# The icosahedral group and the homotopy of the stable mapping class group

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**Abstract**

We construct an explicit generator for $\pi_3(B\Gamma_\infty^+)$, the third homotopy group of the classifying space of stable mapping class group after Quillen's plus construction. The construction relies on a very classical action of the binary icosahedral group on some hyperelliptic Riemann surface.

## 1 Introduction and results

Let $F_g$ be an oriented closed connected surface of genus $g$, let $\mathrm{Diff}^+(F_g)$ be the topological group of orientation-preserving diffeomorphisms of $F_g$ and let $\Gamma_g := \pi_0(\mathrm{Diff}^+(F_g))$ be the *mapping class group*. If $g \geq 2$, then all components of $\mathrm{Diff}^+(F_g)$ are contractible [3] and so $B\Gamma_g \simeq B\,\mathrm{Diff}^+(F_g)$, in other words, $B\Gamma_g$ is the base space of the universal surface bundle $E_g \to B\Gamma_g$ of genus $g$.

It is known that $\Gamma_g$ is a perfect group [17] if $g \geq 3$ and so Quillen's plus construction - a familiar construction from algebraic K-theory, compare [18], p. 266 - can be applied to $B\Gamma_g$. The result is a simply-connected space $B\Gamma_g^+$

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and a homology equivalence $B\Gamma_g \to B\Gamma_g^+$. By definition, $B\Gamma_g$ is aspherical (i.e. all homotopy groups but the fundamental group are trivial), but $B\Gamma_g^+$ has many nontrivial higher homotopy groups.
How can one construct elements in $\pi_n(B\Gamma_g^+)$? Suppose that $M$ is an $n$-dimensional homology sphere and suppose that $\pi_1(M)$ acts on $F_g$ by orientation-preserving diffeomorphisms. Such an action yields a homomorphism $\rho : \pi_1(M) \to \Gamma_g$. Let $\lambda : M \to B\pi_1(M)$ be a classifying map for the universal covering of $M$. Consider the composition $B\rho \circ \lambda : M \to B\Gamma_g$. Quillen's plus construction gives a map $M^+ \to B\Gamma_g^+$. On the other hand, any degree one map $M \to \mathbb{S}^n$ is a homology equivalence and so it induces a homotopy equivalence $M^+ \simeq \mathbb{S}^n$. We end up with a map $\mathbb{S}^n \to B\Gamma_g^+$, giving an element $\theta_{M;\rho} \in \pi_n(B\Gamma_g^+)$ which only depends on $M$ and $\rho$.
The purpose of this note is to study an example of such an action when $M = \mathbb{S}^3/\hat{G}$ is the Poincaré sphere (the quotient of $\mathbb{S}^3$ by the binary icosahedral group $\hat{G}$) and $g = 14$. Here is our main result.

**Theorem 1.1.** *There exists an action of the binary icosahedral group $\hat{G} = \pi_1(M)$ on $F_{14}$, such that $\theta_{M;\rho}$ is a generator of $\pi_3(B\Gamma_{14}^+) \cong \mathbb{Z}/24$.*

Before we delve into the details, we explain our motivation for this apparently very specialized result. It comes from the recent homotopy theory of the *stable* mapping class group. To define the stable mapping class groups, we need to consider the mapping class groups of surfaces with boundary components. Let $F$ be an oriented surface of genus $g$ with $b$ boundary components, let $\mathrm{Diff}(F, \partial F)$ be the topological group of orientation-preserving diffeomorphisms which restrict to the identity on the boundary and put $\Gamma_{g,b} := \pi_0 \mathrm{Diff}(F; \partial)$. These groups are related by stabilization maps $\Gamma_{g,b} \to \Gamma_{g,b+1}$, $\Gamma_{g,b} \to \Gamma_{g,b-1}$, $\Gamma_{g,b} \to \Gamma_{g+1,b}$ for $b > 0$, which are obtained by gluing in pairs of pants, a disc or a torus in an appropriate way. The homological stability theorems of Harer [7], as improved by Ivanov [9], [10], assert that these maps induce isomorphisms on group homology $H_k(B\Gamma_{g,b}; \mathbb{Z})$ as long as $g \geq 2k + 2$. Let $\Gamma_{\infty,b} := \mathrm{colim}(\ldots \to \Gamma_{g,b+1} \to \Gamma_{g+1,b+1} \to \ldots)$. Abbreviate $\Gamma_\infty := \Gamma_{\infty,0}$. Quillen's plus construction gives maps $B\Gamma_{g,1}^+ \to B\Gamma_\infty^+$ and $B\Gamma_{g,1}^+ \to B\Gamma_g^+$. These maps are highly connected because of Harer stability. In particular, we get an isomorphism $\phi : \pi_n(B\Gamma_g^+) \to \pi_n(B\Gamma_\infty^+)$ for all pairs $(g, n)$ with $g \geq 2n+2$, for example for $(g, n) = (14, 3)$. So Theorem 1.1 is really about $\pi_3(B\Gamma_\infty^+)$. Note that, however, the isomorphism $\phi$ is *not* induced from a group homomorphism $\Gamma_g \to \Gamma_\infty$. There is a map $B\Gamma_g \to B\Gamma_\infty^+$

