THE ADAMS CONJECTURE, AFTER EDGAR BROWN

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1. Introduction

Let X be a finite CW-complex. Denote by $\mathrm{Sph}(X)$ the abelian group of stable fibre-homotopy classes of spherical fibrations on X. Let $F(n)$ be the monoid of self-homotopy-equivalences of $\mathbb{S}^{n-1}$ and let $F := \mathrm{colim}_{n \to \infty} F(n)$. The classifying space BF represents $\mathrm{Sph}(X)$, i.e. $\mathrm{Sph}(X) \cong [X; BF]$. For any vector bundle ξ on X, let $J(\xi) \in \mathrm{Sph}(X)$ be the stable fibre homotopy class of the spherical fibration $\xi \setminus 0 \to X$ (cut out the zero-section). This construction defines a homomorphism $J : KO^0(X) \to \mathrm{Sph}(X)$.

Theorem 1.1. ***(The Adams Conjecture)***

Let X be a finite CW-complex and $k \in \mathbb{Z}$. Let ξ be a real vector bundle on X. Then there exists $t \in \mathbb{N}$ (which depends on ξ and k), such that $J(k^t(\Psi^k - 1)\xi) = 0$.

Alternatively, one can formulate the theorem as the statement that the composition

$$KO^0(X) \stackrel{\Psi^k - 1}{\to} KO^0(X) \stackrel{J}{\to} \mathrm{Sph}(X) \to \mathrm{Sph}(X)[\frac{1}{k}]$$

is zero. There are several proofs in the literature. The case when the rank of ξ is less or equal than 2 was established by Adams [1], which led Adams to conjecture Theorem 1.1. Adams' result is the basis for all proofs of 1.1 in the general case.

Quillen [8] uses the technique of Brauer lifting and algebraic computations of the cohomology of general linear groups and orthogonal groups over finite fields. Sullivan [11] gave a proof using étale homotopy theory and Friedlander [7], following ideas of Quillen [9], another one. A purely topological proof was given by Becker and Gottlieb [4] depending on the transfer and the fact that the classifying space BF of spherical fibrations is an infinite loop space. The latter fact can only proven using infinite loop space machines, the quickest methods being Segal's theory of Γ-spaces [10].

In an unpublished manuscript[1] [5], Edgar Brown introduced a clever argument which avoids the use of the fact that BF is an infinite loop space. Apart from that, his approach is similar to Becker-Gottlieb's. In this note, we follow Brown's method in a somewhat modernized notation.

Here is an outline of this note. In section 2, we discuss generalities about spherical fibrations and the "mod k Dold theorem" [1], which is the basis for the proof of the conjecture for bundles of small rank, which is given in 3 (there are two other places where this theorem is used). Apart from the mod k Dold theorem, one uses the representation theory of the group $O(2)$. In 4, we introduce the Becker-Gottlieb transfer and show the crucial Proposition 4.7. Based on this proposition and the

[1]The author of this note is indebted to Karlheinz Knapp for sending him a copy.

Hopf-Samelson theorem, we show in 5 how the Adams conjecture for general vector bundles can be derived from that for bundles with structural group $\Sigma_n \wr O(2)$. In the last section 6, we show that the Adams conjecture for $O(2)$ bundles (established in section 3) implies the conjecture for the structural group $\Sigma_n \wr O(2)$. For that we need a geometric description for the transfer in KO-theory for finite coverings. We also show that both transfers agree in KO-theory.

We fix the following notations. All vector bundles are real. The tautological line bundle over $\mathbb{RP}^n$ or $\mathbb{PR}^\infty$ will be denoted by γ. The trivial n-dimensional bundle over a space X is denoted by ϵ^n_X or simply ϵ^n if X is understood. For a locally compct space X, let X^+ be the one-point compactification of X. The sphere spectrum is denoted by $\mathbf{S}$.

2. Spherical fibrations and the mod k Dold theorem

We say that a *spherical fibration* of fibre dimension n is a fibration $E \to X$ such that the fibres are homotopy equivalent to a sphere $\mathbb{S}^{n-1}$ (sic). Given two spherical fibrations $E_0, E_1 \to X$, we can form its fibrewise join $E_0 * E_1$, which is a spherical fibration of fibre dimension $n_0 + n_1$. We say that two spherical fibrations E_0 and E_1 are *stably fibre homotopy equivalent* if there exist $n_0, n_1 \in \mathbb{N}$ and a fibre homotopy equivalence $E_0 * \epsilon^{n_0} \to E_1 * \epsilon^{n_1}$.

The stable fibre homotopy equivalence classes of spherical fibrations form an Abelian semigroup and its Grothendieck group is denoted by $\mathrm{Sph}(X)$.

Let $F := \operatorname{colim} F(\mathbb{S}^k; \mathbb{S}^k)$, where the latter space denotes base-point preserving homotopy equivalences of $\mathbb{S}^k$. F is a topological grouplike monoid and its classifying space BF is a classifying space for equivalence classes of spherical fibrations, in other words for any space X there is a natural bijection $\mathrm{Sph}(X) \cong [X, BF]$.

