to \pi_1(B\mathrm{hAut}_*((U^n_{g,1})_\mathbb{Q})) \to \pi_{2n}((U^n_{g,})_\mathbb{Q}). \end{aligned}$$

Now $\pi_0(\mathrm{hAut}_*(U^n_{g,1})_\mathbb{Q}) \cong \mathrm{GL}(N(g))\times\mathrm{GL}(N(g)^\vee)$ and $\pi_0(\mathrm{hAut}_\partial(U^n_{g,1})_\mathbb{Q}) \cong \mathrm{GL}(N(g))$, and the map between these two groups is injective as we saw in the proof of Lemma 4.15. Thus from the above sequence, we obtain a short exact sequence

$$0 \to \pi_2(B\mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})) \to \pi_2(B\mathrm{hAut}_*((U^n_{g,1})_\mathbb{Q})) \to \pi_{2n+1}((U^n_{g,1})_\mathbb{Q}) \to 0. \tag{4.18}$$

The vector space $\pi_{2n+1}((U^n_{g,1})_\mathbb{Q})$ is generated by the Whitehead brackets $[y_i, y_j]$, modulo the relations that are universally satisfied by Whitehead brackets of two elements of degree $n+1$, that is

$$[y_i, y_j] = (-1)^{(n+1)^2}[y_j, y_i] = -(-1)^n[y_j, y_i].$$

It follows that

$$\pi_{2n+1}((U^n_{g,1})_\mathbb{Q}) \cong \begin{cases} S^2(N(g)^\vee) & n \text{ odd} \\ \Lambda^2(N(g)^\vee) & n \text{ even.} \end{cases} \tag{4.19}$$

The sequence (4.18) is a sequence of $\pi_0(\mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})) = \mathrm{GL}(N(g))$-modules, and as $S^2(N(g)^\vee)$, as well as $\Lambda^2(N(g)^\vee)$ are irreducible $\mathrm{GL}(N(g))$-modules, the second map in (4.18) must, up to multiplication by a nonzero constant, agree with the natural projection from $\mathrm{Hom}(N(g); N(g)^\vee) = (N(g)^\vee)^{\otimes 2}$ to $S^2(N(g)^\vee)$ or $\Lambda^2(N(g)^\vee)$. Therefore, the kernel of these maps are as asserted. □

Lemma 4.20. *Provided that $n \geq 3$, we have*

$$\mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})^{\cong} = \mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})_\mathbb{Z}.$$

Proof. By Lemma 4.15, $\mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})_\mathbb{Z}$ is isomorphic to $\mathrm{GL}(N(g))_\mathbb{Z} \cong \mathrm{GL}(N(g)_\mathbb{Z})$ via the action on $H_n(U^n_{g,1};\mathbb{Z})$. Hence we need to check that every linear automorphism of $H_n(U^n_{g,1};\mathbb{Z})$ can be realized by a diffeomorphism fixing the boundary pointwise. It is explained in the proof of [29, Proposition 5.3] how to deduce this from [73, Lemma 17]. □

Lemma 4.20 and Proposition 3.27 prove the following.

Corollary 4.21. *The algebra map*

$$H^*(\mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; BO_\mathbb{Q})^0 /\!\!/ \mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})_\mathbb{Z}) \to H^*(B\widetilde{\mathrm{Diff}}_\partial(U^n_{g,1}))$$

is surjective if $n \geq 3$, and the kernel is the ideal generated by the classes κ_{L_m}, $m \in \mathbb{N}$. □

4.4. **The spectral sequence for tangential homotopy automorphisms.** Corollary 4.21 paves the way for the calculation of $H^*(B\widetilde{\mathrm{Diff}}_\partial(U^n_{g,1}))$. We approach this through the spectral sequences for the two fibrations

$$\mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; BO_\mathbb{Q})^0 /\!\!/ \mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})^{\mathrm{id}} \to \mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; BO_\mathbb{Q})^0 /\!\!/ \mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})_\mathbb{Z} \to B\mathrm{GL}(N(g)_\mathbb{Z}), \tag{4.22}$$

(where we identified the base space using $\pi_0(\mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})_\mathbb{Z}) \cong \mathrm{GL}(N(g)_\mathbb{Z})$ which is valid when $n \geq 3$ by Lemma 4.15), and

$$\mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; BO_\mathbb{Q})^0 \to \mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; BO_\mathbb{Q})^0 /\!\!/ \mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})^{\mathrm{id}} \to B\mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})^{\mathrm{id}}. \tag{4.23}$$

In this section, we are concerned with (4.23). Let us start with a crucial observation, deduced from Corollary 3.24.

Observation 4.24. The whole fibre sequence (4.23) is $\pi_0(\mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})) \cong \mathrm{GL}(N(g))$-equivariant; hence its Leray–Serre spectral sequence is a spectral sequence of $\mathrm{GL}(N(g))$-representations.

Combining or work so far, the E_2-term of the spectral sequence of (4.23) is readily computed in a range of degrees. Let us introduce some notation.

Notation 4.25. (1) We denote

$$L^2(N(g)) := \begin{cases} S^2(N(g)) & n \text{ even} \\ \Lambda^2(N(g)) & n \text{ odd.} \end{cases}$$

(2) We fix an integer M; everything that matters is that M is large enough (see 4.29).

(3) Let $V(n)$ be the graded vector space

$$V(n) := \bigoplus_{4m-2n-1>0, m \leq M} \mathbb{Q}[4m-2n-1],$$

(4) let $U(n)$ be the graded vector space

$$U(n) := \bigoplus_{4m-n-1>0, m \leq M} \mathbb{Q}[4m-n-1]$$

(5) and let $W(n)$ be the graded vector space

$$W(n) := \bigoplus_{4m-n>0, m\leq M} \mathbb{Q}[4m-n].$$

Definition 4.26. Let $n \geq 6$ be even. We define a map

$$S^*(S^2(N(g))) \otimes \Lambda^*(V(n)) \otimes S^*(N(g) \otimes W(n)) \otimes \Lambda^*(N(g)^\vee \otimes U(n)) \to E_2^{*,*} \quad (4.27)$$

of bigraded algebras as follows.

- On $S^2(N(g))$, which is in bidegree $(2,0)$, it is the isomorphism $S^2(N(g)) \cong H^2(B\mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})^{\mathrm{id}}$ from Proposition 4.17.
- On $V(n)_{4m-n-1} = \mathbb{Q}$, which is in bidegree $(0, 4m-n-1)$, it is the map which sends 1 to the class $\lambda_{[U^n_{g,1}],L_m} \in H^{4m-2n-1}(\mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; BO_\mathbb{Q})^0) = E_2^{0,4m-2n-1}$ (using the notations introduced in §4.1 and Definition 3.16).
- On $N(g) \otimes W(n)_{4m-n} = N(g)$, which is in bidegree $(0, 4m-n)$, it is the map which sends $a \in N(g)$ to $\lambda_{a,L_m} \in H^{4m-n}(\mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; BO_\mathbb{Q})^0) = E_2^{0,4m-n}$.
- On $N(g)^\vee \otimes U(n)_{4m-n-1} = N(g)^\vee$, which is in bidegree $(0, 4m-n-1)$, it sends $b \in N(g)^\vee$ to $\lambda_{b,L_m} \in H^{4m-n-1}(\mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; BO_\mathbb{Q})^0) = E_2^{4m-n-1}$.

We similarly define for odd $n \geq 5$ a map

$$S^*(\Lambda^2(N(g))) \otimes \Lambda^*(V(n)) \otimes \Lambda^*(N(g) \otimes W(n)) \otimes S^*(N(g)^\vee \otimes U(n)) \to E_2^{*,*} \quad (4.28)$$

by the analogous formulas.

Proposition 4.29. *The maps* (4.27) *and* (4.28) *are* $\pi_0(\mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})) \cong \mathrm{GL}(N(g))$*-equivariant. If* $n \geq 5$*, they are isomorphisms in bidegrees* (p,q) *with* $p \leq n-3$ *and* $q \leq 4M - 2n + 2$.

Proof. Equivariance follows from the naturality of the λ-classes that was recorded after Definition 3.16, and from the equivariance statement of Proposition 4.17. The statement about the maps being isomorphisms follows from Proposition 4.17 and Corollary 3.23. ☐

Proposition 4.30. *Assume* $n \geq 5$ *and* $g \geq 2$*. With respect to the isomorphism of Proposition 4.29, the* d_2*-differential in the spectral sequence of the fibre sequence* (4.23) *is given on generators as follows.*

1. *On* $L^2(N(g))$*, it is zero.*
2. *On* $V(n)$*, it is zero.*
3. *On* $N(g)^\vee \otimes U(n)$*, it is zero.*
4. *On* $N(g) \otimes W(n)$*, it is of the form*

$$N(g) \otimes W(n) \overset{p\otimes S}{\to} L^2(N(g)) \otimes N(g)^\vee \otimes U(n)$$

where $p : N(q) \to L^2(N(g)) \otimes N(g)^\vee$ *is adjoint to the projection* $N(g)^{\otimes 2} \to L^2(N(g))$ *and* $S : W(n) \to U(n)$ *is a degree* -1 *map whose restriction* $W(n)_{4m-n} \to U(n)_{4m-n-1}$ *is an isomorphism unless* $4m-n-1 = 0$.

Proof. Item (1) holds for degree reasons. For item (2), note that the inclusion map $\mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; BO_\mathbb{Q})^0 \to \mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; BO_\mathbb{Q})^0 /\!\!/ \mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})^{\mathrm{id}}$ pulls back κ_{L_m} to the class $\lambda_{[U^n_{g,1}],L_m}$. Therefore, the latter class is a permanent cycle, which verifies (2).

For the other two claims, we use the $\mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})$-equivariant homotopy equivalence

$$\mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; BO_\mathbb{Q})^0 \simeq \prod_{m\geq 1} \mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; K(\mathbb{Q}, 4m))^0$$

given by the cohomology classes $(L_m)_{m\geq 1}$. Since the latter homotopy equivalence, combined with the projection onto the mth factor, pulls back $\lambda_{a,u_m} \in H^*(\mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; K(\mathbb{Q}, 4m))^0; \mathbb{Q})$ to λ_{a,L_m}, the proposition follows from the next Lemma. □

Lemma 4.31. *The E_2-term of the spectral sequence of the fibration*

$$\mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; K(\mathbb{Q}, k))^0 /\!\!/ \mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})^{\mathrm{id}} \to B\mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})^{\mathrm{id}} \tag{4.32}$$

is (in the columns up to degree $n-3$, by Proposition 4.17) given by

$$\mathbb{F}(L^2(N(g)))[2,0] \oplus N(g)[0, k-n] \oplus N(g)^\vee[0, k-n-1])$$

(if $k-n-1 > 0$; if $k-n-1 = 0$, the last summand is dropped and if $k-n \leq 0$, the last two summands are dropped). The differential d_2 vanishes on $N(g)^\vee[0, k-n-1]$, and on $N(g)[0, k-n]$, it is, up to a sign, the map adjoint to the projection $N(g)^{\otimes 2} \to L^2(N(g))$, if $k-n-1 > 1$.