which induces $\phi$, but its construction requires the Madsen-Weiss theorem [13].
Due to the work of Tillmann, Madsen, Weiss and Galatius [20], [12], [13], [5], the homology of $B\Gamma_\infty$ is completely understood, at least in theory and one also gets some information about the homotopy groups of $B\Gamma_\infty^+$. Let us describe this story briefly.
Madsen and Tillmann [12] constructed a spectrum $\mathbf{MTSO}(2)$ and maps $\alpha_{g,b} : B\Gamma_{g,b} \to \Omega_0^\infty \mathbf{MTSO}(2)$ ([12] uses the notation $\mathbb{CP}^\infty_{-1}$ for $\mathbf{MTSO}(2)$). The maps $\alpha_{g,b}$ are compatible with the gluing maps and so they define a limit map $\alpha_\infty : B\Gamma_\infty \to \Omega_0^\infty \mathbf{MTSO}(2)$. The rational homology of $\Omega_0^\infty \mathbf{MTSO}(2)$ is easy to determine (it is isomorphic to the rational homology of $BU$, see [12]). The very complicated homology with $\mathbb{F}_p$-coefficients for all primes $p$ was computed by Galatius [5].
Madsen and Weiss showed that $\alpha_\infty$ is a homology equivalence. Therefore, the plus construction $(\alpha_\infty)^+ : B\Gamma_\infty^+ \to \Omega_0^\infty \mathbf{MTSO}(2)$ is a homotopy equivalence. Then $\beta_g = (\alpha_\infty^+)^{-1} \circ \alpha_g : B\Gamma_g \to \Omega_0^\infty \mathbf{MTSO}(2) \to B\Gamma_\infty^+$ induces the isomorphism $\phi$ above after plus construction.
The first few homotopy groups of $\Omega_0^\infty \mathbf{MTSO}(2)$ and hence of $B\Gamma_\infty^+$ are given by

**Theorem 1.2.**

$$\pi_k(B\Gamma_\infty^+) \cong \pi_k(\Omega_0^\infty \mathbf{MTSO}(2)) \begin{cases} 0 & \textit{if } k = 1 \\ \mathbb{Z} & \textit{if } k = 2 \\ \mathbb{Z}/24 & \textit{if } k = 3. \end{cases}$$

This is proven in [12], p.537, but the proof of the third isomorphism is only sketched. The proof can easily been completed using the computations of the first few stable homotopy groups of $\mathbb{CP}^\infty$ from [16], p. 199.
We should remark that the idea of defining elements of $\pi_n(X^+)$ using homology spheres is not new. For $n = 3$ and $X = B\,\mathrm{Gl}_\infty(\mathbb{Z})$, Jones and Westbury [11] give detailed computations. Their results could be used to prove Theorem 1.1, but our computations are more elementary and do not depend on [11].
**Outline of the paper:** In section 2, we construct a homomorphism $\eta : \pi_3(B\Gamma_\infty^+) \to \mathbb{Q}/\mathbb{Z}$ which will be used later on to detect homotopy classes. It is closely related to the classical $e$-invariant $K_3(\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ in algebraic K-theory, as we will explain in section 3.

In section 4, we recall some facts about the icosahedral groups and we construct the action of $\hat{G}$ on $F_{14}$ of Theorem 1.1. The computations leading to Theorem 1.1 are given in the last section 5.
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# 2 An invariant $\eta : \pi_3(B\Gamma_\infty^+) \to \mathbb{Q}/\mathbb{Z}$

Our procedure to detect elements in $\pi_3(B\Gamma_\infty^+)$ is based on two theorems by Harer. The first one [8] is the isomorphism (for $g \geq 8$)