Now we will present two theorems which are used in the proof of1.1 and we shall give plausibility arguments. Both are based on the following result of Dold [6], Theorem 6.1. Let X be a CW-complex and let $E_0, E_1 \to X$ be two fibrations. If $f : E_0 \to E_1$ is a map over X which is a homotopy equivalence, then f is a fibre homotopy equivalence.

If $q : E \to X$ is a spherical fibration and $g : E \to \mathbb{S}^{n-1}$ is a map which has degree 1 when restricted to any fibre (i.e. $f \circ \iota_x$ is a homotopy equivalence), then the map $(q, g) : E \to X \times \mathbb{S}^{n-1}$ satisfies the assumption of Dold's result. Therefore, in order to show that a spherical fibration E is stably trivial, it suffices to construct a homotopy left inverse of the inclusion ι_x of the fibre (after stabilization). A first application of that reasoning is:

Theorem 2.1. *(Dold's theorem on spherical fibrations, stable homotopy version) Let X be a finite complex and $E \to X$ be a spherical fibration. Then $[E] = 0 \in \mathrm{Sph}(X)$ if and only if E is $\mathbf{S}$-orientable.*

For a vector bundle ξ, we can phrase Theorem 2.1 as the statement that ξ is $\mathbf{S}$-orientable if and only if $J(\xi) = 0$.

Proof. Let $[E] = 0 \in \mathrm{Sph}(X)$. This means that there is an $n \in \mathbb{N}$ and a fibre homotopy equivalence $E * \epsilon^n \to \epsilon^{r+n}$. The composition $\Sigma^n \mathbb{T}\mathbf{h}(E) = \mathbb{T}\mathbf{h}(E * \epsilon^n) \to \mathbb{T}\mathbf{h}(\epsilon^{n+r}) = \Sigma^{n+r+\infty} X_+ \to \Sigma^{r+n}\mathbf{S}$ clearly desuspends to an $\mathbf{S}$-orientation of E.

Conversely, if E is $\mathbf{S}$-oriented, the orientation is realized by a map $u : \Sigma^n \operatorname{Th}(E) \to \mathbb{S}^{r+n}$ of spaces for some $n \in \mathbb{N}$. The composition $E * \epsilon^n * \epsilon^1 \to \operatorname{Th}(E * \epsilon^n) = \Sigma^n \operatorname{Th}(E) \to \mathbb{S}^{r+n}$ is a homotopy left-inverse of the fibre inclusion. □

Theorem 2.2. *(The mod k Dold theorem) Let $E_0, E_1 \to X$ be two spherical fibrations over a finite complex. Assume that there exists a map $f : E_0 \to E_1$ which has degree k in any fibre. Then for some $t \in \mathbb{N}$, $k^t[E_0] = k^t[E_1] \in \mathrm{Sph}(X)$.*

Sketch of Proof. A different, but complete proof can be found in [1]. First choose a complement E_2 of E_1. Then the map $F * \mathrm{id} : E_0 * E_2 \to E_1 * E_2 = \epsilon^r$ has degree k in any fibre. Thus it is enough to show the result when E_1 is trivial and has very large rank.

Assume that $\operatorname{rk} E >> \dim X$. We show the theorem by induction on the cells of X. For the 0-skeleton, the result is trivial, with $t = 0$. So assume that there is a $t \in \mathbb{N}$ and a map $g : k^t E|_{X^{(n-1)}} \to \mathbb{S}^{N-1}$ which has fibrewise degree 1. The problem we have to solve can be summarized in the diagram

$$
\begin{array}{ccc}
k^t E|_{X^{(n-1)}} & \xrightarrow{\;g\;} & \mathbb{S}^{N-1} \\
\downarrow & \nearrow & \downarrow \varphi_{k^u} \\
k^t E|_{X^{(n)}} & \xrightarrow{\;f\;} & \mathbb{S}^{N-1},
\end{array}
\tag{2.3}
$$

where $\varphi_m : \mathbb{S}^{N-1} \to \mathbb{S}^{N-1}$ is a map of degree m. Here $N := k^t r$ and $u \geq t$ is yet to be determined. The obstructions to the existence of the dotted arrow lie in the groups

$$H^p(k^t E|_{X^{(n)}}, k^t E|_{X^{(n-1)}}; \pi_{p-1}(\mathrm{hofib}(\varphi_{k^u})),$$

which are all trivial except in the case $p = N - 1 + n$. If the map ϕ_k induced multiplication by k on homotopy groups, it would follow that $\pi_{p-1}(\mathrm{hofib}(\varphi_{k^u}))$ is k-torsion. However, ϕ_k does *not* induce multiplication by k. But the homology of $\mathrm{hofib}(\varphi_{k^u})$ is k^u-torsion by a simple application of the Leray-Serre spectral sequence and by Serre class theory, we see that $\pi_{p-1}((\mathrm{hofib}(\varphi_{k^u})$ *is* k-torsion.

