THEOREMS OF GROTHENDIECK-SERRE AND BASS-HELLER-SWAN

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1. Introduction

Let R be a ring. Recall that $K_0(R)$ is the Grothendieck group of the semigroup of all isomorphism classes of finitely generated projective R-modules. The assignment $R \mapsto K_0(R)$ is a covariant functor from the category of rings to the category of abelian groups; for a ring homorphism $f : R \to S$, one defines $f_* : K_0(R) \to K_0(S)$ by $[P] \mapsto [P \otimes_R S]$. Any ring has a canonical homomorphism $j : \mathbb{Z} \to R$, and we define the reduces K_0-group of R to be

$$\widetilde{K}_0(R) := \operatorname{coker}(j_* : \mathbb{Z} \cong K_0(\mathbb{Z}) \to K_0(R)).$$

If R admits a unital homomorphism to a field F, then j_* is split-injective.

Theorem A (Grothendieck–Serre)**.** *The inclusion $j : \mathbb{Z} \to \mathbb{Z}[\mathbb{Z}^n]$ induces an isomorphism*

$$j_* : K_0(\mathbb{Z}) \cong K_0(\mathbb{Z}[\mathbb{Z}^n]).$$

Hence $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}^n]) = 0$.

This was first proven by Grothendieck, and the proof that we shall discuss was given by Serre [15]. I was unable to locate a proof written down by Grothendieck, but both proofs are discussed in [4, ch. XII, §3, §4].

For the next result, let us recall the definition of the first algebraic K-theory group $K_1(R)$ of a ring and the Whitehead group $\mathrm{Wh}(G)$ of a group. If R is a ring, the general linear groups are ordered by inclusion:

$$R^\times = \mathrm{GL}_1(R) \subset \mathrm{GL}_2(R) \subset \ldots,$$

and the union $\mathrm{GL}(R)$ is the infinite general linear group. The subgroup $E(R) \subset \mathrm{GL}(R)$ generated by the elementary matrices is equal to the commutator subgroup $[\mathrm{GL}(R), \mathrm{GL}(R)]$, by the Whitehead lemma [11, Lemma 3.1]. One defines

$$K_1(R) := \frac{\mathrm{GL}(R)}{E(R)} = \mathrm{GL}(R)^{\mathrm{ab}}.$$

A ring homomorphism $f : R \to S$ induces a group homomorphism $\mathrm{GL}(R) \to \mathrm{GL}(S)$ and hence $f_* K_1(R) \to K_1(S)$.

Now let G be a group and let $\mathbb{Z}[G]$ be the integral group ring of G. If $g \in G$, then $\pm g \in \mathbb{Z}[G]$ is a unit (a so called *trivial unit*. The set of trivial units is a subgroup $\pm G \cong \mathbb{Z}/2 \times G$ of $\mathrm{GL}_1(\mathbb{Z}[G])$. We obtain a homomorphism

$$i : \pm G \subset \mathrm{GL}_1(\mathbb{Z}[G]) \to \mathrm{GL}(\mathbb{Z}[G]) \to K_1(\mathbb{Z}[G]),$$

whose cokernel is the *Whitehead group* $\mathrm{Wh}(G)$ of G. The following result was first proven in [3].

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Theorem B (Bass-Heller-Swan)**.** *The Whitehead group of* $\mathbb{Z}^n$ *is trivial;* $\mathrm{Wh}(\mathbb{Z}^n) = 0$.

The case $n = 1$ is a good deal more elementary and was first shown by Higman [7].

The groups $\widetilde{K}_0(\mathbb{Z}[G])$ and $\mathrm{Wh}(G)$ play an important role in geometric topology. The group $\widetilde{K}_0(\mathbb{Z}[G])$ is the home of the Wall finiteness obstruction for finitely dominated spaces with fundamental group G [16], while $\mathrm{Wh}(G)$ is the home for the Whitehead torsion of a homotopy equivalence between finite complexes with fundamental group G [10], and of the Whitehead torsion of an h-cobordism between closed manifolds with fundamental group G [8]. Together with the results from the quoted papers, Theorems A and B show that

(1) If X is a finitely dominated space with fundamental group $\mathbb{Z}^n$, then X is homotopy equivalent to a finite CW-complex.
(2) If $f : X \to Y$ is a homotopy equivalence between finite CW-complexes with fundamental group $\mathbb{Z}^n$, then f is a simple homotopy equivalence.
(3) If $W : M \rightsquigarrow N$ is an h-cobordism between closed smooth manifolds with fundamental group $\mathbb{Z}^n$, and if $\dim(W) \geq 6$, then $W \cong M \times [0,1]$ relative to M.