$$H^3(B\Gamma_g; \mathbb{Q}) = 0, \tag{2.1}$$

which can also be derived from 1.2. The second one is that $H^2(B\Gamma_\infty; \mathbb{Z}) \cong \mathbb{Z}$, [6]. We need a precise description of the second cohomology group.
The mapping class group $\Gamma_g$ acts on the cohomology of the surface $F_g$ preserving the intersection form. The action defines a group homomorphism $\Gamma_g \to \mathrm{Sp}_{2g}(\mathbb{Z}) \to \mathrm{Sp}_{2g}(\mathbb{R})$. Because $U(g) \subset \mathrm{Sp}_{2g}(\mathbb{R})$ is a maximal compact subgroup, $B\,\mathrm{Sp}_{2g}(\mathbb{R}) \simeq BU(g)$. We obtain a map $\Phi_g : B\Gamma_g \to BU(g)$.
This map classifies a complex vector bundle $V_1 \to B\Gamma_g$, which has the following alternative description. Consider the universal surface bundle $\pi : E_g \to B\Gamma_g$. One can choose a complex structure on the vertical tangent bundle $T_v E_g$ of this bundle, turning $\pi : B\Gamma_g^1 \to B\Gamma_g$ into a family of Riemann surfaces. Let $V_1$ be the vector bundle whose fiber over $x \in B\Gamma_g$ is the dual of the $g$-dimensional vector space of holomorphic 1-forms on $\pi^{-1}(x)$. This is indeed a vector bundle once the topology is appropriately chosen, see [2]. The isomorphism class of $V_1$ does not depend on the choice of the complex structure because that complex structure is unique up to homotopy. The Hodge decomposition theorem shows that $\Phi_g$ is a classifying map for $V_1$.
There exists a universal map $\Phi : \Omega^\infty \mathbf{MTSO}(2) \to BU$, such that $\iota \circ \Phi_g \sim \Phi \circ \alpha_g$ ($\iota : BU(g) \to BU$ is the inclusion). This is shown in [12], p. 539 f. (they denote $\Phi$ by $\eta'$). Details can also be found in [4], ch. 6.

The map $\Phi$ can be used to obtain cohomology classes of $\Omega^\infty\mathbf{MTSO}(2)$ and hence, via $\alpha_g$, on each $B\Gamma_g$. We put $\zeta_i := \alpha_g^*\Phi^* c_i$ and $\gamma_i := \alpha_g^*\Phi^* s_i$; $s_i = i!\,\mathrm{ch}_i \in H^{2i}(BU;\mathbb{Z})$ the $i$th component of the integral Chern character. The following is the main result of [6][1], which also can be derived from 1.2.

**Theorem 2.2.** *The group $H^2(B\Gamma_g;\mathbb{Z})$ is infinite cyclic, generated by $\zeta_1$.*

We remark that $\zeta_1$ is the first Chern class of the determinant bundle $\det(V_1)$ of the bundle $V_1$. Another result we will need was proven by Morita [15], p. 555, as a consequence of Chern-Weil theory.

**Lemma 2.3.** *The class $\gamma_{2i} \in H^{2i}(B\Gamma_\infty^+)$ is a torsion class.*

Now we want to construct a homomorphism

$$\eta : \pi_3(B\Gamma_\infty^+) \to \mathbb{Q}/\mathbb{Z}. \tag{2.4}$$

For any space $X$, let $q_n^X : \tau_{\geq n}X \to X$ be the $(n-1)$-connected cover of $X$. The space $\tau_{\geq n}X$ is $(n-1)$-connected and $q_n^X$ induces an isomorphism $\pi_k(\tau_{\geq n}X) \to \pi_k(X)$ for all $k \geq n$. If $X$ is understood, we write $q_n := q_n^X$. The assignment $X \mapsto \tau_{\geq n}X$ is a functor on the homotopy category of spaces and the maps $q_n^X$ define a natural transformation of functors $q_n : \tau_{\geq n} \to \mathrm{id}$. If $X = B\Gamma_\infty^+$ and $n = 3$, then $\tau_{\geq 3}B\Gamma_\infty^+$ is the homotopy fiber of $\zeta_1 : B\Gamma_\infty^+ \to K(\mathbb{Z};2)$ by 1.2 and 2.2 and we can view $\tau_{\geq 3}B\Gamma_\infty^+$ as the total space of the circle bundle associated with $\det(V_1)$.

**Lemma 2.5.** *The cohomology class $q_3^*\zeta_2 \in H^4(\tau_{\geq 3}B\Gamma_\infty^+;\mathbb{Z})$ is a torsion class.*

*Proof.* Recall the relation

$$s_2 = c_1^2 - 2c_2 \in H^4(BU;\mathbb{Z}). \tag{2.6}$$

Applying $\Phi^*$ yields $\gamma_2 = \zeta_1^2 - 2\zeta_2 \in H^4(B\Gamma_\infty^+;\mathbb{Z})$. Consequently, $q_3^*(\gamma_2) = q_3^*(\zeta_1^2) - 2q_3^*(\zeta_2) = 0 - 2q_3^*(\zeta_2)$. By 2.3, $\gamma_2$ is a torsion class. $\square$

Because of 1.2, $H^3(\tau_{\geq 3}B\Gamma_\infty^+;\mathbb{Q}) = 0$. Therefore we get an exact sequence

$$0 \longrightarrow H^3(\tau_{\geq 3}B\Gamma_\infty^+;\mathbb{Q}/\mathbb{Z}) \xrightarrow{\delta} H^4(\tau_{\geq 3}B\Gamma_\infty^+;\mathbb{Z}) \longrightarrow H^4(\tau_{\geq 3}B\Gamma_\infty^+;\mathbb{Q}) \tag{2.7}$$