So we know that the obstructions to the existence of the dotted arrow in 2.3 are all k-torsion. Therefore, when we take the Whitney sum of the diagram with itself k^u times for some u, the obstruction problem becomes solvable. This finishes the sketch of proof. □

3. The Adams conjecture for bundles of rank ≤ 2

The Adams conjecture 1.1 is trivially true if $k = 0, 1$, since $\Psi^1 = \mathrm{id}$. For all k, $\Psi^k = \Psi^{-k}$. Therefore we can assume that $k > 1$.

The proof of 1.1 begins with a trivial observation.

Lemma 3.1. *If the Adams conjecture 1.1 holds for all vector bundles of even rank, then it holds for all vector bundles.*

Proof. Let ξ be a bundle of odd rank. By assumption $k^t J(\Psi^k(\xi \oplus \epsilon^1) - (\xi \oplus \epsilon^1)) = 0$ for some $t \in \mathbb{N}$. But $k^t J(\Psi^k(\xi \oplus \epsilon^1) - (\xi \oplus \epsilon^1)) = k^t J(\Psi^k \xi - \xi) + k^t J(\epsilon^1 - \epsilon^1) = k^t J(\Psi^k \xi - \xi)$. □

Theorem 3.2. *The Adams conjecture holds for all bundles of rank two.*

We need a fact about line bundles.

Proposition 3.3. *Let $\xi \in KO^0(X)$ be a real line bundle on a finite complex X. Then for some $t \in \mathbb{N}$, $2^t(1-\xi) = 0$.*

Proof. Because X is a finite complex, there exists a map $f : X \to \mathbb{RP}^{2n}$ for some n such that $f^*\gamma = \xi$. But the group $\widetilde{KO}^0(\mathbb{RP}^{2n})$ is a finite 2-group. Therefore $2^e(1-\gamma) = 0$ for some $t = t(n)$.

To see the assertion about $\widetilde{KO}^0(\mathbb{RP}^{2n})$, look at the Atiyah-Hirzebruch spectral sequence. The group $\widetilde{H}^*(\mathbb{RP}^{2n})$ is a finite 2-group, as well as $\widetilde{H}^*(\mathbb{RP}^{2n}; \mathbb{Z})$. Therefore $KO^0(\mathbb{RP}^{2n}) = \mathbb{Z} \oplus A$, A a finite 2-group. □

Let $\xi \to X$ be a 2-dimensional bundle. The first task we have to accomplish is the computation of the Adams operation $\Psi^k \xi$. Let $RO(O(2))$ be the real representation ring of the group $O(2)$. The bundle ξ induces a homomorphism $RO(O(2)) \to KO^0(X)$, which is a homomorphism of λ-rings. Therefore we need to compute the Adams operations on $RO(O(2))$.

Any element of $O(2)$ is conjugate to either an element in $SO(2)$ or to the nontrivial element in $\mathbb{Z}/2$. Therefore the restriction map

$$\iota : RO(O(2)) \to RO(SO(2)) \times RO(\mathbb{Z}/2) \tag{3.4}$$

is injective. Let λ_2 be the determinant representation, 1 be the trivial representation and μ_1 be the defining representation of $O(2)$. The irreducible representation of $SO(2)$ defined by $g \mapsto g^k$ is denoted ν_k, $k \geq 0$; the nontrivial one-dimensional representation of $\mathbb{Z}/2$ is η.

Lemma 3.5. *The Adams operations on ν_1 are given by $\Psi^k \nu_1 = \nu_k$.*

Proof. Recall that Ψ^k is defined using the Newton polynomials Q_k:

$$\Psi^k(x) = Q_k(\lambda^1(x), \lambda^2(x), \ldots \lambda^k(x)).$$

If x is 2-dimensional, $\lambda^1(x) = x$, $\lambda^0(x) = \lambda^2(x) = 1$, $\lambda^k(x) = 0$. The recursion formula for the Newton polynomials gives $Q_k - \lambda^1 Q_{k-1} + \lambda^2 Q_{k-2} = 0$. Note that $\nu_1 \nu_k = \nu_{k-1} + \nu_{k+1}$. By induction, the result is true if $k = 0, 1$.

□

Lemma 3.6. *There exists a unique representation μ_k of $O(2)$, such that $\iota(\mu_k) = (\nu_k, 1+\eta)$.*

Proof. Uniqueness follows from the injectivity of 3.4. To show the existence, recall that $O(2)$ is the semidirect product $\mathbb{Z}/2 \ltimes SO(2)$, where the nontrivial element of $\mathbb{Z}/2$ acts by $g \mapsto g^{-1}$ on $SO(2)$. The homomorphism $\nu_k : SO(2) \to SO(2)$, $g \mapsto g^k$ is $\mathbb{Z}/2$-equivariant and therefore it extends to $O(2)$. The extension is the desired homomorphism $\mu_k : O(2) \to O(2)$, alias representation of $O(2)$. □

Identify $\mathbb{R}^2$ with $\mathbb{C}$. The $\mathbb{C} \to \mathbb{C}$, $z \mapsto z^k$ is $O(2)$-equivariant when $O(2)$ acts by μ_1 on the source and by μ_k on the target.