Some of the fundamental results about topological manifolds are proven by a *torus argument*, which brings a torus (a space with fundamental group $\mathbb{Z}^n$!) into play in a somewhat unnatural manner. Theorems A and B show that the possible obstructions in $\widetilde{K}_0(\mathbb{Z}[\pi_1(T^n)])$ and $\mathrm{Wh}(\pi_1(T^n))$ are trivial, and this if often one of the key ingredients to a torus argument. Among the results which are proven (or can be proven) by a torus argument are the following:

(1) The topological invariance of rational Pontrjagin classes by Novikov [12].
(2) Several pivotal proofs in Kirby-Siebenmann's theory of topological manifolds [9] (the proof of the stable homeomorphism theorem by Kirby makes only implicit use of Theorems A and B.
(3) The topological invariance of Whitehead torsion (originally proven by Chapman [5]), and West's theorem [17] that a compact ENR has the homotopy type of a finite complex, have fairly accessible proofs by torus arguments in [6, §17,18].

Theorems A and B are proven in the books [4] and [14]. However, these texts prove more general versions, and the actual arguments are scattered over a large number of pages. In this note, we attempt to present the argument in a geodesic way, designed for the reader who just wants to know why A and B are true and then wants to return his attention to geometric topology. The reader who wants to understand these results in the wider context of higher algebraic K-theory should follow Quillen [13, §6].

2. Some commutative algebra

For the rest of this note, R will always be a commutative ring with unit (this assumption is mostly made for convenience, to avoid disctinction between left and right modules). There are two finiteness conditions on a ring which will play an important role in the proofs.

Definition 2.1. A ring R is *noetherian* if each ideal $I \subset R$ is finitely generated.

Equivalently, R is noetherian if each submodule of a finitely generated R-module is again finitely generated [1, Proposition 6.5]. The following famous theorem of Hilbert is a standard result of commutative algebra.

Theorem 2.2 (Hilbert's basis theorem)**.** *If R is noetherian, the so are the polynomial ring $R[t]$ and the Laurent polynomial ring $R[t^{\pm}]$.*

The proof of the first assertion can be found in [1, Theorem 7.5], and this implies the second one by virtue of [1, Proposition 7.3] since $R[t^{\pm}]$ is a localization of $R[t]$. By induction, it follows that the ring

$$\mathbb{Z}[\mathbb{Z}^n] = \mathbb{Z}[\mathbb{Z}^{n-1}][t_n^{\pm}] = \mathbb{Z}[t_1^{\pm}, \ldots, t_n^{\pm}]$$

is noetherian. We denote by

$$\mathbf{Mod}(R),\ \mathbf{Fin}(R),\ \mathbf{Proj}(R)$$

the categories of R-modules, finitely generated R-modules and projective finitely generated R-modules.

Definition 2.3. Let R be a ring and let $M \in \mathbf{Mod}(R)$ be an R-module. We say that the *projective dimension* of M is at most n, $\operatorname{projdim}_R(M) \leq n$, if there is a projective resolution

$$0 \to P_n \to \ldots \to P_0 \to M \to 0$$

of length n. A ring R is *regular* if $\operatorname{projdim}_R(M) < \infty$ for each R-module M.

Example 2.4. M is projective if and only if $\operatorname{projdim}_R(M) \leq 0$. If R is a principal ideal domain, then $\operatorname{projdim}_R(M) \leq 1$ for each R-module M. Hence principal ideal domains are regular.

Example 2.5. The rings $\mathbb{Z}/4$ and $\mathbb{Z}[\mathbb{Z}/n]$ are not regular.