[1] This is stated falsely in [6], but Harer corrected the mistake in [7].

from the Bockstein exact sequence. Thus there exists a unique

$$\bar{\zeta_2} \in H^3(\tau_{\geq 3} B\Gamma^+_\infty; \mathbb{Q}/\mathbb{Z}) \text{ such that } \delta(\bar{\zeta_2}) = q_3^* \zeta_2.$$

Let $x : \mathbb{S}^3 \to B\Gamma^+_\infty$ be a map representing an element $x \in \pi_3(B\Gamma^+_\infty)$. There is a unique lift $\tilde{x} : \mathbb{S}^3 \to \tau_{\geq 3} B\Gamma^+_\infty$ of $x$. We define a homomorphism $\eta : \pi_3(B\Gamma^+_\infty) \to \mathbb{Q}/\mathbb{Z}$ by

$$\eta(x) := \langle \tilde{x}^* \bar{\zeta_2}; [\mathbb{S}^3] \rangle \in \mathbb{Q}/\mathbb{Z}. \tag{2.8}$$

# 3 Comparison with the $e$-invariant

Let us briefly describe the Adams-Quillen $e$-invariant, following [19]; it is a homomorphism $K_{2n-1}(\mathbb{C}_d) \to \mathbb{C}/\mathbb{Z}$. Here $\mathbb{C}_d$ denotes the complex numbers with the discrete topology and the $K$-theory group is $K_n(\mathbb{C}_d) = \pi_n(B\,\mathrm{Gl}_\infty(\mathbb{C}_d)^+)$. Let $F\mathbb{C}$ be the homotopy fiber of the Chern character $\mathrm{ch} : BU \to \prod_{n \geq 1} K(\mathbb{C}, 2n)$. By Bott periodicity, $\pi_{2k}(F\mathbb{C}) = 0$ and $\pi_{2k+1}(F\mathbb{C}) \cong \mathbb{C}/\mathbb{Z}$. The "identity" $\mathbb{C}_d \to \mathbb{C}$ is continuous and induces a map $T : B\,\mathrm{Gl}_\infty(\mathbb{C}_d)^+ \to B\,\mathrm{Gl}(\mathbb{C}) \simeq BU$. By Chern-Weil theory, the induced map in complex cohomology $T^* : H^*(BU; \mathbb{C}) \to H^*((B\,\mathrm{Gl}_\infty(\mathbb{C}_d)^+; \mathbb{C})$ is trivial in positive degrees, compare Lemma 12 in [14] (the argument in loc. cit. generalizes easily to reductive connected groups as $\mathrm{Gl}_n(\mathbb{C})$ and the passage to the limit $n \to \infty$ is also straightforward). It follows that there exists a lift in the diagram

$$\begin{array}{ccc} & & F\mathbb{C} \\ & \overset{\tilde{T}}{\nearrow} & \downarrow \\ B\,\mathrm{Gl}_\infty(\mathbb{C}_d)^+ & \xrightarrow{T} & BU. \end{array} \tag{3.1}$$

This lift is not uniquely determined, but Chern-Weil theory defines a specific choice, compare [11], chapter 2. On homotopy groups, $\tilde{T}$ induces $e : K_{2n-1}(\mathbb{C}_d) \to \pi_{2n-1}(F\mathbb{C}) \cong \mathbb{C}/\mathbb{Z}$, which is the Adams-Quillen $e$-invariant. There is the following alternative description of the $e$-invariant $K_3(\mathbb{C}_d) \to \mathbb{C}/\mathbb{Z}$. Abbreviate $X := \tau_{\geq 3} B\,\mathrm{Gl}_\infty(\mathbb{C}_d)^+$ and consider the Bockstein exact sequence

$$H^3(X; \mathbb{C}) \longrightarrow H^3(X; \mathbb{C}/\mathbb{Z}) \xrightarrow{\delta} H^4(X; \mathbb{Z}) \longrightarrow H^4(X; \mathbb{C}) \tag{3.2}$$

for the coefficients $\mathbb{Z} \to \mathbb{C} \to \mathbb{C}/\mathbb{Z}$.