Proof. The differential vanishes on $N(g)^\vee[0, k-n-1]$ for degree reasons, and because the fibration (4.32) has a section. The claim about $N(g)[0, k-n]$ is true for the same reason if $k-n-1 = 0$, and so we may suppose $k-n-1 > 0$. We pair the spectral sequence of (4.32) with the spectral sequence of the universal fibration

$$(U^n_{g,1})_\mathbb{Q} /\!\!/ \mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})^{\mathrm{id}} \to B\mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})^{\mathrm{id}}. \tag{4.33}$$

We write $\tilde{E}^{*,*}_*$ for the rational cohomological spectral sequence of the latter. The E_2-term is of the form

$$\tilde{E}_2 = S^*(L^2 N(g)) \otimes H^*(U^n_{g,1}; \mathbb{Q})$$

(in the columns up to degree $(n-3)$, by Proposition 4.17). Let $x_{ij} \in L^2(N_g)$ the image of $a_i \otimes a_j \in N_g^{\otimes 2}$ under the projection map; these elements satisfy $x_{ji} = (-1)^n x_{ij}$. By Proposition 4.17), the differential $\tilde{d}_2 : \tilde{E}_2^{0,n+1} \to \tilde{E}_2^{2,n}$ is given on basis elements by the formulas

$$\tilde{d}_2(\beta_i) = \sum_j x_{ij} \otimes \alpha_j \tag{4.34}$$

and

$$\tilde{d}_2(\alpha_i) = 0. \tag{4.35}$$

To transfer this knowledge about $\tilde{E}^{*,*}_*$ to information about $E^{*,*}_*$, observe that the evaluation map $\mathrm{ev} : U^n_{g,1} \times \mathrm{map}_\partial(U^n_{g,1}; K(\mathbb{Q}, k))^0 \to K(\mathbb{Q}, k)$ is $\mathrm{hAut}_\partial(U^n_{g,1})$-equivariant and hence induces a map

$$(U^n_{g,1} \times \mathrm{map}_\partial(U^n_{g,1}; K(\mathbb{Q}, k))^0) /\!\!/ \mathrm{hAut}_\partial(U^n_{g,1}) \to K(\mathbb{Q}, k).$$

It follows that $\mathrm{ev}^* u_k \in H^k(U^n_{g,1} \times \mathrm{map}_\partial(U^n_{g,1}; K(\mathbb{Q}, k))^0)$ lies in the image of the map induced by the inclusion

$$U^n_{g,1} \times \mathrm{map}_\partial(U^n_{g,1}; K(\mathbb{Q}, k))^0 \to (U^n_{g,1} \times \mathrm{map}_\partial(U^n_{g,1}; K(\mathbb{Q}, k))^0) /\!\!/ \mathrm{hAut}_\partial(U^n_{g,1})$$

of the fibre in the total space. This forces $\mathrm{ev}^* u_k \in \hat{E}_2^{0,k}$ to be a permanent cycle in the spectral sequence $\hat{E}_*^{*,*}$ of

$$(U_{g,1}^n \times \mathrm{map}_\partial(U_{g,1}^n; K(\mathbb{Q},k))^0) /\!\!/ \mathrm{hAut}_\partial((U_{g,1}^n)_\mathbb{Q})^{\mathrm{id}} \to B\mathrm{hAut}_\partial((U_{g,1}^n)_\mathbb{Q})^{\mathrm{id}},$$

so $\hat{d}_2(\mathrm{ev}^* u_k) = 0$. By formula (3.17)

$$\mathrm{ev}^* u_k = \sum_{i=1}^g \alpha_i \times \lambda_{a_i,u_k} + \sum_{i=1}^g \beta_i \times \lambda_{b_i,u_k}.$$

Using (4.34) and (4.35), and that $d_2(\lambda_{b_i,u_k}) = 0$ for degree reasons, we obtain

$$0 = \sum_{i=1}^g \hat{d}_2(\alpha_i \times \lambda_{a_i,u_k}) + \hat{d}_2(\beta_i \times \lambda_{b_i,u_k}) =$$

$$= \sum_{i=1}^g (d_2\alpha_i) \times \lambda_{a_i,u_k} + \sum_{i=1}^g (-1)^n \alpha_i \times (d_2 \lambda_{a_i,u_k}) + \sum_{i=1}^g (\tilde{d}_2 \beta_i) \times \lambda_{b_i,u_k} + \sum_{i=1}^g (-1)^{n+1} \beta_i \times (\tilde{d}_2 \lambda_{b_i,u_k}) =$$

$$= \sum_{j=1}^g (-1)^n \alpha_j \times (d_2 \lambda_{a_j,u_k}) + \sum_{i,j=1}^g x_{ij} \otimes (\alpha_j \times \lambda_{b_i,u_k}).$$

Comparing coefficients leads to the identity

$$d_2 \lambda_{a_j,u_k} = (-1)^{n+1} \sum_{i=1}^g x_{ij} \otimes \lambda_{b_i,u_k}$$

which finishes the proof. □

Propositions 4.29 and 4.30 were the goals of this section. Before we can use these to evaluate the two spectral sequences for (4.22) and (4.23), we need to switch gears and introduce some more algebraic background.

5. A representation-theoretic calculation

5.1. **Generalities.**

Rational representations. Let $\mathbb{K}$ be a field of characteristic 0, and let N be a finite-dimensional $\mathbb{K}$-vector space. Recall that a representation $\rho : \mathrm{GL}(N) \to \mathrm{GL}(W)$ on some other $\mathbb{K}$-vector space is *rational* if the matrix entries (after choice of a basis) of $\rho(g)$ are polynomial functions of the matrix entries of g and of $\det(g)^{-1}$. A similar definition applies to representations of $\mathrm{SL}(N)$.

Let $\mathbb{K} \subset \mathbb{L}$ be a field extension, let N and W be $\mathbb{K}$-vector spaces and let $\rho : \mathrm{GL}(N) \to \mathrm{GL}(W)$ be rational. Then ρ extends to a rational representation $\rho_\mathbb{L} : \mathrm{GL}(N_\mathbb{L}) \to \mathrm{GL}(W_\mathbb{L})$. A similar statement is true for $\mathrm{SL}(N)$.

If there is $w \in \mathbb{Z}$ such that each element $\lambda \in \mathbb{K}^\times \subset \mathrm{GL}(N)$ in the centre acts by multiplication with λ^w on W, we say that ρ has *weight* w. We shall use the notation $T^{k,l}(N) := N^{\otimes k} \otimes (N^\vee)^{\otimes l}$.

The rational representations are described in terms of Schur functors, whose definition we briefly recall. Let $\mathcal{P}_n$ be the set of partitions of n, thought of as Young diagrams. If λ is a partition of n, we also write $|\lambda| = n$. The *height* $\mathrm{ht}(\lambda)$ of $\lambda \in \mathcal{P}_n$ is the number of rows of λ. To a partition λ of n, there is associated the Young symmetrizer $c_\lambda \in \mathbb{Q}[\Sigma_n] \subset \mathbb{K}[\Sigma_n]$ and the associated irreducible Σ_n-representation M_λ (over $\mathbb{K}$; it is irreducible since it is irreducible when the scalars are extended to the algebraic closure $\overline{\mathbb{K}}$). The tensor power $T^{n,0}(N)$ has a canonical

$\mathrm{GL}(N) \times \Sigma_n$-representation, and the Schur functor $S_\lambda(N)$ is defined as the $\mathrm{GL}(N)$-representation

$$S_\lambda(N) := c_\lambda \cdot T^{n,0}(N).$$

The theory of rational representations can be summarized in the following result.

Theorem 5.1. *Let N be a finite-dimensional $\mathbb{K}$-vector space.*

(1) *As $\mathrm{GL}(N) \times \Sigma_n$-modules, we have*

$$T^{n,0}(N) = \bigoplus_{\lambda \in \mathcal{P}_n} S_\lambda(N) \otimes M_\lambda.$$

If $\mathrm{ht}(\lambda) > \dim(N)$, then $S_\lambda(N) = 0$, and if $\mathrm{ht}(\lambda) \leq \dim(N)$, $S_\lambda(N)$ is nonzero and irreducible. The Schur functors $S_\lambda(N)$ and $S_\mu(N)$ are isomorphic only if they both vanish or if $\lambda = \mu$.

(2) *After taking the tensor product with a suitable power of the determinant representation $\det(N)$, each $\mathrm{GL}(N)$-representation embeds into a sum of copies of $T^{n,0}(N)$, for some n. The same is true for $\mathrm{SL}(N)$-representations instead.*

(3) *Rational $\mathrm{GL}(N)$- and $\mathrm{SL}(N)$-representations are completely reducible, and the Schur functors give a complete list of the irreducible representations.*

(4) *Let N, V, W be $\mathbb{K}$ vector spaces and let $\mathrm{GL}(N) \to \mathrm{GL}(V)$ and $\mathrm{GL}(N) \to \mathrm{GL}(W)$ be rational representations. Let $\mathbb{K} \subset \mathbb{L}$ be a field extension. Then W is irreducible if and only if $W_\mathbb{L}$ is irreducible; V and W are isomorphic if and only if $V_\mathbb{L}$ and $W_\mathbb{L}$ are isomorphic, and furthermore*

$$(W^{\mathrm{GL}(N)})_\mathbb{L} = (W_\mathbb{L})^{\mathrm{GL}(N_\mathbb{L})}.$$

The same is true for SL in place of GL.

References. Statements (1)–(3) are well-known when $\mathbb{K}$ is algebraically closed. We explain why the proof given in the textbook [58] carries over to arbitrary $\mathbb{K}$.

(1) the decomposition is shown for algebraically closed fields (of characteristic 0) in [58, Theorem 9.3.1.4], but it exists over any ground field of characteristic 0 because the Young symmetrizer have rational coefficients. For a field extension $\mathbb{K} \subset \mathbb{L}$, we have $S_\lambda(N_\mathbb{L}) \cong (S_\lambda(N))_\mathbb{L}$ as $\mathrm{GL}(N_\mathbb{L})$-representations. The statement about the (non)vanishing of the Schur functors follows immediately. Since $S_\lambda(N)$ is irreducible when the ground field is algebraically closed, irreducibility follows for an arbitrary field. The statement about isomorphisms $S_\lambda(N) \cong S_\mu(N)$ follows from the fact that the character of $S_\lambda(N)$ at $x \in \mathrm{GL}(N)$ is the Schur polynomial s_λ, evaluated on the eigenvalues of x.

(2) The proof of [58, Lemma 7.1.4] does not use that $\mathbb{K}$ is algebraically closed. (3) follows from (2), since $T^{n,0}(N)$ is completely reducible, and by generalities on completely reducible representations, e.g. [19, Proposition 3.1.4]. (4) If $W_\mathbb{L}$ is irreducible, then clearly W is irreducible. Conversely, if W is irreducible, it must be a Schur functor, hence $W_\mathbb{L}$ is also a Schur functor, hence irreducible. By what we just saw, the identity $(W^{\mathrm{GL}(N)})_\mathbb{L} = (W_\mathbb{L})^{\mathrm{GL}(N_\mathbb{L})}$ is true for irreducible W, and follows for all W by complete reducibility. The statement about isomorphisms is also clear from all the other statements. □

To see how $T^{k,l}(N)$ for $l > 0$ fits in, one uses the isomorphism

$$N^\vee \cong \det(N)^{-1} \otimes \Lambda^{g-1}(N)$$

of $\mathrm{GL}(N)$-representations for g-dimensional N.