The Adams operations on the representation μ_1 are given by the next lemma.

Lemma 3.7. *$\Psi^k\mu_1 = \mu_k$ for odd k and $\Psi^k\mu_1 = \mu_k - \lambda_2 + 1$ for even k.*

Proof. This follows from an application of the injective ι to both sides of the equations. If k is odd, then

$$\iota\Psi^k\mu_1 = \Psi^k(\nu_1, 1+\eta) = (\nu_k, 1+\eta) = \iota\mu_k.$$

If k is even, then

$$\iota\Psi^k\mu_1 = \Psi^k(\nu_1, 1+\eta) = (\nu_k, 2),$$

while

$$\iota(\mu_k - \lambda_2 + 1) = (\nu_k - 1 + 1, 1 + \eta - \eta + 1).$$

□

Corollary 3.8. *For an $O(2)$-bundle ξ, denote by $\xi^{(k)}$ the bundle associated to ξ and the $O(2)$-representation μ_k. Then there exists a nonlinear fibrewise map $f : \xi \to \xi^{(k)}$ which has degree k in any fibre.*

Proof of Theorem 3.2. Let ξ be an $O(2)$-bundle. If k is odd, then

$$\Psi^k(\xi) - \xi = \xi^{(k)} - \xi$$

by 3.7. By Corollary 3.8, there is a fibrewise map $\xi \to \Psi^k\xi$ of degree k. By the mod k Dold theorem 2.2, the proof is complete in that case.

If k is even, then

$$\Psi^k(\xi) - \xi = (\xi^{(k)} - \xi) + (1 - \det(\xi)).$$

The first summand becomes trivial after multiplication with k^t and application of J by the same argument as before. The second summand is annihilated by some large power k^s since 2^s divides k^s (Proposition 3.3).

□

4. The transfer

An important tool for the proof of the Adams conjecture is the *Becker-Gottlieb transfer* [4]. We give a simplified account, which is sufficient for the proof of the Adams conjecture. Note that any finite CW-complex X can be embedded into a euclidean space and a neighborhood of the image is a manifold which is homotopy equivalent to X. Therefore the study of homotopy properties of vector bundles over finite complexes can be reduced to the study of smooth vector bundles on smooth manifolds. Let E and B be smooth manifolds and $p : E \to B$ be a proper submersion (alias fibre bundle with closed fibre). Let $T_v E := \ker Tp$ be the vertical tangent bundle of E.

Choose an embedding $j : E \to B \times \mathbb{R}^n$ over B for some n. Let

$$\nu(j) := p^* TB \oplus \epsilon^n_E / TE \cong p^* TB \oplus \epsilon^n_E/(p^*TB \oplus T_v E) \cong \epsilon^n_E / T_v E$$

be the normal bundle of j. Choose a tubular neighborhood of $j(E)$, i.e. an open embedding of the normal bundle $\nu(j)$ of j onto an open neighborhood U of $j(E)$, which is the identity along the zero section of $\nu(j)$. Let $c : (B \times \mathbb{R}^n)^+ \to U^+$ be the collapse map, i.e. c is the identity on U and it sends all point outside U to the additional point of U^+. Clearly $\Sigma^n B_+ \cong (B \times \mathbb{R}^n)^+$ and $U^+ \cong \mathrm{Th}(\nu(j))$. The inclusion $\nu(j) \subset \epsilon^n_E$ induces a map $i : \mathrm{Th}(\nu(j)) \to \Sigma^n E_+$. Consider the composition $i \circ c : \Sigma^n B_+ \to \sigma^n E_+$. The dimension n was arbitrary, and when n tends to ∞, we obtain a map of spectra

$$\mathrm{trf}_p : \Sigma^\infty B_+ \to \Sigma^\infty E_+,$$

the *transfer* of the fiber bundle $p : E \to B$. By the Whitney embedding theorem and the parameterized tubular neighborhood theorem, the construction of trf_p is unique up to a contractible space of choices.

We note the following facts: The transfer is natural with respect to pullbacks. In other words, if $f : B' \to B$ is a map and $p' : E' := E \times_B B' \to B'$ is the induced bundle, then the diagram

(4.1)
$$\begin{array}{ccc} \Sigma^\infty E'_+ & \longrightarrow & \Sigma^\infty E_+ \\ \uparrow{\scriptstyle \mathrm{trf}_{p'}} & & \uparrow{\scriptstyle \mathrm{trf}_p} \\ \Sigma^\infty B'_+ & \longrightarrow & \Sigma^\infty B_+ \end{array}$$

is homotopy-commutative (even on the nose when the choices are appropriate). We note two consequences of this fact. First of all, if $f : A \to B$ is the inclusion of a subspace, $E_A := E|_A$, then we get a commutative diagram

(4.2)
$$\begin{array}{ccccc} \Sigma^\infty E_{A,+} & \longrightarrow & \Sigma^\infty E_+ & \longrightarrow & \Sigma^\infty E/E_A \\ \uparrow{\scriptstyle \mathrm{trf}_{p|_A}} & & \uparrow{\scriptstyle \mathrm{trf}_p} & & \vdots{\scriptstyle \mathrm{trf}_{p|_{B/A}}} \\ \Sigma^\infty A_+ & \longrightarrow & \Sigma^\infty B_+ & \longrightarrow & \Sigma^\infty B/A. \end{array}$$

In other words, the relative transfer is defined.