Lemma 2.6. *An R-module M is projective if and only if $\operatorname{Ext}^1_R(M; N) = 0$ for all R-modules N.*

Proof. If M is projective, then $0 \to M \xrightarrow{\mathrm{id}} M \to 0$ is a projective resolution, which shows that $\operatorname{Ext}^1_R(M; N) = 0$. Vice versa, assume $\operatorname{Ext}^1(M; N) = 0$ for all N and let $f : N \to M$ be an epimorphism. The exact sequence

$$\operatorname{Hom}(M; N) \xrightarrow{f\circ -} \operatorname{Hom}(M; M) \to \operatorname{Ext}^1_R(M; \ker(f)) = 0$$

shows that f is split surjective and hence that M is projective. □

Lemma 2.7. *Let M be an R-module. The following are equivalent:*

(1) There is a projective resolution

$$0 \to P_n \xrightarrow{d_n} P_{n-1} \to \ldots \to P_0 \xrightarrow{d_0} M \to 0$$

of length n.

(2) For each exact sequence

$$0 \to Q \to P_{n-1} \xrightarrow{d_{n-1}} \ldots \to P_0 \xrightarrow{d_0} M \to 0$$

of R-modules with $P_0, \ldots, P_{n-1}$ projective, the module Q is projective.

(3) $\operatorname{Ext}^{n+1}_R(M; N) = 0$ for each R-module N.

Proof. The implications (2) $\Rightarrow$ (1) $\Rightarrow$ (3) are clear. To show (3) $\to$ (2), consider an exact sequence as in (2) and put

$$M_k := \begin{cases} M & k = 0 \\ \ker(d_{k-1}) = \mathrm{Im}(d_k) & 1 \leq k \leq n-1 \\ Q & k = n. \end{cases}$$

By Lemma 2.6, we need to prove that $\mathrm{Ext}^1_R(M_n; N) = 0$ for each N. There are short exact sequences

$$0 \to M_k \to P_{k-1} \to M_{k-1} \to 0$$

which induce, for each R-module N and $p \in \mathbb{N}$, exact sequences

$$0 = \mathrm{Ext}^p_R(P_{k-1}; N) \to \mathrm{Ext}^p_R(M_k; N) \to \mathrm{Ext}^{p+1}_R(M_{k-1}; N) \to \mathrm{Ext}^{p+1}(P_{k-1}; N) = 0$$

and isomorphism

$$\mathrm{Ext}^1(M_n; N)) \cong \mathrm{Ext}^2(M_{n-1}; N) \cong \ldots \cong \mathrm{Ext}^{n+1}_R(M_0; N) = 0. \qquad \square.$$

Corollary 2.8. *If R is regular and noetherian, then each finitely generated R-module admits a finite projective resolution, that is, a resolution of finite length by finitely generated projective R-modules.* $\square$

We need another famous theorem by Hilbert.

Theorem 2.9 (Hilbert's Syzygy theorem)**.** *If R is regular, then $R[t]$ and $R[t^{\pm}]$ are regular.*

Proof. Let M be an $R[t]$-module. We can view M, by restriction of scalars, as an R-module, together with an endomorphism $\varphi : M \to M$ coming from multiplication by t. The sequence

$$0 \to R[t] \otimes_R M \stackrel{t\otimes 1 - 1 \otimes \varphi}{\to} R[t] \otimes_R M \stackrel{p \otimes m \mapsto pm}{\to} M \to 0$$

is exact by a direct verification. For any $R[t]$-module N, we obtain exact sequences

$$\mathrm{Ext}^n_{R[t]}(R[t] \otimes_R M; N) \to \mathrm{Ext}^{n+1}_{R[t]}(M; N) \to \mathrm{Ext}^{n+1}_{R[t]}(R[t] \otimes_R M; N).$$

Because $R[t]$ is free as an R-module, there is an isomorphism

$$\mathrm{Ext}^n_{R[t]}(R[t] \otimes_R M; N) \cong \mathrm{Ext}^n_R(M; N).$$

It follows that if $n > \mathrm{projdim}_R(M)$, then $\mathrm{Ext}^{n+1}_{R[t]}(M; N)$ for all $R[t]$-modules N. In other words $\mathrm{projdim}_{R[t]} M \leq n$.