Definition 5.2. Let $\rho : \mathrm{SL}(N) \to \mathrm{GL}(W)$ be a rational representation. We say that ρ (or W) has *load* $\leq n$ if each irreducible summand of W is a direct summand of $T^{k,l}(N)$, for some k, l with $k + l \leq n$.

The fundamental theorem of invariant theory. We need the fundamental theorem of invariant theory of $\mathrm{GL}(N)$. We can identify $T^{m,m}(N) \cong \mathrm{End}(N)^{\otimes m} \cong \mathrm{End}(N^{\otimes m})$. There is a natural map

$$\sigma_{N,m} : \mathbb{K}[\Sigma_m] \to \mathrm{End}(N^{\otimes m})^{\mathrm{GL}(N)} \cong T^{m,m}(N)^{\mathrm{GL}(N)} \tag{5.3}$$

given by the Σ_m-action permuting the factors.

Theorem 5.4. *Let N be a finite-dimensional $\mathbb{K}$-vector space. The map $\sigma_{N,m}$ is surjective, and it is also injective if $m \leq \dim(N)$.*

Surjectivity is the content of the first fundamental theorem which is proven in e.g. [58, §9.1.2], [46, Theorem 9.1.2] or [28, Theorem 5.3.1] (and is a key ingredient for Theorem 5.1). Injectivity is the second fundamental theorem. The treatment of that result in [58] or [28] has a slightly different layout; the version as stated above is shown in [46, Theorem 9.1.3].

Let $(a_1, \ldots, a_g)$ be a basis of N and let $(a^1, \ldots, a^g)$ be the dual basis of $N^\vee$. Under the identification $T^{m,m}(N) \cong \mathrm{End}(N^{\otimes m})$, the map $\sigma_{N,m}$ is given by the formula

$$s \mapsto \sum_{i_1,\ldots,i_m} a_{i_1} \otimes \ldots \otimes a_{i_m} \otimes a^{i_{s^{-1}(1)}} \otimes \ldots \otimes a^{i_{s^{-1}(m)}}. \tag{5.5}$$

Special linear groups. For technical reasons, we have to consider the special linear groups as well.

Lemma 5.6. *Let N be a finite-dimensional $\mathbb{K}$-vector space. Then*

(1) $T^{k,l}(N)^{\mathrm{GL}(N)} = 0$ *unless* $k = l$.
(2) $T^{k,l}(N)^{\mathrm{SL}(N)} = 0$ *unless* $\dim(N)$ *divides* $k - l$.
(3) $T^{k,k}(N)^{\mathrm{SL}(N)} = T^{k,k}(N)^{\mathrm{GL}(N)}$.

Proof. (1) is easy; just look at the action of a scalar matrix. (2) Use Theorem 5.1 (4) to replace $\mathbb{K}$ by its algebraic closure. Then $\mathrm{SL}(N)$ contains a primitive gth root of unity ζ_g, where $g = \dim(N)$, which acts by ζ_g^{k-l} on $T^{k,l}(N)$. Hence $T^{k,l}(N)^{\mathrm{SL}(N)} = 0$ unless g divides $k-l$. (3) Use Theorem 5.1 (4) to replace $\mathbb{K}$ by its algebraic closure. The canonical isomorphism $T^{k,k}(N) \cong \mathrm{End}(N^{\otimes k})$ identifies $T^{k,k}(N)^{\mathrm{GL}(N)}$ with the commutant algebra of the image of $\rho_{\mathrm{GL}} : \mathbb{K}[\mathrm{GL}(N)] \to \mathrm{End}(N^{\otimes k})$. Similarly, $T^{k,k}(N)^{\mathrm{SL}(N)}$ is the commutant of the image of $\rho_{\mathrm{SL}} : \mathbb{K}[\mathrm{SL}(N)] \to \mathrm{End}(N^{\otimes k})$, so it is enough to prove that ρ_{GL} and ρ_{SL} have the same image. Each element $A \in \mathrm{GL}(N)$ can be written as $A = \lambda B$ with $\lambda \in \mathbb{K}^\times$ and $B \in \mathrm{SL}(V)$ (here we are using that $\mathbb{K}$ is algebraically closed). It follows that $\rho_{\mathrm{GL}}(A) = \lambda^k \rho_{\mathrm{SL}}(B) \in \mathrm{im}(\rho_{\mathrm{SL}})$. □

5.2. **A special invariant calculation.** In this section, we carry out a representation-theoretic calculation that will be used later on. The main ideas for the proof were communicated to us by Jerzy Weyman, and we thank him for allowing to reproduce his argument here.

Let $\mathbb{K}$ be a field of characteristic 0, and let N, W and U be finite-dimensional $\mathbb{K}$-vector spaces. We let $g := \dim(N)$ and fix a basis $(a_1, \ldots, a_g)$ of N, with dual basis $(a^1, \ldots, a^g)$. We consider the algebras

$$A := S^*(S^2(N)) \otimes S^*(N \otimes W) \otimes \Lambda^*(N^\vee \otimes U)$$

and

$$C := S^*(\Lambda^2(N)) \otimes \Lambda^*(N \otimes W) \otimes S^*(N^\vee \otimes U).$$

These algebras have an obvious trigrading and actions of the group $\mathrm{GL}(N) \times \mathrm{GL}(W) \times \mathrm{GL}(U)$. We want to determine the algebras $A^{\mathrm{GL}(N)}$ and $C^{\mathrm{GL}(N)}$ of $\mathrm{GL}(N)$-invariants. There are some obvious invariants. We define

$$\varphi_A : \Lambda^2(U) \to (S^1(S^2(N)) \otimes S^0(N \otimes W) \otimes \Lambda^2(N^\vee \otimes U))^{\mathrm{GL}(N)} = A^{\mathrm{GL}(N)}_{1,0,2}$$

by

$$u_1 \wedge u_2 \mapsto \sum_{i,j=1}^{g} (a_i \cdot a_j) \otimes 1 \otimes ((a^i \otimes u_1) \wedge (a^j \otimes u_2)),$$

and we define

$$\psi_A : W \otimes U \to (S^0(S^2(N)) \otimes S^1(N \otimes W) \otimes \Lambda^1(N^\vee \otimes U))^{\mathrm{GL}(N)} = A^{\mathrm{GL}(N)}_{0,1,1}$$

by

$$w \otimes u \mapsto \sum_{i=1}^{g} 1 \otimes (a_i \otimes w) \otimes (a^i \otimes u).$$

For $x_0, x_1 \in \Lambda^2(U)$ and $y_0, y_1 \in W \otimes U$, the relations

$$\varphi_A(x_0)\varphi_A(x_1) = \varphi_A(x_1)\varphi_A(x_0),$$
$$\psi_A(y_0)\psi_A(y_1) = \psi_A(y_1)\psi_A(y_0)$$

and

$$\varphi_A(x_0)\psi_A(x_1) = \psi_A(x_1)\varphi_A(x_0)$$

hold, and these imply that $\varphi_A \oplus \psi_A$ extends to an algebra map

$$G : S^*(\Lambda^2(U)) \otimes \Lambda^*(W \otimes U) \to A^{\mathrm{GL}(N)}.$$

Similarly let us define

$$\varphi_C : \Lambda^2(U) \to (S^1(\Lambda^2(N)) \otimes \Lambda^0(N \otimes W) \otimes S^2(N^\vee \otimes U))^{\mathrm{GL}(N)} = C^{\mathrm{GL}(N)}_{1,0,2} \subset C^{\mathrm{GL}(N)}$$

by

$$u_1 \wedge u_2 \mapsto \sum_{1 \leq i < j \leq g} (a_i \wedge a_j) \otimes 1 \otimes ((a^i \otimes u_1) \cdot (a^j \otimes u_2))$$

and

$$\psi_C : W \otimes U \to (S^0(\Lambda^2(N)) \otimes \Lambda^1(N \otimes W) \otimes S^1(N^\vee \otimes U))^{\mathrm{GL}(N)} = C^{\mathrm{GL}(N)}_{0,1,1} \subset C^{\mathrm{GL}(N)}$$

by

$$w \otimes u \mapsto \sum_{i=1}^{g} 1 \otimes (a_i \otimes w) \otimes (a^i \otimes u).$$

A similar argument as above shows that $\varphi_C \oplus \psi_C$ extends to an algebra map

$$H : S^*(\Lambda^2(U)) \otimes \Lambda^*(W \otimes U) \to C^{\mathrm{GL}(N)}.$$

Proposition 5.7. *We have* $A^{\mathrm{GL}(N)}_{p,q,r} = C^{\mathrm{GL}(N)}_{p,q,r} = 0$ *unless* $2p + q - r = 0$, *and* $A^{\mathrm{SL}(N)}_{p,q,r} = C^{\mathrm{SL}(N)}_{p,q,r} = 0$ *unless* $g := \dim(N)$ *divides* $2p + q - r$. *The maps*

$$S^p(\Lambda^2(U)) \otimes \Lambda^q(W \otimes U) \xrightarrow{G} A^{\mathrm{GL}(N)}_{p,q,2p+q} \subset A^{\mathrm{SL}(N)}_{p,q,2p+q}$$

and

$$S^p(\Lambda^2(U)) \otimes \Lambda^q(W \otimes U) \xrightarrow{H} C^{\mathrm{GL}(N)}_{p,q,2p+q} \subset C^{\mathrm{SL}(N)}_{p,q,2p+q}$$

are surjective, and isomorphisms as long as $2p + q \leq g$.

We use Proposition 5.7 in conjunction with Proposition 5.8 below. To state it, let us assume that $\mathbb{K} = \mathbb{Q}$ and that N has an *integral form*, i.e. a subgroup $N_{\mathbb{Z}} \subset N$ such that $N_{\mathbb{Z}} \otimes \mathbb{Q} = N$. In that case, we have the subgroup $\mathrm{SL}(N_{\mathbb{Z}}) \subset \mathrm{SL}(N)$ of automorphisms preserving $N_{\mathbb{Z}}$; note that $\mathrm{SL}(N_{\mathbb{Z}}) \cong \mathrm{SL}_g(\mathbb{Z})$.

Proposition 5.8. *If $\mathbb{K} = \mathbb{Q}$ and N has an integral form $N_{\mathbb{Z}}$, the inclusions*

$$A^{\mathrm{SL}(N)}_{p,q,2p+q} \subset A^{\mathrm{SL}(N_{\mathbb{Z}})}_{p,q,2p+q}$$

and

$$C^{\mathrm{SL}(N)}_{p,q,2p+q} \subset C^{\mathrm{SL}(N_{\mathbb{Z}})}_{p,q,2p+q}$$

are equalities.

Proof of Proposition 5.7, surjectivity. The first sentence is a straightforward application of Lemma 5.6. It also follows from Lemma 5.6 that $A^{\mathrm{GL}(N)}_{p,q,2p+q} = A^{\mathrm{SL}(N)}_{p,q,2p+q}$ and $C^{\mathrm{GL}(N)}_{p,q,2p+q} = C^{\mathrm{SL}(N)}_{p,q,2p+q}$. So we must only show that G and H are isomorphism, and we start with surjectivity.