By 2.6, the class $q_3^*T^*\operatorname{ch}_2 = \frac{1}{2}q_3^*T^*s_2$ lies in $H^4(X;\mathbb{Z})$ and it maps to zero in $H^4(X;\mathbb{C})$ by Chern-Weil theory. Therefore there is a class $\epsilon \in H^3(X;\mathbb{C}/\mathbb{Z})$ such that $\delta\epsilon = \frac{1}{2}q_3^*T^*s_2$. If $x \in K_3(\mathbb{C}_d) = \pi_3(X)$, then $e(x) = \langle x^*\epsilon; [\mathbb{S}^3]\rangle$.
Let $\psi_g$ be the composition $B\Gamma_g \to B\operatorname{Sp}_{2g}(\mathbb{Z}) \to B\operatorname{Gl}_{2g}(\mathbb{Z}) \to B\operatorname{Gl}_{2g}(\mathbb{C}_d)$ given by the action on the first cohomology of the surface. There is an induced map $\Psi : B\Gamma_\infty^+ \to B\operatorname{Gl}_\infty(\mathbb{C}_d)^+$ in the stable range. The composition $T \circ \Psi$ differs from the map $\Phi$ described in section 2. More precisely, let $c : BU \to BU$ be the map which sends a complex vector bundle $V$ to $V \otimes_{\mathbb{R}} \mathbb{C} = V \oplus \bar{V}$. Then

$$T \circ \Psi \simeq c \circ \Phi \tag{3.3}$$

by the Hodge decompostion theorem.

**Proposition 3.4.** *The homomorphisms $\eta$ and $e \circ (\Psi)_* : \pi_3(B\Gamma_\infty^+) \to \mathbb{C}/\mathbb{Z}$ coincide.*

*Proof.* Because we are only interested in $\pi_3$, we may as well pass to the 2-connected covers of $B\Gamma_\infty^+$ and $BU$. We have to show that $(\tau_{\geq 3}\Psi)^*\epsilon = \bar{\zeta}_2$, where $\tau_{\geq 3}\Psi : \tau_{\geq 3}B\Gamma_\infty^+ \to \tau_3 B\operatorname{Gl}_\infty(\mathbb{C}_d)^+$ is induced from $\Psi$. Because $\bar{\zeta}_2$ is uniquely characterized by $\delta\bar{\zeta}_2 = q_3^*\zeta_2$, it suffices to show $\delta(\tau_{\geq 3}\Psi)^*\epsilon = q_3^*\zeta_2$. By the definition of $\epsilon$ and the functoriality of the connected covers, we obtain:

$$\delta(\tau_{\geq 3}\Psi)^*\epsilon = (\tau_{\geq 3}\Psi)^*\frac{1}{2}q_3^*T^*s_2 = (\tau_{\geq 3}\Psi)^*(\tau_{\geq 3}^*T)^*(\frac{1}{2}q_3^*s_2)$$

Using 2.6, it is easy to see that $(\tau_{\geq 3}c)^*q_3^*s_2 = 2s_2$. This relation, together with 3.3 and the definition of $\zeta_2$ shows that

$$(\tau_{\geq 3}\Psi)^*(\tau_{\geq 3}^*T)^*(\frac{1}{2}q_3^*s_2) = (\tau_{\geq 3}\Phi)^*(q_3^*s_2) = q_3^*\zeta_2.$$

□

# 4 An action of the icosahedral group on a surface

Consider a regular icosahedron $\mathcal{I}$ in Euclidean 3-space, centered at 0 and such that all vertices lie on $\mathbb{S}^2$. It has 20 faces (which are triangles), 12 vertices (at every vertex, exactly 5 edges and 5 faces meet) and 30 edges. Let $G \subset SO(3)$

be the symmetry group of the icosahedron. We choose a vertex, an edge and a face of the icosahedron. We denote the isotropy group of the midpoint of the edge by $G_2$, the isotropy group of the midpoint of the face by $G_3$ and the isotropy group of the vertex by $G_5$. These groups are cyclic of order $2, 3, 5$, respectively, and we choose generators $y_2, y_3, y_5$ of these groups. The group $G$ has order 60, it is generated by $y_2$, $y_3$ and $y_5$ and $G$ is a perfect group.
Let $\hat{G}$ the preimage of $G$ under the 2-fold covering $\mathbb{S}^3 \to SO(3)$. The kernel of $\hat{G} \to G$ agrees with the center $Z(\hat{G})$ of $\hat{G}$. We denote the unique nontrivial element in $Z(\hat{G})$ by $h$. It is known that the extension $\mathbb{Z}/2 \to \hat{G} \to G$ is the universal central extension of $G$. The group $\hat{G}$ is the *binary icosahedral group.*
We choose preimages $x_i$ of the generators $y_i$. These set $\{x_2, x_3, x_5\}$ generates $\hat{G}$. The quotient $\hat{G}\backslash\mathbb{S}^3$ is the famous Poincaré homology 3-sphere.
Our analysis of $G$ and $\hat{G}$ begins with the description of the Sylow-subgroups. Clearly $\langle x_3^2\rangle$ and $\langle x_5^2\rangle$ are the 3- and 5-Sylow subgroups of $\hat{G}$, respectively. The quaternion group $Q_8 = \{\pm1, \pm i, \pm j, \pm j\} \subset \mathbb{S}^3 \subset \mathbb{H}$ is a 2-Sylow-subgroup of $\hat{G}$. The Abelianization is $Q_8/\{\pm 1\} \cong V_4 := \mathbb{Z}/2 \times \mathbb{Z}/2$. Let $U$ be the standard representation of $\hat{G} \subset \mathbb{S}^3 = \mathrm{SU}(2)$ as a subgroup and denote the restriction to $Q_8$ by the same symbol. Let $u \in H^4(B\hat{G}; \mathbb{Z})$ be the second Chern class of $U$. Again, the restriction to $BQ_8$ will be denoted by $u$ as well. Furthermore $\mathrm{Hom}(Q_8; U(1)) \cong \mathrm{Hom}(V_4; U(1)) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ (the last isomorphism is not canonical). Let $a, b \in H^2(BQ_8; \mathbb{Z})$ be the first Chern classes of a pair of generators of $\mathrm{Hom}(Q_8; U(1))$.