For any space X, let $\mathrm{diag}_X : X \to X \times X$ be the diagonal map. The obvious pullback diagram

(4.3)
$$\begin{array}{ccccc} E & \xrightarrow{\mathrm{diag}_E} & E \times E & \xrightarrow{p \times \mathrm{id}} & B \times E \\ \downarrow{\scriptstyle p} & & & & \downarrow{\scriptstyle \mathrm{id} \times p} \\ B & & \xrightarrow{\mathrm{diag}_B} & & B \times B \end{array}$$

leads to the commutative diagram of spectra

(4.4)
$$\begin{array}{ccccc} \Sigma^\infty E_+ & \xrightarrow{\mathrm{diag}_E} & \Sigma^\infty E_+ \wedge \Sigma^\infty E_+ & \xrightarrow{p \wedge \mathrm{id}} & \Sigma^\infty B_+ \wedge \Sigma^\infty E_+ \\ {\scriptstyle \mathrm{trf}_p}\uparrow & & & & \uparrow{\scriptstyle \mathrm{id} \wedge \mathrm{trf}_p} \\ \Sigma^\infty B_+ & & \xrightarrow{\mathrm{diag}_B} & & \Sigma^\infty B_+ \wedge \Sigma^\infty B_+. \end{array}$$

Proposition 4.5. *Let $p : E \to B$ be a fibre bundle and let A be a commutative ring spectrum with multiplication map $\mu : A \wedge A \to A$. Then for any $(x, y) \in A^*(B) \times A^*(E)$, we have the equality $\mathrm{trf}_p^*(p^*x \cdot y) = x\, \mathrm{trf}_p^*(y)$.*

Proof. This follows immediately from the diagram 4.4. More precisely

(4.6)
$$\mathrm{trf}_p^*(p^*x \cdot y) = \mu \circ (x \wedge y) \circ (p \wedge \mathrm{id}) \circ \mathrm{diag}_E \circ \mathrm{trf}_p = \mu \circ (x \wedge y) \circ (\mathrm{id} \wedge \mathrm{trf}_p) \circ \mathrm{diag}_B = x \cdot \mathrm{trf}_p^*(y).$$
□

Proposition 4.7. *Let $p : E \to X$ be a smooth proper fibre bundle with fibre F. Then the composition $\mathrm{trf}_p^* p^* : H^*(X) \to H^*(X)$ is multiplication by the Euler number $\chi(F)$.*

Proof. By 4.5, we have $\mathrm{trf}_p^* p^*(x) = \mathrm{trf}_p^*(p^*(x) \cdot 1) = x\, \mathrm{trf}_p^*(1)$. Without loss of generality, we can assume that X is connected. Let $x \to X$ be the inclusion of a base-point; the diagram

(4.8)
$$\begin{array}{ccc} F & \longrightarrow & E \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle p} \\ x & \longrightarrow & X \end{array}$$

is a pullback-diagram. Moreover, $H^0(X) \to H^0(x)$ is an isomorphism. Therefore it is enough to show that $\mathrm{trf}_c^*(1) = \chi(F)$. This is the same as showing that the degree of the composition

(4.9)
$$\tau : \mathbb{S}^n \to \mathrm{Th}\, \nu(j) \to \mathbb{S}^n F_+ \to \mathbb{S}^n$$

is $\chi(F)$. Here F is embedded into $\mathbb{R}^n$ via $j : F \to \mathbb{R}^n$. The first map is the collapse map, the second one is induced by the inclusion $\nu(j) \to F \times \mathbb{R}^n$ and the last map is the projection. Let U be an ϵ-tubular neighborhood of F, $f : U \to F$ the retraction map. Then the map $U \to \mathbb{D}^n_\epsilon$ which sends x to $x - f(x)$ is proper and extends to $U^+ \to (\mathbb{D}^n_\epsilon)^+ \cong \mathbb{S}^k$. The composition with the collapse $\mathbb{S}^n \to U^+$ is τ. Let $\phi_t : F \to F$ be an isotopy of F with $\phi_0 = \mathrm{id}$. Instead of $x \mapsto x - f(x)$ we can consider $x \mapsto \phi_t(f(x))$: they are homotopic. If the fixed points of ϕ_t are nondegenerate, we can compute the degree from the preimages of 0, which are precisely the fixed points of ϕ_t. Let ϕ_t be generated by a nondegenerate vector field V, it follows that

$$\deg(\tau) = \sum_{x \in F} \mathrm{ind}_x V = \chi(F);$$

the latter equality is true by the Poincaré-Hopf theorem.