Consider the case of G first. We establish a commutative diagram

$$\begin{array}{ccc} \mathbb{Q}[\Sigma_{2p+q}] \otimes W^{\otimes q} \otimes U^{\otimes 2p+q} & \xrightarrow{F} & (N^{\otimes 2p+q} \otimes (N^{\vee})^{\otimes 2p+q} \otimes W^{\otimes q} \otimes U^{\otimes 2p+q})^{\mathrm{GL}(N)} \\ \downarrow Q & & \downarrow S \\ S^p(\Lambda^2(U)) \otimes \Lambda^q(W \otimes U) & \xrightarrow{G} & A^{\mathrm{GL}(N)}_{p,q,2p+q}, \end{array} \tag{5.9}$$

and show that F and S are surjective. The map S is the restriction of the ($\mathrm{GL}(N)$-equivariant) quotient map

$$N^{\otimes 2p+q} \otimes (N^{\vee})^{\otimes 2p+q} \otimes W^{\otimes q} \otimes U^{\otimes 2p+q} \cong N^{\otimes 2p} \otimes (N \otimes W)^{\otimes q} \otimes (N^{\vee} \otimes U)^{\otimes 2p+q} \to A_{q,p,2p+q}$$

to the invariant subspace. As the quotient map is surjective and as the category of rational representations of $\mathrm{GL}(N)$ is semisimple, S is surjective.

The map F sends

$$s \otimes w_1 \otimes \ldots \otimes w_q \otimes u_1 \otimes \ldots \otimes u_{2p+q} \in \Sigma_{2p+q} \otimes W^{\otimes q} \otimes U^{\otimes 2p+q} \subset \mathbb{Q}[\Sigma_{2p+q}] \otimes W^{\otimes q} \otimes U^{\otimes 2p+q}$$

to

$$\sum_{i_1,\ldots,i_{2p+q}} a_{i_1} \otimes \ldots \otimes a_{i_{2p+q}} \otimes a^{i_{s^{-1}(1)}} \otimes \ldots \otimes a^{i_{s^{-1}(2p+q)}} \otimes w_1 \otimes \ldots \otimes w_q \otimes u_1 \otimes \ldots \otimes u_{2p+q},$$

and is clearly $\mathrm{GL}(N)$-invariant. Formula (5.5) shows that (upon identification of its target) F is the tensor product of the map $\sigma_{N,2p+q}$ defined in (5.3) and the identity on $W^{\otimes q} \otimes U^{\otimes 2p+q}$. Therefore, by Theorem 5.4, F is surjective.

We define the map Q on $\Sigma_{2p+q} \otimes W^{\otimes q} \otimes U^{\otimes 2p+q}$ by

$$s \otimes w_1 \otimes \ldots \otimes w_q \otimes u_1 \otimes \ldots \otimes u_{2p+q} \mapsto$$

$$\mathrm{sgn}(s)(u_{s(1)} \wedge u_{s(2)}) \cdots (u_{s(2p-1)} \wedge u_{s(2p)}) \otimes (w_1 \otimes u_{s(2p+1)}) \wedge \ldots \wedge (w_q \otimes u_{s(2p+q)}).$$

It remains to prove that (5.9) commutes, but this follows from

$$SF(s \otimes w_1 \otimes \ldots \otimes w_q \otimes u_1 \otimes \ldots \otimes u_{2p+q}) =$$

$$\sum_{i_1,\ldots,i_{2p+q}} (a_{i_1} \cdot a_{i_2}) \cdots (a_{i_{2p-1}} \cdot a_{i_{2p}}) \otimes (a_{i_{2p+1}} \otimes w_1) \cdots (a_{i_{2p+q}} \otimes w_q) \otimes (a^{i_{s^{-1}(1)}} \otimes u_1) \wedge \ldots \wedge (a^{i_{s^{-1}(2p+q)}} \otimes u_{2p+q})$$

and

$$GQ(s \otimes w_1 \otimes \ldots \otimes w_q \otimes u_1 \otimes \ldots \otimes u_{2p+q}) =$$

$$\operatorname{sgn}(s) \sum_{i_1,\ldots,i_{2p+q}} (a_{i_1}\cdot a_{i_2})\cdots(a_{i_{2p-1}}\cdot a_{i_{2p}})\otimes(a_{i_{2p+1}}\otimes w_1)\cdots(a_{i_{2p+q}}\otimes w_q)\otimes(a^{i_1}\otimes u_{s(1)})\wedge\ldots\wedge(a^{i_{2p+q}}\otimes u_{s(2p+q)}).$$

This finishes the proof that G is surjective. The case of H is almost identical. In that case, we consider

$$\begin{array}{ccc} \mathbb{Q}[\Sigma_{2p+q}] \otimes W^{\otimes q} \otimes U^{\otimes 2p+q} & \xrightarrow{F} & (N^{\otimes 2p+q} \otimes (N^{\vee})^{\otimes 2p+q} \otimes W^{\otimes q} \otimes U^{\otimes 2p+q})^{\mathrm{GL}(N)} \\ \downarrow{\scriptstyle Q'} & & \downarrow{\scriptstyle S'} \\ S^p(\Lambda^2(U)) \otimes \Lambda^q(W \otimes U) & \xrightarrow{G} & C^{\mathrm{GL}(N)}_{p,q,2p+q}. \end{array} \tag{5.10}$$

The map F is the same map as before. The map Q' is defined just as Q, the only difference being that the factor $\operatorname{sgn}(s)$ in front of the definition of Q is dropped, and the map S' is again the quotient map. □

For the proof of injectivity of G and H, we need some classical results of invariant theory. Here is some notation: the conjugate partition to $\lambda \in \mathcal{P}_n$ is denoted $\tilde{\lambda} \in \mathcal{P}_n$. By $\mathcal{P}^{\mathrm{er}}_{2p}$, we denote the set of partitions of $2p$ with even rows, and by $\mathcal{P}^{\mathrm{ec}}_{2p}$ the set of partitions of $2p$ with even columns. The first ingredient we shall use are the Cauchy formulas which state that [58, §9.6.3, p. 271]

$$S^q(V \otimes W) = \bigoplus_{\lambda \in \mathcal{P}_q} S_\lambda(V) \otimes S_\lambda(W), \tag{5.11}$$

[58, §9.8.4]

$$\Lambda^r(V \otimes W) = \bigoplus_{\lambda \in \mathcal{P}_r} S_\lambda(V) \otimes S_{\tilde{\lambda}}(W) \tag{5.12}$$

and [58, §11.4.5]

$$S^p(S^2(V)) = \bigoplus_{\lambda \in \mathcal{P}^{\mathrm{er}}_{2p}} S_\lambda(V), \tag{5.13}$$

as well as [58, §11.4.5]

$$S^p(\Lambda^2(V)) = \bigoplus_{\lambda \in \mathcal{P}^{\mathrm{ec}}_{2p}} S_\lambda(V). \tag{5.14}$$

Furthermore, we need the formula [58, §12.5.1]

$$S_\lambda(V) \otimes S_\mu(V) = \bigoplus_{|\kappa|=|\lambda|+|\mu|} c^\kappa_{\lambda,\mu} S_\kappa(V). \tag{5.15}$$

The coefficients $c^\kappa_{\lambda,\mu} \in \mathbb{N}_0$ are the well-known *Littlewood–Richardson coefficients*. These are the structure constants of the ring $\bigwedge$ of symmetric functions (over the integers) when one takes the Schur functions s_λ, $\lambda \in \mathcal{P}$, as a basis. From this, it follows that the coefficients in (5.15) do not depend on $\dim(V)$. The symmetry

$$c^\kappa_{\lambda,\mu} = c^\kappa_{\mu,\lambda};$$

of the Littlewood–Richardson coefficients is obvious; we also need to know the relation

$$c^\kappa_{\lambda,\mu} = c^{\tilde{\kappa}}_{\tilde{\lambda},\tilde{\mu}}. \tag{5.16}$$

To see (5.16), let $\omega : \bigwedge \to \bigwedge$ be the involutive (ring) automorphism which is constructed in [47, p.21]; formula (3.8) on p.42 of [47] shows that $\omega(s_\lambda) = s_{\tilde{\lambda}}$. Since the Littlewood–Richardson coefficients are the structure constants with respect to the Schur functions, (5.16) follows. An alternative proof of (5.16) can be found in [22, p. 62].

Proof of Proposition 5.7, injectivity. Since we already saw that G and H are surjective, it suffices to show that the dimensions of the two vector spaces agree (degreewise, and in the range of degrees we claimed). It is therefore of no danger to write $S = S'$ for isomorphic representations S and S', and $nS := S^{\oplus n}$.

We first turn to the map G. Its components are maps

$$S^p(\Lambda^2 U) \otimes \Lambda^q(W \otimes U) \to (S^p(S^2(N)) \otimes S^q(N \otimes W) \otimes \Lambda^{2p+q}(N^\vee \otimes U))^{\mathrm{GL}(N)}.$$

As a $\mathrm{GL}(N) \times \mathrm{GL}(W) \times \mathrm{GL}(U)$-representation, we have by (5.13), (5.11) and (5.12)

$$S^p(S^2(N)) \otimes S^q(N \otimes W) \otimes \Lambda^{2p+q}(N^\vee \otimes U) = \bigoplus_{\lambda \in \mathcal{P}^{\mathrm{er}}_{2p}, |\mu|=q, |\nu|=2p+q} S_\lambda(N) \otimes S_\mu(N) \otimes S_\mu(W) \otimes S_\nu(N^\vee) \otimes S_{\tilde{\nu}}(U).$$

By (5.15), the latter is isomorphic to

$$\bigoplus_{\lambda \in \mathcal{P}^{\mathrm{er}}_{2p}, |\mu|=q, |\nu|=|\kappa|=2p+q} c^\kappa_{\lambda,\mu} S_\kappa(N) \otimes S_\mu(W) \otimes S_\nu(N^\vee) \otimes S_{\tilde{\nu}}(U).$$

Since the $S_\lambda(N)$ are mutually nonisomorphic irreducible $\mathrm{GL}(N)$-representations or trivial, we have, with $g := \dim(N)$,

$$(S_\kappa(N) \otimes S_\nu(N^\vee))^{\mathrm{GL}(N)} \cong \begin{cases} \mathbb{Q} & \kappa = \nu \text{ and } \mathrm{ht}(\nu) = \mathrm{ht}(\kappa) \leq g \\ 0 & \kappa \neq \nu \text{ or } \mathrm{ht}(\nu) > g \text{ or } \mathrm{ht}(\kappa) > g. \end{cases} \tag{5.17}$$

Therefore

$$(S_\kappa(N) \otimes S_\mu(W) \otimes S_\nu(N^\vee) \otimes S_{\tilde{\nu}}(U))^{\mathrm{GL}(N)} = \begin{cases} S_\mu(W) \otimes S_{\tilde{\nu}}(U) & \kappa = \nu \text{ and } \mathrm{ht}(\nu) = \mathrm{ht}(\kappa) \leq g \\ 0 & \kappa \neq \nu \text{ or } \mathrm{ht}(\nu) > g \text{ or } \mathrm{ht}(\kappa) > g, \end{cases}$$

and so

$$\begin{aligned} &(S^p(S^2(N)) \otimes S^q(N \otimes W) \otimes \Lambda^{2p+q}(N^\vee \otimes U))^{\mathrm{GL}(N)} \\ &= \bigoplus_{\lambda \in \mathcal{P}^{\mathrm{er}}_{2p}, |\mu|=q, |\nu|=2p+q, \mathrm{ht}(\nu) \leq g} c^\nu_{\lambda,\mu} S_\mu(W) \otimes S_{\tilde{\nu}}(U) \\ &= \bigoplus_{\lambda \in \mathcal{P}^{\mathrm{er}}_{2p}, |\mu|=q, |\nu|=2p+q, \mathrm{ht}(\nu) \leq g} c^{\tilde{\nu}}_{\tilde{\lambda},\tilde{\mu}} S_\mu(W) \otimes S_{\tilde{\nu}}(U) \quad \text{(by (5.16))}. \end{aligned}$$