The second part follows from the first and the fact that $R[t^{\pm}]$ is a flat $R[t]$-module. Let M be an $R[t^{\pm}]$-module and let $(x_i)_i$ be a generating set of M, and let $M_1 \subset M$ be the $R[t]$-module generated by $(x_i)_i$. Then (because $R[t^{\pm}]$ is a flat $R[t]$-module), there is an isomorphism $M_1 \otimes_{R[t]} R[t^{\pm}] \cong M$. It follows that (again using flatness)

$$\mathrm{Ext}^n_{R[t^{\pm}]}(M; N) \cong \mathrm{Ext}^n_{R[t^{\pm}]}(M_1 \otimes_{R[t]} R[t^{\pm}]; N) = \mathrm{Ext}^n_{R[t]}(M_1; N),$$

and so $\mathrm{projdim}_{R[t^{\pm}]} M \leq \mathrm{projdim}_{R[t]} M_1$. $\square$

The basis theorem and the Syzygy theorem together imply that $\mathbb{Z}[\mathbb{Z}^n]$ is regular noetherian.

3. Proof of the Grothendieck–Serre theorem

We shall prove a more precise version:

Theorem 3.1. *Let R be a ring and let $R \xrightarrow{i} R[t] \xrightarrow{j} R[t^{\pm}]$ be the inclusions. Then if R is regular noetherian, the induced maps*

$$K_0(R) \xrightarrow{i_*} K_0(R[t]) \xrightarrow{j_*} K_0(R[t^{\pm}])$$

are isomorphisms.

Remark 3.2. Theorem 3.1 implies Theorem A by induction (using the basis and syzygy theorem in the induction step). In order to prove Theorem 3.1, it is enough to prove that i_* and j_* are surjective. This is because the map $r : R[t^{\pm}] \to R$, $t^{\pm} \mapsto 1$ is a right-inverse to $j \circ i$.

Lemma 3.3. *If R is regular noetherian, then $j_* : K_0(R[t]) \to K_0(R[t^{\pm}])$ is surjective.*

In the proof, we use the following simple observation: If

$$0 \to N_n \xrightarrow{d_n} N_{n-1} \to \ldots \to N_1 \xrightarrow{d_0} N_0 \to 0$$

is an exact sequence of projective R-modules, then

$$\sum_{i=0}^{n} (-1)^i [N_i] = 0 \in K_0(R). \tag{3.4}$$

To see this, one picks a split of d_0 to write $N_1 \cong \operatorname{Im}(d_2) \oplus N_0$, observes that $\operatorname{Im}(d_2)$ is projective and argues by induction.

Proof of Lemma 3.3. Let P be a finitely generated projective $R[t^{\pm}]$-module. Since $R[t^{\pm}]$ is noetherian, we can write P as a quotient

$$P = R[t^{\pm}]^n / N,$$

where the submodule N is finitely generated. Hence we may write

$$N = \langle x_1, \ldots, x_r \rangle$$

where $x_i \in R[t^{\pm}]^n$ is a vector whose entries are Laurent polynomials. There is $s \gg 0$ such that $t^s x_i \in R[t]^n$ for all i. Since t^s is a unit in $R[t^{\pm}]$, it follows that

$$P = R[t^{\pm}]^n / N \overset{t^s \cdot}{\cong} R[t^{\pm}]^n / t^s N = \frac{R[t^{\pm}]^n}{\langle t^s x_1, \ldots, t^s x_r \rangle} \cong \frac{R[t]^n}{\langle t^s x_1, \ldots, t^s x_r \rangle} \otimes_{R[t]} R[t^{\pm}].$$

Hence we can find a finitely generated $R[t]$-module M such that

$$P \cong M \otimes_{R[t]} R[t^{\pm}].$$

Because $R[t]$ is regular noetherian, there is a finite projective resolution

$$0 \to Q_n \to \ldots \to Q_0 \to M \to 0$$

by Corollary 2.8. Since $R[t^{\pm}]$ is a localization of $R[t]$, the functor $- \otimes_{R[t]} R[t^{\pm}]$ is exact, and hence we get a short exact sequence

$$0 \to Q_n \otimes_{R[t]} R[t^{\pm}] \to \ldots \to Q_0 \otimes_{R[t]} R[t^{\pm}] \to P \to 0 \tag{3.5}$$

of projective $R[t^{\pm}]$-modules. Apply (3.4) to the sequence (3.5) to obtain

$$[P] = \sum_{i=0}^{n} (-1)^i [Q_i \otimes_{R[t]} R[t^{\pm}]] = j_* \Big(\sum_{i=0}^{n} (-1)^i [Q_i] \Big). \qquad \square$$

For the proof that $i_* : K_0(R) \to K_0(R[t])$ is an isomorphism, we need to introduce *graded rings*.