□

5. Reduction of the general conjecture to bundles with structural group $\Sigma_n \wr O(2)$

Proposition 5.1. *(Brown's clever trick) Let $F \to E \xrightarrow{p} B$ be a smooth fibre bundle such that the fibre F is connected*[2] *and has $\chi(F) = 1$. Let ξ be a real vector bundle of rank r on B. If $J(p^*\xi) = 0$, then $J(\xi) = 0$.*

Proof. Let $u : \mathbb{T}\mathbf{h}(p^*\xi) \to \Sigma^r \mathbf{S}$ be a Thom class whose existence is asserted by Dold's theorem 2.1 and the assumption that $J(p^*\xi) = 0$. Let $\tilde{B}$ and $\tilde{E}$ be the sphere bundles of $\xi \oplus \epsilon^1$ and $p^*\xi \oplus \epsilon^1$, respectively. Observe that $\tilde{E} = E \times_B \tilde{B}$, whence there is a fibration $\tilde{p} : \tilde{E} \to \tilde{B}$ with fibre F. Let τ be the relative transfer

$$\tau : \Sigma^\infty(\tilde{B}/B) \to \Sigma^\infty(\tilde{E}/E).$$

But $\tilde{B}/B \cong \mathrm{Th}(\xi)$ and $\tilde{E}/E \cong \mathrm{Th}(p^*\xi)$. Thus τ is a map

$$\mathbb{T}\mathbf{h}(\xi) \to \mathbb{T}\mathbf{h}(p^*\xi).$$

We want to show that the composition $u \circ \tau : \mathbb{T}\mathbf{h}(\xi) \to \Sigma^r \mathbf{S}$ is an $\mathbf{S}$-Thom class for ξ. Let $x \in B$ be an arbitrary base-point, let $i_x : \mathbb{S}^r \to \tilde{B}$ be the inclusion of a fibre and let $j_x : \mathbb{S}^r \to \mathrm{Th}(\xi)$ be the composition of i_x with the quotient map $\tilde{B} \to \mathrm{Th}(\xi)$. It induces $j_x : \Sigma^r \mathbf{S} \to \mathbb{T}\mathbf{h}(\xi)$. We have to show that $u \circ \tau \circ j_x$ has degree ± 1. We clearly have a pullback diagram

$$\begin{array}{ccc} F \times \mathbb{S}^r & \longrightarrow & \tilde{E} = \mathrm{Th}(p^*\xi) \\ \downarrow & & \downarrow \\ \mathbb{S}^r & \xrightarrow{j_x} & \tilde{B} = \mathrm{Th}(\xi). \end{array} \tag{5.2}$$

[2]Removing this hypothesis requires an only marginally more complicated proof.

Therefore the diagram of transfers

(5.3)
$$\begin{array}{ccccc} \Sigma^r\Sigma^\infty F_+ & \xrightarrow{\text{inc}} & \mathbb{T}\mathbf{h}(p^*\xi) & \xrightarrow{u} & \Sigma^r\mathbf{S} \\ \uparrow{\scriptstyle \text{trf}} & & \uparrow{\scriptstyle \tau} & & \\ \Sigma^r\mathbf{S} & \xrightarrow{j_x} & \mathbb{T}\mathbf{h}(\xi) & & \end{array}$$

commutes. Moreover, the composition of the inclusion map inc : $\Sigma^k\Sigma^\infty F_+ \to \mathbb{T}\mathbf{h}(p^*\xi)$ with u is an $\mathbf{S}$-orientation of the trivial bundle $F \times \mathbb{R}^k$. Therefore its degree (note that $\mathbb{Z} \cong \pi_0(\Sigma^\infty F_+)$) is ± 1. Therefore the degree of the composition $u \circ \tau \circ j_x = u \circ \text{inc} \circ \text{trf}$ is $\pm\chi(f) = \pm 1$ by Proposition 4.7. □

Theorem 5.4. *If the Adams conjecture holds for all bundles with structural group $\Sigma_n \wr O(2) := \Sigma_n \ltimes O(2)^n$, (for all n, on all spaces X) then it holds for all vector bundles.*

Proof. First we collect a few facts about Lie groups. Let G be a connected compact Lie group with maximal torus T and Weyl group $W = NT/T$. A well-known theorem (Hopf-Samelson) says that the Euler number $\chi(G/T)$ of the quotient is equal to $|W|$, the order of the Weyl group. Therefore $\chi(G/NT) = 1$. We apply this to the group $SO(2n)$. Let $T = SO(2)^n \subset SO(2n)$ be the standard maximal torus. Let N_0T be the normalizer in $SO(2)$ and NT be the normalizer in $O(2n)$. Because $SO(2n)/N_0T = O(2n)/NT$, it follows that $\chi(O(2n)/NT) = 1$.