Under the hypothesis that $g \geq 2p + q$, $\mathrm{ht}(\nu) \leq g$ holds for all $\nu \in \mathcal{P}_{2p+q}$. Using (5.15) again, the latter is isomorphic to

$$\bigoplus_{\lambda \in \mathcal{P}^{\mathrm{er}}_{2p}, |\mu|=q} S_\mu(W) \otimes S_{\tilde{\lambda}}(U) \otimes S_{\tilde{\mu}}(U),$$

and by (5.12), this agrees with

$$\bigoplus_{\lambda \in \mathcal{P}^{\mathrm{er}}_{2p}} \Lambda^q(W \otimes U) \otimes S_{\tilde{\lambda}}(U) = \Lambda^q(W \otimes U) \otimes \bigoplus_{\lambda \in \mathcal{P}^{\mathrm{ec}}_{2p}} S_\lambda(U) =$$

$$= \Lambda^q(W \otimes U) \otimes S^p(\Lambda^2(U)).$$

The proof for H is almost identical. The components of H are maps

$$S^p(S^2 U) \otimes \Lambda^q(W \otimes U) \to (S^p(\Lambda^2(N)) \otimes \Lambda^q(N \otimes W) \otimes S^{2p+q}(N^\vee \otimes U))^{\mathrm{GL}(N)}.$$

We compute, by (5.14), (5.11) and (5.12) and (5.15),

$$S^p(\Lambda^2(N)) \otimes \Lambda^q(N \otimes W) \otimes S^{2p+q}(N^\vee \otimes U) =$$

$$= \bigoplus_{\lambda \in \mathcal{P}^{\mathrm{ec}}_{2p}, |\mu|=q, |\nu|=2p+q} S_\lambda(N) \otimes S_\mu(N) \otimes S_{\tilde{\mu}}(W) \otimes S_\nu(N^\vee) \otimes S_\nu(U) =$$

$$= \bigoplus_{\lambda \in \mathcal{P}^{\mathrm{ec}}_{2p}, |\mu|=q, |\nu|=|\kappa|=2p+q} c^\kappa_{\lambda,\mu} S_\kappa(N) \otimes S_{\tilde{\mu}}(W) \otimes S_\nu(N^\vee) \otimes S_\nu(U).$$

Taking $\mathrm{GL}(N)$-invariants and using (5.17) yields

$$(S^p(\Lambda^2(N)) \otimes \Lambda^q(N \otimes W) \otimes S^{2p+q}(N^\vee \otimes U))^{\mathrm{GL}(N)} =$$

$$= \bigoplus_{\lambda \in \mathcal{P}^{\mathrm{ec}}_{2p}, |\mu|=q, |\nu|=2p+q, \mathrm{ht}(\nu) \leq g} c^\nu_{\lambda,\mu} S_{\tilde{\mu}}(W) \otimes S_\nu(U).$$

If $g \geq 2p+q$, (5.15) shows that this is equal to

$$\bigoplus_{\lambda \in \mathcal{P}^{\mathrm{ec}}_{2p}, |\mu|=q} S_{\tilde{\mu}}(W) \otimes S_\lambda(U) \otimes S_\mu(U) =$$

$$(\bigoplus_{\lambda \in \mathcal{P}^{\mathrm{ec}}_{2p}} S_\lambda(U)) \otimes (\bigoplus_{|\mu|=q} S_{\tilde{\mu}}(W) \otimes S_\mu(U)) \overset{(5.14),(5.12)}{=}$$

$$S^p(\Lambda^2(U)) \otimes \Lambda^q(W \otimes U)$$

as claimed. □

Finally, we give the short proof of Proposition 5.8. This is an immediate consequence of a more general result.

Lemma 5.18. *Let $N_{\mathbb{Z}}$ be a finitely generated free abelian group, write $N := N_{\mathbb{Z}} \otimes \mathbb{Q}$ and let $\rho : \mathrm{SL}(N) \to \mathrm{GL}(W)$ be a rational representation. Then*

$$W^{\mathrm{SL}(N)} = W^{\mathrm{SL}(N_{\mathbb{Z}})}.$$

Proof. Assume $N = \mathbb{Z}^g$. A special case of Borel's density theorem [4] states that $\mathrm{SL}_g(\mathbb{Z}) \subset \mathrm{SL}_g(\mathbb{Q})$ is Zariski dense (a very short and elementary proof for the special linear group has been written down by Putman [59]). For $v \in V^{\mathrm{SL}_g(\mathbb{Z})}$ and $\ell \in V^*$, the function $\mathrm{SL}_g(\mathbb{Q}) \to \mathbb{Q}$, $A \mapsto \ell(v - Av)$, is polynomial and vanishes on $\mathrm{SL}_g(\mathbb{Z})$, hence on $\mathrm{SL}_g(\mathbb{Q})$, whence $v = Av$ for all $A \in \mathrm{SL}_g(\mathbb{Q})$. □

6. The cohomology of the block diffeomorphism space

6.1. **Using invariant theory.** In this section, we finish our partial evaluation of the spectral sequence of the fibration

$$\mathrm{map}_\partial((U^n_{g,1})_{\mathbb{Q}}; BO_{\mathbb{Q}})^0 \to \mathrm{map}_\partial((U^n_{g,1})_{\mathbb{Q}}; BO_{\mathbb{Q}})^0 /\!\!/ \mathrm{hAut}_\partial((U^n_{g,1})_{\mathbb{Q}})^{\mathrm{id}} \to B\mathrm{hAut}_\partial((U^n_{g,1})_{\mathbb{Q}})^{\mathrm{id}}. \tag{6.1}$$

Before we state the result, let us fix some bounds that the various parameters have to fulfil.

Assumption 6.2. (1) We assume throughout that $n \geq 5$.
(2) We pick M in (4.25) large enough so that

$$4M \geq 3n - 5.$$

(3) We furthermore choose g large enough to satisfy

$$g > n - 3,$$

which implies also that $g \geq 3$.

Using the number M, we define the graded vector spaces $U(n)$, $V(n)$ and $W(n)$ as in (4.25). We let $v_m \in V(n)_{4m-2n-1}$ and $w_m \in W(n)_{4m-n}$ be the obvious generators, and let $u_m := S(w_m) \in U(n)_{4m-n-1}$ be the image under the map S from Proposition 4.30. To formulate the result we are aiming at, some more notation is necessary.

Definition 6.3. We let $K(n)$ be the following graded vector space. It has basis elements $k_m \in K(n)_{4m-2n-1}$ for $m \leq M$ and $4m - 2n - 1 > 0$, and it has basis elements $k_{m_0,m_1} \in K(n)_{4(m_0+m_1)-2n-1}$ for $m_0 \leq m_1 \leq M$, $4m_0 \geq n+1$, $4(m_0 + m_1) - 2n - 1 > 0$.

Note that all generators in $K(n)$ are in odd degrees. We define a (degree-preserving) map

$$\xi : \Lambda^*(K(n)) \to H^*(\mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; BO_\mathbb{Q})^0 /\!\!/ \mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})_\mathbb{Z} \tag{6.4}$$

by

$$\xi(k_m) := \kappa_{L_m},\ \xi(k_{m_0,m_1}) := \kappa_{L_{m_0}L_{m_1}}.$$

The composition of ξ with the pullback map

$$H^*(\mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; BO_\mathbb{Q})^0 /\!\!/ \mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})_\mathbb{Z}; \mathbb{Q}) \to H^*(\mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; BO_\mathbb{Q})^0 /\!\!/ \mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})^{\mathrm{id}}; \mathbb{Q})$$

goes into the $\mathrm{SL}(N(g)_\mathbb{Z})$-invariant part. Here is the goal of this subsection.

Proposition 6.5. *The map*

$$\xi : \Lambda^*(K(n)) \to H^*(\mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; BO_\mathbb{Q})^0 /\!\!/ \mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})^{\mathrm{id}}; \mathbb{Q})^{\mathrm{SL}(N(g))_\mathbb{Z}}$$

is an isomorphism in degrees $* \leq (n-4)$, provided that M and g satisfy the bounds from (6.2).

Recall from Observation 4.24 that the spectral sequence $E^{*,*}_*$ of (6.1) is a spectral sequence of $\mathrm{GL}(N(g)) \cong \pi_0(\mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q}))$-modules. Let us elaborate this a little.

Proposition 6.6. *Let $n \geq 5$ and $g \geq 3$.*

1. *For $q \leq (n-3)$, $H^q(\mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; BO_\mathbb{Q})^0 /\!\!/ \mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})^{\mathrm{id}}; \mathbb{Q})$ is a rational representation of $\mathrm{SL}(N(g)_\mathbb{Z})$, of load $\leq q$.*
2. *Let $E^{*,*}_*$ denote the spectral sequence of* (6.1)*, and define*
$$\overline{E}^{p,q}_r := (E^{p,q}_r)^{\mathrm{SL}(N(g)_\mathbb{Z})}.$$
Then $\overline{E}^{*,*}_*$ is a spectral sequence, and it converges to $H^*(\mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; BO_\mathbb{Q})^0 /\!\!/ \mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})^{\mathrm{id}}; \mathbb{Q})^{\mathrm{SL}(N(g))_\mathbb{Z}}$.

Proof. (1) We have to invoke a deep result by Bass–Milnor–Serre [1, Corollary 16.6]. The quoted result implies that a homomorphism $\mathrm{SL}(N(g)) \to \mathrm{GL}(V)$, where V is a finite-dimensional $\mathbb{Q}$-vector space and $g \geq 3$, is actually rational (see also the discussion on p. 63 f and p. 134 of loc.cit.). Hence $H^q(\mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; BO_\mathbb{Q})^0 /\!\!/ \mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})^{\mathrm{id}}; \mathbb{Q})$ is a rational $\mathrm{SL}(N(g))$-representation.

From the description of the E_2-term given in Proposition 4.29, it is apparent that (in total degrees $* \leq (n-3)$) the $\mathrm{GL}(N(g))$-representations in the E_2-term are all rational. Since subquotients of rational representations are again rational (and the load does not increase), the representations occuring in $E_\infty^{*,*}$ with total degree q are all rational, of load $\leq q$.

(2): Theorem 5.1 and Lemma 5.18 together imply that taking $\mathrm{SL}(N(g)_\mathbb{Z})$-invariants is an exact functor from rational $\mathrm{GL}(N(g))$-representations to $\mathbb{Q}$-vector spaces. Hence $\overline{E}_*^{*,*}$ is also a spectral sequence. The claim about convergence follows from Theorem 6.17. □

Hence in order to prove Proposition 6.5, we can focus our attention completely on $\overline{E}_*^{*,*}$.