**Proposition 4.1.** *There is an isomorphism of rings $H^*(BQ_8; \mathbb{Z}) \cong \mathbb{Z}[a, b, u]/I$, where $I$ is the ideal generated by $8u$, $2a$, $2b$, $a^2$, $b^2$ and $ab - 4u$.*
*The cohomology ring $H^*(B\hat{G}; \mathbb{Z})$ is isomorphic to $\mathbb{Z}[u]/(120u)$.*

*Proof.* By Poincaré duality, the cohomology of the closed oriented manifold $\mathbb{S}^3/Q_8$ is given by $H^*(\mathbb{S}^3/Q_8; \mathbb{Z}) \cong \mathbb{Z}$ for $* = 0, 3$, $H^2(\mathbb{S}^3/Q_8; \mathbb{Z}) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ and $H^*(\mathbb{S}^3/Q_8; \mathbb{Z}) = 0$ in all other cases. Now consider the commutative diagram of fibrations

$$\begin{array}{ccc} \mathbb{S}^3 & \longrightarrow & \mathbb{S}^3/Q_8 \\ \downarrow & & \downarrow \\ E\mathbb{S}^3 & \longrightarrow & BQ_8 \\ \downarrow & & \downarrow \\ B\mathbb{S}^3 & = & B\mathbb{S}^3. \end{array} \tag{4.2}$$

The top horizontal map has degree 8. An application of the Leray-Serre spectral sequence shows that the additive structure of $H^*(BQ_8)$ is as asserted and that multiplication by $u$ is an isomorphism $H^p(BQ_8) \to H^{p+4}(BQ_8)$ for all $p > 0$. The products on $H^2(BQ_8)$ are computed in [1], p. 60.
The computation for $\hat{G}$ is similar, but easier, because the full structure follows from the spectral sequence. □

As a consequence of Proposition 4.1, the sum of the restriction maps to the Sylow subgroups $H^4(B\hat{G}) \to H^4(BQ_8) \oplus H^4(B\mathbb{Z}/3) \oplus H^4(B\mathbb{Z}/5) \cong \mathbb{Z}/8 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/5 \cong \mathbb{Z}/120$ is an isomorphism.

**Remark 4.3.** We will need a property of the representations of $Q_8$. The representation $U$ is (up to isomorphism) the only irreducible representation of $Q_8$ in which the central element $h$ acts as $-1$ and not as the identity. It follows: If $W$ is a $Q_8$-representation, on which the central element acts by $-1$, then $V$ is a direct sum of copies of $U$.

**Remark 4.4.** The last fact we need is that any automorphism of the 3- and 5-Sylow subgroups of $\hat{G}$ is induced by conjugation with an element in $\hat{G}$. The quickest way to see this uses the well-known isomorphism $\hat{G} \cong \mathrm{Sl}_2(\mathbb{F}_5)$. This observation has the following consequence. If $W$ is any representation of $\hat{G}$ and $p = 3$ or $p = 5$, then the restriction of $W$ to $\mathbb{Z}/p \subset \hat{G}$ is isomorphic to $n\mathbb{C} \oplus m(\bigoplus_{1 \leq r \leq p-1} L_{p,r})$ for appropriate $n, m \in \mathbb{N}$, where $L_{p,r}$ denotes the representation $1 \mapsto \exp(\frac{2\pi i r}{p})$ of $\mathbb{Z}/p$ on $\mathbb{C}$.

Now we construct certain actions of $\hat{G}$ on Riemann surfaces which in the end will give interesting elements in $\pi_3(B\Gamma_g^+)$. The construction is based on an easy lemma. Let $\mathcal{O}(k)$ be the $k$th tensor power of the Hopf bundle on $\mathbb{CP}^1$. The bundle $\mathcal{O}(k)$ is an $\mathrm{Sl}_2(\mathbb{C})$-equivariant bundle over the $\mathrm{Sl}_2(\mathbb{C})$-space $\mathbb{CP}^1$. If $k$ is odd, then the central element $-1 \in \mathrm{Sl}_2(\mathbb{C})$ acts as $-1$ on $\mathcal{O}(k)$ (i.e. on any fiber of the bundle). If $k$ is even, then $-1$ acts trivially on $\mathcal{O}(k)$, i.e. the action descends to an action of $\mathbb{P}\mathrm{Sl}_2(\mathbb{C})$. Recall that the vector space of global holomorphic sections of $\mathcal{O}(k)$ can be identified with the vector space of homogeneous polynomials of degree $k$ in $\mathbb{C}[t_1, t_2]$ (as $\mathrm{Sl}_2(\mathbb{C})$-representations).