The normalizer $NT \subset O(2n)$ can be identified with the wreath product $\Sigma_n \wr O(2)$.

Now let $\xi \to X$ be an arbitrary vector bundle on a finite complex X. We want to show that the Adams conjecture holds for ξ, under the assumption of the theorem. By 3.1, we can assume that the rank of ξ is even, say $\operatorname{rk}\xi = 2n$.

Let $P \to X$ be the associated $O(2n)$-principal bundle and let $E := P/(\Sigma_n \wr O(2)) \xrightarrow{p} P/O(2n) = X$. This is a smooth fibre bundle with fibre $O(2n)/(\Sigma_n \wr O(2))$, which has Euler number 1.

On the other hand, $p^*\xi$ has structural group $\Sigma_n \wr O(2)$. By assumption, the Adams conjecture holds for $p^*(\xi)$. In other words, there exists a $t \in \mathbb{N}$ such that $J(p^*k^t(\Psi^k - 1)\xi) = 0$. Now represent the element $k^t(\Psi^k - 1)\xi \in KO^0(X)$ as a difference $\zeta - \epsilon^m$ for some vector bundle ζ and some $m \in \mathbb{N}$. Therefore $J(p^*\eta) = 0$ and by Proposition 5.1 it follows that $J(\eta) = 0$, which is what we had to show. □

6. Reduction to $O(2)$-bundles

For the last step, we need to introduce the *geometric transfer* in KO-theory for finite coverings. Let $f : X \to Y$ be a finite covering of degree n and $\xi \to X$ a vector bundle of rank m. Let $f_!\xi$ be the vector bundle on Y of rank mn whose fibre at $y \in Y$ is the vector space of sections in $\xi|_{f^{-1}(y)}$, which is the same as $\bigoplus_{x\in f^{-1}(y)} \xi_x$. This construction defines a map

$$f_! : KO^0(X) \to KO^0(Y),$$

the *geometric transfer* of f.

Proposition 6.1. *For any $\Sigma_n \wr O(2)$-bundle ξ on a space Y, there exists an n-fold covering $f : X \to Y$ and an $O(2)$-bundle λ such that $f_! \lambda \cong \xi$.*

Proof. We need the notion of induced representations. Let $H \subset G$ be two groups, where H has finite index in G. Let V be an H-representation. It defines a G-vector bundle $G \times_H V \to G/H$. The space of its sections is a G-representation denoted $\mathrm{ind}_H^G V$.

In our specific situation, let $H := O(2) \times (\Sigma_{n-1} \wr O(2)) \subset \Sigma_n \wr O(2) =: G$; this is an index n subgroup. The projection on the first factor is a homomorphism $\rho : H \to O(2)$ and the induced representation $\mathrm{ind}_H^G \rho$ is isomorphic to the standard representation W of G.

Let P be a G-principal bundle for ξ and put $X := P/H$. Clearly there is an n-sheeted covering $p : X \to Y$ and $P \to X$ is an H-principal bundle. Let $\lambda := P \times_{H,\rho} \mathbb{R}^2$. The assertion that $p_! \lambda \cong \xi$ is a simple consequence of the fact about induced representations. □

Proposition 6.2. *The geometric transfer commutes with Adams operations after inverting k, more precisely, for a finite covering $f : X \to Y$, a vector bundle ξ on X, there exists $t \in \mathbb{N}$ such that $k^t(\Psi^k f_! \xi - f_! \Psi^k \xi) = 0$.*

The proof of 6.2 is deferred, because it needs a further detour. First we show how to finish the proof of the Adams conjecture.

Proof of the Adams conjecture. By 5.4, it remains to show the Adams conjecture for an $\Sigma_n \wr O(2)$-bundle, assuming (3.2) that it holds for $O(2)$-bundles. Let ξ be a $\Sigma_n \wr O(2)$-bundle on Y. By 6.1, write $\xi = f_! \lambda$. Then for some sufficient large power k^t

$$k^t J((\Psi^k - 1)\xi) = J(k^t(\Psi^k - 1) f_! \lambda) = J(f_! k^t (\Psi^k - 1)\lambda) \tag{6.3}$$

by Proposition 6.2. By 3.2, there exists an $s \in \mathbb{N}$ such that there is a degree k^s fibrewise map $h : \Psi^k \lambda \to \lambda$.

By taking fibrewise joins, it follows that there is a degree k^{sn} fibrewise map $f_! \Psi^k \lambda \to f_! \lambda$. □

Now we turn to the proof of 6.2. First note that inversion of k is indeed necessary. Consider the covering $f : E\mathbb{Z}/2 \to B\mathbb{Z}/2$. Clearly $f_!(\epsilon^1) = \gamma + \epsilon^1$. Thus $\Psi^2 f_!(1) - f_!(\Psi^2 1) = \Psi^2(1 * \gamma) - 1 - \gamma = 1 - \gamma \neq 0$. The situation does not improve by passing to finite subcomplexes.