Definition 6.7. Let $D^{*,*}$ be the bigraded algebra

$$D^{*,*} := \Lambda^*(V(n) \oplus W(n) \otimes U(n)) \otimes S^*(\mathbb{Q}[2,0] \otimes \Lambda^2(U(n)))$$

where the graded vector spaces $W(n), U(n)$ and $V(n)$ are as in (4.25), but sit in bidegrees $(0,*)$. Let $\delta : D^{*,*} \to D^{*,*}$ be the differential of degree $(2,-1)$ which is a derivation, and is given on the generators as follows:

(1) $\delta|_{V(n)} = 0$,
(2) $\delta|_{W(n)\otimes U(n)}$ is the map $W(n) \otimes U(n) \overset{S\otimes 1}{\to} U(n) \otimes U(n) \to \Lambda^2(U(n)) \cong \mathbb{Q}[2,0] \otimes \Lambda^2(U(n))$,
(3) $\delta|_{\mathbb{Q}[2,0]\otimes\Lambda^2(U(n))} = 0$.

We define a graded algebra homomorphism

$$\eta : D^{*,*} \to \overline{E}_2^{*,*} = (E_2^{*,*})^{\mathrm{SL}(N_\mathbb{Z}(g))} \tag{6.8}$$

to the invariant part of the spectral sequence of (6.1), by sending

$$\begin{aligned} v_m &\mapsto \lambda_{[U^n_{g,1}],L_m} \in E_2^{0,4m-2n-1} \\ w_{m_0} \otimes u_{m_1} &\mapsto \sum_i \lambda_{a_i,L_{m_0}} \lambda_{b_i,L_{m_1}} \in E_2^{0,4m_0+4m_1-2n-1} \\ u_{m_0} \wedge u_{m_1} &\mapsto \sum_{ij} x_{ij} \otimes \lambda_{b_i,L_{m_0}} \lambda_{b_j,L_{m_1}} \in E_2^{2,4m_0+4m_1-n-2}. \end{aligned}$$

Here $x_{ij} \in E_2^{2,0}$ are the generators in $E_2^{2,0} \cong L^2(N(g))$.

Proposition 6.9. *The map η is a map of differential bigraded algebras, and it is an isomorphism in total degrees $\leq (n-3)$, provided that the bounds of 6.2 are satisfied.*

Proof. That η is an isomorphism in the indicated range of degrees follows from Propositions 4.29, 5.7 and 5.8. The first two conditions of 6.2 are needed for 4.29, and the third for 5.7.

Proposition 4.30 and the definition of the maps in Proposition 5.7 show that η is compatible with the differential. □

Let us next compute the cohomology of the differential graded algebras $D^{*,*}$. This is in terms of the following well-known construction.

Definition 6.10. Let $F : Y \to X$ be a linear map of finite-dimensional $\mathbb{Q}$-vector spaces. The *Koszul complex* of the map F is the graded commutative differential graded algebra

$$D_F := \Lambda^*(Y) \otimes S^*(X),$$

where $S^1(X)$ has degree 2, $\Lambda^1(Y)$ has degree 1, and the differential $d_F : D_F \to D_F$ is the derivation of degree $+1$ given by the condition that $d_F|_{S^1(X)} = 0$ and that $d_F : \Lambda^1(Y) \to S^1(X)$ is the map F.

Up to a different grading, $D^{*,*}$ is D_F, where F is the map

$$V(n) \oplus W(n) \otimes U(n) \overset{0 \oplus S \otimes 1_{U(n)}}{\to} U(n)^{\otimes 2} \to \Lambda^2(U(n)). \tag{6.11}$$

In order to compute $H^*(D^{*,*})$ and hence $(E_3^{*,*})^{\mathrm{SL}(N_{\mathbb{Z}}(g))}$ in a range of degrees, we compute $H^*(D_F)$ in general. There are obvious linear maps

$$\ker(F) \cong \ker\Big(d_F : \Lambda^1(Y) \to S^1(X)\Big) \to H^1(D_F)$$

and

$$\operatorname{coker}(F) \cong \operatorname{coker}\Big(d_F : \Lambda^1(Y) \to S^1(X)\Big) \to H^2(D_F).$$

They give a map of graded commutative algebras

$$\eta_F : \Lambda^*(\ker(F)) \otimes S^*(\operatorname{coker}(F)) \to H^*(D_F).$$

Lemma 6.12. *The map η_F is an isomorphism.*

Proof. It follows from the Künneth formula that if η_{F_0} and η_{F_1} are isomorphisms, then so is $\eta_{F_0 \oplus F_1}$.

By elementary linear algebra, we can write F as a direct sum of an isomorphism and a zero map, and so it suffices to treat these two cases separately. If F is a zero map, the claim is obvious. If F is an isomorphism, we can assume without loss of generality that F is an identity map. In that case, D_{id_V} is the Koszul complex of the vector space V which is well-known to be acyclic (since we are over a field of characteristic 0), see e.g. [30, §3.1]. □

Proof of Proposition 6.5. The map in (6.11) is surjective. Lemma 6.12 and Proposition 6.9 prove that

$$\overline{E}_3^{p,q} = 0$$

if $p + q \leq n - 3$ and $p \neq 0$. Hence, using Proposition 6.6, the natural map

$$A : H^q(\mathrm{map}_\partial((U_{g,1}^n)_{\mathbb{Q}}; BO_{\mathbb{Q}})^0 /\!\!/ \mathrm{hAut}_\partial((U_{g,1}^n)_{\mathbb{Q}})^{\mathrm{id}}; \mathbb{Q})^{\mathrm{SL}(N(g))_{\mathbb{Z}}} \to \overline{E}_\infty^{0,q} \subset \overline{E}_3^{0,q}$$

is an isomorphism if $q \leq n - 4$. It is therefore sufficient to show that $A \circ \xi : \Lambda^*(K(n)) \to \overline{E}_3^{0,*}$ is an isomorphism in the indicated range of degrees. By the definition of ξ and by Lemma 3.25, $A \circ \xi$ is given by

$$k_m \mapsto \lambda_{[U_{g,1}^n], L_m}$$

and

$$k_{m_0,m_1} \mapsto \lambda_{[U_{g,1}^n], L_{m_0} L_{m_1}}.$$

We must therefore check that $\overline{E}_3^{0,*}$ is the free graded-commutative algebra on the listed generators. Lemma 6.12 tells us how to do that. We distinguish to cases.

First let $n \not\equiv 3 \pmod 4$. Then the map $S : W(n) \to U(n)$ is an isomorphism. We deduce that the following set is a basis for the kernel of (6.11):

$$\{v_m, (w_{m_0} \otimes u_{m_1} + w_{m_1} \otimes u_{m_0})|\ 4m-2n-1 > 0, m \leq M, m_0 \leq m_1 \leq M, 4m_0 - n - 1 > 0\}. \tag{6.13}$$

Under the map η from (6.8), the element v_m is mapped to $\lambda_{[U^n_{g,1}],L_m}$. The element $w_{m_0} \otimes u_{m_1} + w_{m_1} \otimes u_{m_0}$ is mapped to

$$\begin{aligned} &\sum_i \lambda_{a_i,L_{m_0}} \lambda_{b_i,L_{m_1}} + \lambda_{a_i,L_{m_1}} \lambda_{b_i,L_{m_0}} \\ =&\sum_i \lambda_{a_i,L_{m_0}} \lambda_{b_i,L_{m_1}} + \lambda_{b_i,L_{m_0}} \lambda_{a_i,L_{m_1}} \text{(for degree reasons)} \\ &= \lambda_{[U^n_{g,1}],L_{m_0}L_{m_1}} \text{(by (4.4))}. \end{aligned}$$

This completes the proof if $n \not\equiv 3 \pmod 4$.

If $n \equiv 3 \pmod 4$, the kernel of (6.11) is larger: one obtains a basis by adding the elements

$$w_{\frac{n+1}{4}} \otimes u_m,\ 4m - n - 1 > 0, m \leq M$$

to the list of elements in (6.13). Using (6.8) and (4.4) again, these elements go to $\lambda_{[U^n_{g,1}],L_{\frac{n+1}{4}}L_m}$. So the proof is complete. □

6.2. **Using Borel's vanishing theorem.** We now look at the spectral sequence of the fibration

$$\mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; BO_\mathbb{Q})^0 /\!\!/ \mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})^{\mathrm{id}} \to \mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; BO_\mathbb{Q})^0 /\!\!/ \mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})_\mathbb{Z} \to B\mathrm{GL}(N(g)_\mathbb{Z}). \tag{6.14}$$

Recall the graded vector space $K(n)$ from Definition 6.3 and from (6.4) the map

$$\xi : \Lambda^*(K(n)) \to H^*(\mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; BO_\mathbb{Q})^0 /\!\!/ \mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})_\mathbb{Z}; \mathbb{Q}).$$

Let moreover B be the graded vector space $B := \bigoplus_{k\geq 1} \mathbb{Q}[4k+1]$. Mapping the generators to the Borel classes gives furthermore a map

$$\beta : \Lambda^*(B) \to H^*(\mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; BO_\mathbb{Q})^0 /\!\!/ \mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})_\mathbb{Z}; \mathbb{Q}).$$

Proposition 6.15. *Assume the bounds from* (6.2) *(which in particular means $n \geq 5$), but strengthened by*

$$g \geq 2n - 4.$$

Then the map

$$\xi \otimes \beta : \Lambda^*(K(n) \oplus B) \to H^*(\mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; BO_\mathbb{Q})^0 /\!\!/ \mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})_\mathbb{Z}; \mathbb{Q})$$

is an isomorphism in degrees $* \leq (n-4)$.

When combined with Proposition 3.27, we obtain the following result as a corollary. This establishes Theorem 1.8 from the introduction.

Corollary 6.16. *Assume the bounds of Proposition 6.15. Then in degrees $\leq (n-4)$, $H^*(B\widetilde{\mathrm{Diff}}_\partial(U^n_{g,1}); \mathbb{Q})$ is the exterior algebra on the Borel classes and on the tautological classes $\kappa_{L_{m_0}L_{m_1}}$ with $m_0 \leq m_1$, $4m_0 - n > 0$ and $4(m_0+m_1) - 2n - 1 > 0$.* □

The proof of Proposition 6.15 relies on Borel's vanishing theorem [6] that we shall state first.

Theorem 6.17 (Borel)**.** *Let V be a rational representation of $\mathrm{SL}_g(\mathbb{Q})$ of load at most n. Then the map*

$$H^p(\mathrm{SL}_g(\mathbb{Z}); \mathbb{Q}) \otimes V^{\mathrm{SL}_g(\mathbb{Z})} = H^p(\mathrm{SL}_g(\mathbb{Z}); \mathbb{Q}) \otimes V^{\mathrm{SL}_g(\mathbb{Q})} \to H^p(\mathrm{SL}_g(\mathbb{Z}); V^{\mathrm{SL}_g(\mathbb{Q})}) \to H^p(\mathrm{SL}_g(\mathbb{Z}); V)$$

is an isomorphism, provided that $2p + 2 \leq g - n$ (the first equation holds by Lemma 5.18).

References. This is essentially due to Borel [6, Theorem 4.4], but the ranges are not made explicit in Borel's work, so some more words need to be said here. By complete reducibility, we can assume that $V \subset T^{k,l}(\mathbb{Q}^g)$ with $k+l \leq n$, and finally suppose that $V = T^{k,l}(\mathbb{Q}^g)$. Moreover, it is enough to prove that statement for $\mathbb{Q}$ replaced by $\mathbb{R}$ and $\mathbb{Q}^g$ replaced by $\mathbb{R}^g$.