**Lemma 4.5.** *Let $G \subset \mathbb{P}\mathrm{Sl}_2(\mathbb{C})$ be a finite subgroup and let $\hat{G} \subset \mathrm{Sl}_2(\mathbb{C})$ be preimage of $G$ in $\mathrm{Sl}_2(\mathbb{C})$ (it is a central extension of $G$ by $\mathbb{Z}/2$). Let $m \in \mathbb{N}$ be positive. Let $s$ be a $G$-invariant holomorphic section of $\mathcal{O}(2m)$ having only simple zeroes. Then there exists a connected Riemann surface $F$ with a*

*$\hat{G}$-action and a two-sheeted $\hat{G}$-equivariant branched covering $f : F \to \mathbb{CP}^1$, whose branch points are precisely the zeroes of $s$.*
*If $m$ is odd, then the central element $h \in \hat{G}$ acts as the hyperelliptic involution on $F$, if $m$ is even, then $h$ acts trivially on $F$.*

**Proof:** Let $S \subset \mathcal{O}(2m)$ be the image of the section $s$. It is a surface of genus 0 and it is stable under the $G$-action on $\mathcal{O}(2m)$. Let $q : \mathcal{O}(m) \to \mathcal{O}(2m)$ be the squaring map, let $F := q^{-1}(S)$ and let $f := q|_F$. Clearly, $F$ has a $\hat{G}$-action and $f$ is equivariant. The other statements of the Lemma are trivial to verify. □
From now on, let $G$ again be the icosahedral group. The following example is the basis for the computation leading to Theorem 1.1.

**Example 4.6.** Let $z_1, \ldots, z_{30} \in \mathbb{S}^2$ be the midpoints of the edges of the icosahedron and consider them as points on $\mathbb{CP}^1$ after the choice of a conformal map $\mathbb{S}^2 \cong \mathbb{CP}^1$. The precise value of the points does not play a significant role in this discussion.
Now we take holomorphic sections $s_i$, $i = 1, \ldots, 30$, of the Hopf bundle $\mathcal{O}(1)$ with simple zeroes precisely at $z_i$. Such sections exist and are unique up to multiplication with a complex constant. Put $s := s_1 \otimes \ldots \otimes s_{30} \in H^0(\mathbb{CP}^1, \mathcal{O}(30))$.
For $g \in \hat{G}$, there exists a $c(g) \in \mathbb{C}^\times$ with $gs = c(g)s$, because $gs$ has the same zeroes as $s$. The map $c : g \mapsto c(g)$ is a homomorphism $\hat{G} \to \mathbb{C}^\times$. Since $\hat{G}$ has no Abelian quotient, $c$ is constant. Thus $s$ is a $\hat{G}$-invariant section.
If we apply the construction of Lemma 4.5 to $s$, we obtain a surface of genus 14 (by the Riemann-Hurwitz formula) with a $\hat{G}$-action.

**Proposition 4.7.** *The number of fixed points of the elements $x_2, x_3, x_5 \in \hat{G}$ in Example 4.6 is $2, 0, 0$, respectively. All powers $x_3^{2r}$, $r \not\equiv 0 \pmod 3$ and $x_5^{2r}$, $r \not\equiv 0 \pmod 5$ have precisely 4 fixed points.*

The proof is an elementary consideration, using the fact that the number of fixed points of an automorphism of a Riemann surface is less or equal than $2g + 2$ with equality holding only for hyperelliptic involutions.
Of course, one can apply a similar construction to the midpoints of the faces or to the vertices of the icosahedron. The results are surfaces of lower genus and elements in $\pi_3(B\Gamma_g)^+$ of lower order.

# 5 Computations

The proof of Theorem 1.1 starts with the computation of the characteristic class $\zeta_2$ for the bundle on $B\hat{G}$ given by the action $\rho : \hat{G} \to \Gamma_{14}$ constructed before.

**Proposition 5.1.** *The characteristic class* $B\rho^*\zeta_2 \in H^4(B\hat{G};\mathbb{Z}) \cong \mathbb{Z}/120$ *has order* 24.