Definition 3.6. A *graded ring* is a ring A, together with a decomposition

$$A = \bigoplus_{n \geq 0} A_n$$

such that $A_n A_m \subset A_{m+n}$. The ideal of elements of positive degree is

$$A_+ := \bigoplus_{n>0} A_n \subset A$$

(note that $A/A_+ \cong A_0$). An element $a \in A$ is *homogeneous* if $a \in A_n$ for some n. In that case, $|a| := n$ is the degree of a.

Note that a commutative graded ring is a different thing than a "graded commutative ring". Let us briefly list some important examples of graded rings.

- An (ungraded) ring R considered as a graded ring with $R_0 := R$.
- If A is graded, the polynomial ring $A[t]$ is graded, by declaring the degree of $t^k a$ to be $n + k$, when $|a| = n$.
- In particular, if R is ungraded, then $R[t, s]$ is graded, by declaring $|s| = |t| = 1$.

Definition 3.7. Let A be a graded ring. A *graded A-module* is an A-module M, together with a decomposition

$$M = \bigoplus_{n \in \mathbb{Z}} M_n,$$

such that $A_m M_n \subset M_{m+n}$ for all m, n. A homomorphism $f : M \to N$ between graded modules is graded, if $f(M_n) \subset N_n$ for all n. Such an f can be written in the form $f = \bigoplus_n f_n$, with $f_n : M_n \to N_n$. A submodule $N \subset M$ of a graded module is graded if $N = \bigoplus_n N \cap M_n$. In that case, the quotient module M/N inherits a grading from that of M.

We define

$$\mathbf{Mod}^{\mathrm{gr}}(A), \ \mathbf{Fin}^{\mathrm{gr}}(A), \ \mathbf{Proj}^{\mathrm{gr}}(A)$$

as the categories of all graded A-modules and graded homomorphisms, and the full subcategories such that the underlying A-module is finitely generated (resp. projective). If M is a graded A-module, we define the shift $M[n]$ as the graded A-module with underlying A-module M and grading defined by

$$M[n]_m := M_{m-n}.$$

For a graded A-module M, we define the graded A_0-module

$$T(M) := M/A_+M = M \otimes_A A_0.$$

Lemma 3.8. *Let A be a graded ring and let $M \in \mathbf{Fin}^{\mathrm{gr}}(A)$. Then*

1. *M is generated by finitely many homogeneous elements.*
2. *$M_n = 0$ for $n \ll 0$.*
3. *If $T(M) = 0$, then $M = 0$.*

Proof. The first claim is true, because any generating set of M contains a finite generating set. The minimum degree $n_0 \in \mathbb{Z}$ of an element of a homogeneous generating set has the property that $M_n = 0$ for $n < n_0$, showing claim (2). If $M \neq 0$, it contains a nonzero homogeneous element m of minimal degree, and $m \notin A_+ M$. □

Lemma 3.9. *Let A be a graded ring and let $P \in \mathbf{Proj}^{\mathrm{gr}}(A)$. Then*

(1) *Let $M \in \mathbf{Mod}^{\mathrm{gr}}(A)$ and let $f : M \to P$ be a surjective graded homomorphism. Then there is a graded homomorphism $g : P \to M$ with $f \circ g = \mathrm{id}$.*
(2) *P is a direct summand of a finitely generated free graded A-module, i.e. a sum of graded modules of the form $A[m]$.*