Borel proved in loc.cit. that

$$H^p(\mathrm{SL}_g(\mathbb{Z});\mathbb{R}) \otimes T^{k,l}(\mathbb{R}^g)^{\mathrm{SL}_g(\mathbb{R})} \to H^p(\mathrm{SL}_g(\mathbb{Z}); T^{k,l}(\mathbb{R}^g)) \tag{6.18}$$

is an isomorphism provided that $p \leq \min(M(\mathrm{SL}_g(\mathbb{R}),(k,l)), C(\mathrm{SL}_g(\mathbb{R}),(k,l)))$, where $M(\mathrm{SL}_g(\mathbb{R}),(k,l))$ and $C(\mathrm{SL}_g(\mathbb{R}),(k,l))$ are constants which can be read off from the root system of $\mathfrak{sl}_g$ and the weights of $T^{k,l}(\mathbb{Q}^g)$. In loc.cit., Borel showed that $M(\mathrm{SL}_g(\mathbb{R}),(k,l)) \geq g-2$, but left $C(\mathrm{SL}_g(\mathbb{R}),(k,l))$ implicit. A relatively naive counting argument given in the proof of [42, Theorem 7.3] shows that $C(\mathrm{SL}_g(\mathbb{R}),(k,l)) \geq \frac{1}{8}g^2 - \max(k,l) - 1$, which implies that (6.18) is an isomorphism when $p \leq f_{k,l}(g)$, and $f_{k,l}$ is a function with $\lim_{g\to\infty} f_{k,l}(g) = \infty$.

To get at the range claimed by us, we use Van der Kallen's work on homological stability, more precisely [71, Theorem 5.6]. The latter result implies that for $2q+2 \leq g-(k+l)$ and all $h \geq g$, the map $H_q(\mathrm{SL}_g(\mathbb{Z}); T^{k,l}(\mathbb{R}^g)) \to H_q(\mathrm{SL}_h(\mathbb{Z}); T^{k,l}(\mathbb{R}^h))$ is an isomorphism. This implies a cohomological statement by an instance of the universal coefficient theorem [64, Lemma 3.5], and we can pick h large enough so that $f(h) \geq q$. □

Remark 6.19. Tshishiku [70] showed by a careful analysis of root systems that the corresponding result is true for $\mathrm{SO}_{g,g}(\mathbb{Z})$ and $\mathrm{Sp}_{2g}(\mathbb{Z})$, but in a range that only depends on g, not on the representation V. We have been informed by him that the analogous procedure for $\mathrm{SL}_g(\mathbb{Z})$ does *not* lead to a range independent of V.

Proof of Proposition 6.15. Since Theorem 6.17 is about the special linear group rather than the general linear group, we modify the sequence a bit and look at

$$\mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; BO_\mathbb{Q})^0 /\!\!/ \mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})^{\mathrm{id}} \to \mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; BO_\mathbb{Q})^0 /\!\!/ \mathrm{ShAut}_\partial((U^n_{g,1})_\mathbb{Q})_\mathbb{Z} \to B\mathrm{SL}(N(g)_\mathbb{Z}), \tag{6.20}$$

where $\mathrm{ShAut}_\partial((U^n_{g,1})_\mathbb{Q})_\mathbb{Z} \subset \mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})_\mathbb{Z}$ consists of those homotopy automorphisms whose action on $H_n(U^n_{g,1})$ is by maps of determinant 1. The natural map

$$\mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; BO_\mathbb{Q})^0 /\!\!/ \mathrm{ShAut}_\partial((U^n_{g,1})_\mathbb{Q})_\mathbb{Z} \to \mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; BO_\mathbb{Q})^0 /\!\!/ \mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})_\mathbb{Z}$$

is, up to homotopy, a 2-fold covering. Hence it induces an injective map in rational cohomology by a general argument [34, Proposition 3G.1]. We may therefore show the Proposition with $\mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; BO_\mathbb{Q})^0 /\!\!/ \mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})_\mathbb{Z}$ replaced by $\mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; BO_\mathbb{Q})^0 /\!\!/ \mathrm{ShAut}_\partial((U^n_{g,1})_\mathbb{Q})_\mathbb{Z}$.

Let $E^{*,*}_*$ be the spectral sequence for (6.20). Its E_2-term is

$$E_2^{p,q} = H^p(\mathrm{SL}(N(g)_\mathbb{Z}); H^q(\mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; BO_\mathbb{Q})^0 /\!\!/ \mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})^{\mathrm{id}}; \mathbb{Q})).$$

By Proposition 6.6, the coefficient module is a rational representation of load $\leq q$. Hence if $2p+q+2 \leq g$, we may invoke Theorem 6.17 and see that

$$E_2^{p,q} = H^p(\mathrm{SL}(N(g)_\mathbb{Z}); \mathbb{Q}) \otimes H^q(\mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; BO_\mathbb{Q})^0 /\!\!/ \mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})^{\mathrm{id}}; \mathbb{Q}))^{\mathrm{SL}(N(g)_\mathbb{Z})}. \tag{6.21}$$

Under our bound on g, this holds for all p, q with $p+q \leq n-4$.

It follows from Proposition 6.5 and (6.4) that the map

$$H^*(\mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; BO_\mathbb{Q})^0 /\!\!/ \mathrm{ShAut}_\partial((U^n_{g,1})_\mathbb{Q})_\mathbb{Z}; \mathbb{Q}) \to H^*(\mathrm{map}_\partial((U^n_{g,1})_\mathbb{Q}; BO_\mathbb{Q})^0 /\!\!/ \mathrm{hAut}_\partial((U^n_{g,1})_\mathbb{Q})^{\mathrm{id}}; \mathbb{Q})^{\mathrm{SL}(N(g)_\mathbb{Z})}$$

is surjective if $* \leq (n-4)$, but that can be identified with the edge homomorphism of the spectral sequence, so that all differentials starting in the zeroeth column vanish (in degrees $* \leq n-4$). The isomorphism (6.21) implies that the map $E_2^{*,0} \otimes E_2^{0,*} \to E_2^{*,*}$ of bigraded vector spaces is an isomorphism in total degrees $\leq n-4$. As the spectral sequence is a spectral sequence of algebras, we conclude that all differentials starting in a term $E_r^{p,q}$ with $p+q \leq (n-4)$ are trivial.

An application of Theorem 2.11 and the Leray–Hirsch theorem shows that the map $\xi \otimes \beta$ is an isomorphism in degrees $\leq (n-4)$. □

7. The endgame: from block diffeomorphisms to actual diffeomorphisms

7.1. **Getting the mapping class group under control.** To shorten notation, we introduce the following notations:

$$\mathcal{D}_g := \mathrm{Diff}_\partial(U_{g,1}^n);\ \tilde{\mathcal{D}}_g := \widetilde{\mathrm{Diff}}_\partial(U_{g,1}^n).$$

We let $B\mathcal{D}_\infty := \mathrm{hocolim}_{g\to\infty} B\mathcal{D}_g$ and define $B\tilde{\mathcal{D}}_\infty$ analogously. The spaces

$$B\mathcal{D} := \coprod_{g\geq 0} B\mathcal{D}_g \text{ and } B\tilde{\mathcal{D}} := \coprod_{g\geq 0} B\tilde{\mathcal{D}}_g$$

carry E_{2n+1}-structures, analogous to the E_2-structure described in [23, §3] for diffeomorphisms of surfaces; we refrain from giving any more details here. As a consequence of May's recognition principle [49], the group completions $\Omega BB\mathcal{D}$ and $\Omega BB\tilde{\mathcal{D}}$ have the structures of $(2n+1)$-fold loop spaces.

Lemma 7.1. *The natural maps*

$$\mathbb{Z} \times B\mathcal{D}_\infty \to \Omega BB\mathcal{D} \tag{7.2}$$

and

$$\mathbb{Z} \times B\tilde{\mathcal{D}}_\infty \to \Omega BB\tilde{\mathcal{D}} \tag{7.3}$$

are acyclic (i.e. their homotopy fibres have the integral homology of a point). The commutator subgroups of $\pi_1(B\mathcal{D}_\infty)$ and $\pi_1(B\tilde{\mathcal{D}}_\infty)$ are perfect. The maps (7.2) *and* (7.3) *identify their targets with the Quillen plus construction on their source (here the Quillen plus construction is performed one component at a time, on the maximal perfect normal subgroup of the fundamental group).*

Proof. A straightforward application of the group completion theorem [51] shows that the two maps are integral homology equivalences. An improved version of the group completion theorem, namely [63, Theorem 1.1], proves the (stronger) claim of acyclicity. Perfectness of the commutator subgroups follows from [63, Proposition 3.1], and the statement about the Quillen plus construction is a consequence: by [35, Proposition 3.1], acyclic maps out of a given space are classified up to homotopy equivalence by the kernels of their induced maps on fundamental groups. □

Our next step is to let

$$B\mathcal{D}_g \to \mathcal{E}_g \to B\tilde{\mathcal{D}}_g$$

be the $(2n-5)$th stage of the Moore–Postnikov tower of the natural map $B\mathcal{D}_g \to B\tilde{\mathcal{D}}_g$. Let us motivate this maneuver.

Lemma 7.4.

(1) *The natural map*

$$B\mathcal{D}_g \to B\tilde{\mathcal{D}}_g$$

induces an isomorphism on fundamental groups.

(2) *The diagram*

$$\begin{array}{ccc} \mathcal{E}_g & \longrightarrow & \mathcal{E}_{g+1} \\ \downarrow & & \downarrow \\ B\tilde{\mathcal{D}}_g & \longrightarrow & B\tilde{\mathcal{D}}_{g+1} \end{array}$$

is homotopy cartesian, for each g.

(3) *The fibration $\mathcal{E}_g \to \tilde{\mathcal{D}}_g$ is "very simple" in the following sense: if $\mathcal{F}_g$ denotes its homotopy fibre, then the monodromy action $\mu(\gamma) : \mathcal{F}_g \to \mathcal{F}_g$, for each $\gamma \in \pi_1(B\tilde{\mathcal{D}}_g)$, is homotopic to the identity.*

Proof. (1): Surjectivity of $\pi_1(B\mathrm{Diff}_\partial(M)) \to \pi_1(B\widetilde{\mathrm{Diff}}_\partial(M))$ follows from the very definition of $\widetilde{\mathrm{Diff}}_\partial(M)$, and injectivity follows from Cerf's theorem [11, Théorème 0].

(2): It is an immediate consequence of a version of the so-called *Morlet's lemma of disjunction*, more precisely [10, Corollary 3.2 on page 29], that the comparison maps of vertical homotopy fibres in the

$$\begin{array}{ccc} B\mathcal{D}_g & \longrightarrow & B\mathcal{D}_{g+1} \\ \downarrow & & \downarrow \\ B\tilde{\mathcal{D}}_g & \longrightarrow & B\tilde{\mathcal{D}}_{g+1} \end{array}$$

are $(2n-4)$-connected. The claim then follows from the general properties of the Moore–Postnikov factorization.