There are essentially two possibilities to prove 6.2. The first method is representation-theoretic and the argument can be found in [8], at least for finite structural groups. We give a topological proof.

Proposition 6.4. *For a finite covering $f : X \to Y$, Y compact, the maps $f_! : KO^0(X) \to KO^0(Y)$ and $\mathrm{trf}_f^* : KO^0(X) \to KO^0(Y)$ coincide.*

Proof. There is a short proof based on the Atiyah-Singer family index theorem for elliptic operators. In our case, the covering is considered to be a manifold bundle. Let $V \to X$ be a real vector bundle. The operator from sections of V to 0 is an elliptic operator; $f_! V$ is the analytic index while $\mathrm{trf}_f^* V$ is the topological index.

For those who prefer not to use the index theorem, there is the following more elementary argument.

Let X be a locally compact space. Then elements in the K-theory with compact support $KO_c^0(X)$ are represented by complexes

$$0 \to V_0 \xrightarrow{\phi_1} V_1 \ldots \to V_n \xrightarrow{\phi_n} 0$$

of vector bundles on X, such that the V_i are trivialized outside a compact set, the maps ϕ_i are constant and the complex is exact. For the equivalence relation between such complexes and further details, compare [3]. For any open embedding $j : U \to X$, there is an map $j_! : U \to X$, given by extension of complexes by constant ones. Let $U; W; Z$ be locally compact spaces, $j : U \to W$ be an open embedding, $h : W \to Z$ a finite covering such that $h \circ j : U \to Z$ is an open embedding. We claim that $h_! \circ j_j = (h \circ j)_!$, where, of course $h_!$ is the geometric transfer. This can be seen as follows. If j is a homeomorphism, then the statement can be checked directly. For general j, look at the diagram

$$\begin{array}{ccccc} U & \xrightarrow{j} & h^{-1}(\jmath(U))d & \xrightarrow{\subset} & Y \\ & \searrow^{h\circ j} & & & \downarrow \\ & & j(U) & \xrightarrow{\subset} & Z \end{array}$$

with cartesian right square.

Now return to the situation of the Proposition. Let $j : X \to Y \times \mathbb{R}^{8m}$ be an embedding. Consider the diagram

(6.5)
$$\begin{array}{ccc} KO^0(X) & \xrightarrow{f_!} & KO^0(Y) \\ \downarrow \beta & & \downarrow \beta \\ KO_c^0(X \times \mathbb{R}^{8m}) & \xrightarrow{(f\times \mathrm{id})_!} & KO_c^0(Y \times \mathbb{R}^{8m}). \end{array}$$

The vertical maps are the Bott periodicity maps and the diagram commutes by the definition of the latter ones. Consider the homotopy $h : [0,1] \times X \times \mathbb{R}^{8m} \to Y \times \mathbb{R}^{8m}$ defined by the formula:

$$h_s(x, v) := (f(x), v + sj(x)); \tag{6.6}$$

this is a proper homotopy through coverings starting at $f \times \mathrm{id}$. Therefore $(f \times \mathrm{id})_! = (h_1)_!$. Let $\iota : X \times \mathbb{R}^{8m} \to Y \times \mathbb{R}^{8m}$ be a a tubular neighborhood of the image of j; we can choose it to agree with h_1 on $X \times \mathbb{D}_\delta^{8m}$ for some $\delta > 0$. In other words,

ι is the composition $h_1 \circ \mathrm{inc}$; $\mathrm{inc} : X \times \mathbb{D}^{8m}_{\delta} \to X \times \mathbb{R}^{8m}$. Also $\mathrm{inc}_! = \mathrm{id}$ (suitably interpreted). Therefore

$$(f \times \mathrm{id})_! = (h_1)_! = \iota_!.$$

But the homotopy-theoretic transfer of f is, up to Bott periodicity maps, the same as the extension map $\iota_! : KO^0_c(X \times \mathbb{R}^{8m}) \to KO^0_c(Y \times \mathbb{R}^{8m})$. □

Proof of 6.2. We know that the Adams operations $\Psi^k : \mathbb{Z} \times BO \to \mathbb{Z} \times BO$ are not infinite loop maps. But they are infinite loop maps when k is inverted: $\Psi^k : BO \to BO[\frac{1}{k}]$ is an infinite loop map, in other words, there is a map of spectra $\Psi^k : KO \to KO[\frac{1}{k}]$ which induces Ψ^k on infinite loop spaces. But maps of spectra commute with the homotopy theoretic transfer (for trivial reasons). More precisely, for $x \in KO^0(X)$, both, $\Psi^k(\mathrm{trf}^*_f\, x)$ and $\mathrm{trf}^*_f\, \Psi^k(x)$ are represented by

$$\Sigma^\infty Y_+ \stackrel{\mathrm{trf}_f}{\to} \Sigma^\infty X_+ \stackrel{x}{\to} KO \stackrel{\Psi^k}{\to} KO[\frac{1}{k}]. \tag{6.7}$$

By 6.4, the same is true for the geometric transfers. Spelling this out gives the result. □

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