Proof. For claim (1), write $f = \bigoplus_n f_n$, with $f_n : M_n \to P_n$. Let $\pi_n^P : P \to P_n$ and $\pi_n^M : M \to M_n$ be the projections and let $\iota_n : M_n \to M$ be the inclusion. Pick a split $h : P \to M$ of f of ungraded modules. We define

$$g = \bigoplus_n \pi_n^P \circ h \circ \iota_n : P \to M.$$

Then

$$f_n \circ g_n = f_n \circ \pi_n^P \circ h \circ \iota_n = \pi_n^M \circ f \circ h \circ \iota_n = \pi_n^M \circ \iota_n = \mathrm{id}_{M_n}.$$

For claim (2), use Lemma 3.8 to construct a graded surjection

$$\bigoplus_{i=1}^r A[j_i] \to P$$

and use part (1). □

Lemma 3.10. *Each $P \in \mathbf{Proj}^{\mathrm{gr}}(A)$ is of the form $P = M \otimes_{A_0} A$ for some $M \in \mathbf{Proj}^{\mathrm{gr}}(A_0)$.*

Proof. The quotient map $f : P \to T(P)$ is a surjective homomorphism of graded A_0-modules (P is an A_0 module by forgetting) and $T(P)$ is projective. By Lemma 3.9 (1), there is a graded A_0-module map $g : T(P) \to P$ with $f \circ g = \mathrm{id}$. It induces a homomorphism of graded A-modules

$$h : T(P) \otimes_{A_0} A \to P, \; h(x \otimes y) = g(x)y.$$

Once we can show that h is an isomorphism, the lemma follows. To see this, observe that $T(h) : T(T(P) \otimes_{A_0} A) = T(P) \to T(P)$ is the identity. Because T is right exact, it follows that

$$0 = \mathrm{coker}(T(h)) \cong T(\mathrm{coker}(h)).$$

Since $\mathrm{coker}(h)$ is a finitely generated graded A-module, it follows from Lemma 3.8 that $\mathrm{coker}(h) = 0$, i.e. that h is surjective. On the other hand, since P is projective, there is a graded homomorphism $k : P \to T(P) \otimes_{A_0} A$ with $h \circ k = \mathrm{id}$, by Lemma 3.9. Since $T(h)$ is an isomorphism, so is $T(k)$. From

$$\ker(h) \cong \mathrm{coker}(k)$$

and the right-exactness of T, we obtain

$$T(\ker(h)) = T(\mathrm{coker}(k)) \cong \mathrm{coker}(T(k)) = 0,$$

and another application of Lemma 3.8 proves that $\ker(h) = 0$. □

Lemma 3.11. *Let A be a noetherian graded ring. Then the functor*

$$- \otimes_{A[t]} A : \mathbf{Fin}^{\mathrm{gr}}(A[t]) \to \mathbf{Fin}(A)$$

(induced by the ungraded *ring homomorphism $A[t] \to A$, $t \mapsto 1$) is exact and essentially surjective.*

Proof. Right-exactness is clear. To prove left-exactness, we have to prove that if $N \subset M$ are finitely generated graded $A[t]$-modules, then the induced map $N \otimes_{A[t]} A \to M \otimes_{A[t]} A$ is injective. Because $M \otimes_{A[t]} A \cong M/(1-t)M$, we find that

$$\ker(N \otimes_{A[t]} A \to M \otimes_{A[t]} A) = \frac{N \cap (1-t)M}{(1-t)N}.$$

Thus to prove left-exactness, we have to verify that

$$(1-t)N = N \cap \big((1-t)M\big). \tag{3.12}$$

It is obvious that $(1-t)N \subset N \cap \big((1-t)M\big)$. For the reverse inclusion, let $m \in M$ with $(1-t)m \in N$. To see that $m \in N$, write

$$m = m_i + m_{i+1} + \ldots + m_j \in M$$

as a sum of homogeneous elements $m_k \in M_k$. Then

$$(1-t)m = m_i + (m_{i+1} - tm_i) + \ldots \in N$$

implies $m_i, m_{i+1}, \ldots \in N$ and hence $m \in N$.