(3): We follow the outline of a very similar argument contained in [43, §5.3]. We present the core argument first. Let

$$\sigma : \tilde{\mathcal{D}}_0/\mathcal{D}_0 \to \tilde{\mathcal{D}}_g/\mathcal{D}_g$$

be the stabilization map and let $\gamma \in \tilde{\mathcal{D}}_g$. Note that σ is given by gluing in (block) diffeomorphisms in a fixed disc, and note that γ can be isotoped so that it fixed this disc. It follows that

$$L_\gamma \circ \sigma \sim \sigma : \tilde{\mathcal{D}}_0/\mathcal{D}_0 \to \tilde{\mathcal{D}}_g/\mathcal{D}_g,$$

where L_γ denotes the left translation by γ on the homogeneous space $\tilde{\mathcal{D}}_g/\mathcal{D}_g$.

To turn this observation into an argument concerning the monodromy action, note that the monodromy action of γ on $\tilde{\mathcal{D}}_g/\mathcal{D}_g$ is exactly L_γ, and note that the map $q : \tilde{\mathcal{D}}_g/\mathcal{D}_g \to \mathcal{F}_g$ is the $(2n-5)$th Postnikov truncation, and equivariant with respect to the two monodromy actions. Also, pick a definite CW model for $\mathcal{F}_g$.

Since $\pi_k(\mathcal{F}_g) = 0$ if $k \geq 2n-5$ by construction, two maps $f_0, f_1 : K \to \mathcal{F}_g$ from a CW complex are homotopic if and only if the restrictions $f_j|_{K^{(2n-5)}}$ are homotopic. Apply this to $f_0 = \mathrm{id}_{\mathcal{F}_g}$ and $f_1 = \mu(\gamma)$. The inclusion of the $(2n-5)$-skeleton $\mathcal{F}_g^{(2n-5)} \to \mathcal{F}_g$ can be factored through maps

$$\mathcal{F}_g^{(2n-5)} \xrightarrow{h} \tilde{\mathcal{D}}_0/\mathcal{D}_0 \xrightarrow{\sigma} \tilde{\mathcal{D}}_g/\mathcal{D}_g \xrightarrow{q} \mathcal{F}_g.$$

Let $\gamma \in \tilde{\mathcal{D}}_g$. Then

$$\mu(\gamma)|_{\mathcal{F}_g^{(2n-5)}} \sim \mu(\gamma) \circ q \circ \sigma \circ h \sim q \circ L_\gamma \circ \sigma \circ h \sim q \circ \sigma \circ h = \mathrm{id}|_{\mathcal{F}_g^{(2n-5)}},$$

so that $\mu(\gamma) \sim \mathrm{id}$, as desired. □

To proceed, we need a general property of the Quillen plus construction.

Lemma 7.5. *Let $f : X \to Y$ be a n-connected map of connected spaces, $n \geq 2$, let $P \subset \pi_1(X) = \pi_1(Y)$ be a perfect normal subgroup of the common fundamental group, and let $X \to X^+$, $Y \to Y^+$ be the Quillen plus constructions on P. Then $f^+ : X^+ \to Y^+$ is n-connected.*

Proof. If $P = \pi_1(X)$, there is not much say, besides quoting Hurewicz' theorem. In the general case, let $\tilde{X} \to X$, $\tilde{Y} \to Y$ be the coverings with fundamental group P. Now X^+ can be realized as the homotopy pushout

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & \tilde{X}^+ \\ \downarrow & & \downarrow \\ X & \longrightarrow & X^+, \end{array}$$

see e.g. [34, p.374], and the claim follows. □

Lemma 7.6. *We let $\mathcal{E}_\infty := \mathrm{hocolim}_{g\to\infty}\mathcal{E}_g$. The commutator subgroup of $\pi_1(\mathcal{E}_\infty)$ is perfect, and the diagram*

$$\begin{array}{ccccc} \mathcal{E}_0 & \longrightarrow & \mathcal{E}_\infty & \longrightarrow & \mathcal{E}_\infty^+ \\ \downarrow & & \downarrow & & \downarrow \\ B\tilde{\mathcal{D}}_0 & \longrightarrow & B\tilde{\mathcal{D}}_\infty & \longrightarrow & B\tilde{\mathcal{D}}_\infty^+ \end{array} \tag{7.7}$$

is homotopy cartesian. Furthermore, the natural map

$$B\mathcal{D}_\infty^+ \to \mathcal{E}_\infty^+$$

is $(2n-5)$-connected.

Proof. First note that the composition

$$B\mathcal{D}_\infty \to \mathcal{E}_\infty \to B\tilde{\mathcal{D}}_\infty$$

induces an isomorphism on fundamental groups by Lemma 7.4 (1), and the first map is $(2n-5)$-connected by definition, so that both maps induce isomorphisms on fundamental groups. By Lemma 7.1, the commutator subgroup of $\mathcal{E}_\infty$ is hence perfect.

Lemma 7.4 (2) shows that the left square in (7.7) is homotopy cartesian. The right half of the diagram arises from applying the Quillen plus construction (on the commutator subgroups) to the middle column. For the proof that the right square is homotopy cartesian, we use a theorem by Berrick [3, Theorem 1.1].

Let $\mathcal{F}_\infty$ be the homotopy fibre of $\mathcal{E}_\infty \to B\tilde{\mathcal{D}}_\infty$. The latter map induces an isomorphism on fundamental groups, which three consequences: $\mathcal{F}_\infty$ is connected; $\pi_1(\mathcal{F}_\infty)$ is abelian (because it is a quotient of $\pi_2(B\tilde{\mathcal{D}}_\infty)$); and $\mathcal{F}_\infty$ is nilpotent ($\mathcal{F}_\infty$ is also the homotopy fibre of the map $\widetilde{\mathcal{E}_\infty} \to \widetilde{B\tilde{\mathcal{D}}_\infty}$ induced on universal coverings, so it is the homotopy fibre of a map of 1-connected spaces; it is a general fact that such homotopy fibres are nilpotent, see e.g. [50, Proposition 4.4.1]). Hence $\mathcal{F}^+_\infty = \mathcal{F}_\infty$ is nilpotent, which is part of hypothesis (b) of [3, Theorem 1.1]. Lemma 7.4 (3) shows that the commutator subgroup of $\pi_1(B\tilde{\mathcal{D}}_\infty)$ acts trivially on the homology of $\mathcal{F}_\infty$. It now follows from [3, Theorem 1.1] that

$$\mathcal{F}_\infty \to \mathcal{E}^+_\infty \to B\tilde{\mathcal{D}}^+_\infty$$

is a fibre sequence, which is exactly the statement that the right square in (7.7) is homotopy cartesian.

Finally, Lemma 7.5 implies that $B\mathcal{D}^+_\infty \to \mathcal{E}^+_\infty$ is $(2n-5)$-connected. $\square$

We now arrive at the goal of these constructions.

Lemma 7.8. *Let $\mathcal{G}$ be the homotopy fibre of the map $\Omega BB\mathcal{D} \to \Omega BB\tilde{\mathcal{D}}$ at the basepoint, so that there is a fibre sequence*

$$\mathcal{G} \xrightarrow{j} \Omega_0 BB\mathcal{D} \to \Omega_0 BB\tilde{\mathcal{D}}.$$

Then

(1) *$\mathcal{G}$ has the homotopy type of a connected $(2n+1)$-fold loop space, and its rational homotopy groups in degrees $k \leq 2n-6$ are given by*

$$\pi_k(\mathcal{G}) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & 1 \leq k \leq 2n-6,\ k \equiv 0 \pmod 4 \\ 0 & 1 \leq k \leq 2n-6,\ k \not\equiv 0 \pmod 4. \end{cases}$$

(2) *The map j induces the zero map on (reduced) rational homology up to degree $2n-5$.*

Proof. It is clear that $\mathcal{G}$ has the homotopy type of a connected loop space, so for (1) it suffices to calculate the dimensions of the rational homotopy groups in the indicated range. By Lemma 7.1 and Lemma 7.6, the homotopy groups of $\mathcal{G}$ agree with the homotopy groups of the fibre of $B\mathrm{Diff}_\partial(D^{2n+1}) \to B\widetilde{\mathrm{Diff}}_\partial(D^{2n+1})$, up to degree $2n-6$. The homotopy group $\pi_k(B\widetilde{\mathrm{Diff}}_\partial(D^{2n+1}))$ can be identified with the group Θ_{2n+1+k} of homotopy spheres, which is of course finite by [41]. The rational homotopy groups of $B\mathrm{Diff}_\partial(D^{2n+1})$ are famously related to algebraic K-theory, originally by [20]; the range we need was given by Krannich in [43, Corollary B]. This establishes (1).

For (2), consider the diagram

$$\begin{array}{ccccc} \tilde{\mathcal{D}}_0/\mathcal{D}_0 & \longrightarrow & \longrightarrow & \longrightarrow & \mathcal{G} \\ \downarrow & & & & \downarrow j \\ B\mathcal{D}_0 & \longrightarrow & B\mathcal{D}_\infty & \longrightarrow & B\mathcal{D}^+_\infty \\ \downarrow & & & & \downarrow \\ B\tilde{\mathcal{D}}_0 & \longrightarrow & B\tilde{\mathcal{D}}_\infty & \longrightarrow & B\tilde{\mathcal{D}}^+_\infty. \end{array}$$

The first named author proved in [14, Theorem 1.7], based on [7] and [57] that $B\mathcal{D}_0 \to B\mathcal{D}_\infty$ induces the zero map on (reduced) rational homology, up to degree $2n-3$. The top map is $(2n-5)$-connected, and as $B\tilde{\mathcal{D}}_0$ has finite homotopy groups, $\tilde{\mathcal{D}}_0/\mathcal{D}_0 \to B\mathcal{D}_0$ is a rational homotopy equivalence. Putting these facts together, triviality of $H_*(j)$ follows. $\square$

7.2. **Computation of the cohomology.** Having established Lemma 7.8, the completion of the proof of our main result is straightforward.

Proof of Theorem A. The three spaces in the fibre sequence

$$\mathcal{G} \xrightarrow{j} \Omega_0 BB\mathcal{D} \xrightarrow{p} \Omega_0 BB\tilde{\mathcal{D}}$$

are connected double loop spaces. Hence by the Milnor–Moore theorem [53], their rational homology (with the Pontrjagin product) is the free graded commutative algebra on the rational homotopy. It follows from Lemma 7.8 that the map $\pi_*(j)\otimes\mathbb{Q}$ is also trivial map up to degree $(2n-5)$. Hence $\pi_*(p)\otimes\mathbb{Q}$ is injective up to degree $(2n-5)$, so $H_*(p;\mathbb{Q})$ is injective, and $H^*(P;\mathbb{Q})$ is surjective, both in degrees $\leq (2n-5)$. Using the knowledge about the rational homotopy of $\mathcal{G}$, we see that $\mathrm{coker}(\pi_*(p)\otimes\mathbb{Q})$ is concentrated in degrees $4k+1$, $k\geq 1$ and in these degrees has dimension 1 (again in degrees $\leq (2n-5)$.

Hence the kernel of $H^*(p;\mathbb{Q})$ must, degrees $\leq (2n-5)$, be an ideal generated by classes in each degree $4k+1$, $k\geq 1$. However, we know by Proposition 2.14 that the ideal generated by the Borel classes lies in the kernel of $H^*(p;\mathbb{Q})$, and by a dimension count must be equal to the kernel. This finishes the evaluation of $H^*(\Omega_0 BB\mathcal{D};\mathbb{Q})$, which by Lemma 7.1 gives $H^*(B\mathcal{D}_\infty;\mathbb{Q})$. At the very last, we invoke [56, Corollary 1.3.2] to get the statement for finite values of g. □

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