*Proof.* The $\hat{G}$-action on $F$ is holomorphic by construction and therefore there is an induced linear action $\hat{G} \curvearrowright H^1(F;\mathcal{O})$. The cohomology class $B\rho^*\zeta_2$ of 5.1 is the same as the second Chern class of this linear $\hat{G}$-representation, which will be briefly denoted by $c$. According to the structure of $H^4(B\hat{G})$ described above, we need to show: The restriction of $c$ to $H^4(B\mathbb{Z}/3)$ has order 3, the restriction to $H^4(\mathbb{Z}/5)$ is trivial and the restriction to $H^4(BQ_8)$ has order 8.
By the Dolbeault theorem, there is an isomorphism of complex $\hat{G}$-representations:

$$H^1(F;\mathcal{O}) \otimes_{\mathbb{R}} \mathbb{C} \cong H^1(F;\mathbb{C}). \tag{5.2}$$

The Lefschetz fixed point formula, applied to the result of Proposition 4.7, allows us to determine the decomposition of the representation of $\mathbb{Z}/3 \cong \hat{G}_{(3)} = \langle x_3^2 \rangle$ on the 28-dimensional space $H^1(F;\mathbb{C})$. The result is $H^1(F;\mathbb{C}) \cong 8\mathbb{C} \oplus 10L_1 \oplus 10L_2$. By 5.2 and Remark 4.4, it follows that $H^1(F;\mathcal{O}) \cong 4\mathbb{C} \oplus 5L_1 \oplus 5L_2$ as $\mathbb{Z}/3$-modules. The Chern polynomial of $L_n$ is $1 + nv$; $v \in H^2(\mathbb{Z}/3)$ an appropriate generator. It follows that $c|_{B\mathbb{Z}/3} = v^2$.
The result for the subgroup $\mathbb{Z}/5$ follows by an analogous argument. The decomposition of $H^1(F;\mathcal{O})$ as a $\mathbb{Z}/5$-module is $2\mathbb{C} \oplus 3(L_1 \oplus L_2 \oplus L_3 \oplus L_4)$. We have seen that the central element $h \in Q_8 \subset \hat{G}$ acts as a hyperelliptic involution. Thus it acts on $H^1(F;\mathcal{O})$ by $-1$. By Remark 4.3, this implies that $H^1(F,\mathcal{O})$, as a $Q_8$-module, decomposes into seven copies of the twodimensional representation $U$. The first Chern class of $U$ is zero, while the second is a generator of $H^4(BQ_8)$. Thus $c|_{BQ_8}$ is seven times a generator and thus again a generator. □

Let $M = \mathbb{S}^3/\hat{G}$ be the Poincaré homology sphere and let $\theta_{M;\rho} \in \pi_3(B\Gamma_\infty^+)$ be the element determined by $M$ and the action $\hat{G} \curvearrowright F_{14}$. In view of Theorem 1.2, the following proposition is enough to establish Theorem 1.1.

**Proposition 5.3.** *The order of* $\eta(\theta) \in \mathbb{Q}/\mathbb{Z}$ *is* 24.

*Proof.* Consider the 2-connected covering $q_3 : \tau_{\geq 3}B\Gamma^+_\infty \to B\Gamma^+_\infty$ with homotopy fiber $K(\pi_2(B\Gamma^+_\infty;1) \simeq \mathbb{S}^1$. By elementary obstruction theory (and the knowledge of the cohomology of $B\hat{G}$), there exists a unique (up to homotopy) lift

$$\begin{array}{ccc} & & \tau_{\geq 3}B\Gamma^+_\infty \\ & \overset{\tilde{B\rho}}{\nearrow} & \downarrow \\ B\hat{G}^+ & \xrightarrow{B\rho} & B\Gamma^+_\infty . \end{array} \tag{5.4}$$

Comparing the Bockstein sequences for the spaces and maps $B\hat{G}^+ \to \tau_{\geq 3}B\Gamma^+_\infty \to B\Gamma^+_\infty$, we see that $\tilde{B\rho}^*\bar{\zeta}_2 \in H^3(B\hat{G}^+;\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/120$ has order 24 (use 2.5, 2.1 and 5.1).
Now consider the map $\lambda^+ : \mathbb{S}^3 \simeq M^+ \to B\hat{G}^+$ which is induced from the classifying map $\lambda : M \to B\hat{G}$ of the universal covering of $M$. Observe that $\lambda^* : H^3(B\hat{G}^+;\mathbb{Q}/\mathbb{Z}) \to H^3(\mathbb{S}^3;\mathbb{Q}/\mathbb{Z})$ is injective, which follows from a straightforward consideration of the Leray-Serre spectral sequence for the fibration $\mathbb{S}^3 \to M \to B\hat{G}$.
Thus $\lambda^*\tilde{B\rho}^*\bar{\zeta}_2$ has order 24 in $H^3(\mathbb{S}^3;\mathbb{Q}/\mathbb{Z})$. But $\tilde{B\rho}\circ\lambda$ represents the element $\theta \in \pi_3(B\Gamma^+_\infty)$ which we consider. □

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