To show that $- \otimes_{A[t]} A$ is essentially surjective, let M be a finitely generated A-module, which can be written as

$$M = A^n/N$$

with a finitely generated submodule N (here we use that A is noetherian). Let $N = \langle x_1, \ldots, x_r \rangle$ and write $x_i = (x_{i1}, \ldots, x_{in})$ with $x_{ij} \in A$. For an arbitrary element $x \in A$, write

$$x = x_0 + \ldots + x_d \in A$$

as a sum of homogeneous elements and define the homogenized element as

$$x' := t^d x_0 + \ldots + t^0 x_d \in A[t]_d.$$

Let

$$N' := \langle x_1', \ldots, x_r' \rangle \subset A[t]^n;$$

this is a finitely generated submodule generated by homogeneous elements and hence a graded submodule. Define a finitely generated graded $A[t]$-module

$$M' := A[t]^n/N'.$$

Then

$$M' \otimes_{A[t]} A \cong M,$$

and this finishes the proof. □

The proof of Theorem 3.1 is completed by the following lemma.

Lemma 3.13. *If R is regular noetherian, then the map $i_* : K_0(R) \to K_0(R[t])$ is surjective.*

Proof. Let P be a projective finitely generated $R[t]$-module. By Lemma 3.11, there is a finitely generated graded $R[t,s]$-module M with

$$M \otimes_{R[t,s]} R[t] \cong P$$

(here we use the basis theorem to see that $R[t]$ is noetherian). By the Syzygy theorem and basis theorem, $R[t,s]$ is regular noetherian, and hence there is a finite projective resolution

$$0 \to Q_n \to \ldots \to Q_0 \to M \to 0$$

by *graded* projective $R[t,s]$-modules (use Lemma 3.9). By Lemma 3.11, the tensored sequence

$$0 \to Q_n \otimes_{R[t,s]} R[t] \to \ldots \to Q_0 \otimes_{R[t,s]} R[t] \to M \otimes_{R[t,s]} R[t] \cong P \to 0$$

is again a projective resolution. By Lemma 3.10, there are finitely generated projective R-modules N_j with $N_j \otimes_R R[t,s] \cong Q_j$. Therefore

$$Q_j \otimes_{R[t,s]} R[t] \cong N_j \otimes_R R[t,s] \otimes_{R[t,s]} R[t] \cong N_j \otimes_R R[t].$$

Apply (3.4) to conclude that

$$[P] = \sum_{j \geq 0} (-1)^j [N_j \otimes_R R[t]] \in \operatorname{Im}(j_*). \qquad \square$$

4. Proof of the Bass-Heller-Swan Theorem

Before we can state the version of Theorem B that we actually prove, let us recall a simple fact about $K_1(R)$. Suppose that P is a projective finitely generated R-module and $f : P \to P$ is an automorphism. Pick a complement Q, i.e. a finitely generated projective module such that $P \oplus Q \cong R^n$. The automorphism $f \oplus 1$ of R^n is represented by a matrix F in $\mathrm{GL}_n(R)$. By [11, Lemma 3.2], the class in $K_1(R)$ represented by F does not depend on the choice of P and the isomorphism. We denote this class by

$$[P, f] \in K_1(R).$$

The definitions are made up so that the following relations hold:

$$\begin{aligned} [P, fg] &= [P, f] + [P, g] \\ [P, 1] &= 0 \\ [P_0 \oplus P_1, f_0 \oplus f_1] &= [P_0, f_0] + [P_1, f_1]. \end{aligned}$$

Now we define a homomorphism

$$\beta : K_0(R) \to K_1(R[t^{\pm}]),$$

by sending the class of a projective module P to $[P \otimes_R R[t^{\pm}], t] \in K_1(R[t^{\pm}])$. This could legitimately be called the *Bott map*, due to its similarity with the Bott map in complex K-theory. In fact, a large portion of the proof of Theorem 4.1 below is very similar to one of the standard proofs [2] of the Bott periodicity theorem. Furthermore, we let $\iota : K_0(R) \to K_0(R[t^{\pm}])$ be induced by the inclusion $R \to R[t^{\pm}]$. Here is the version of Theorem B that we actually prove.

Theorem 4.1. *If R is regular noetherian, then $(\iota, \beta) : K_1(R) \oplus K_0(R) \to K_1(R[t^{\pm}])$ is surjective.*

Remark 4.2. One can show that (ι, β) is in fact bijective (but one does not need to know this if one only wants to understand why $\mathrm{Wh}(\mathbb{Z}^n) = 0$). In the special case of interest, it is fairly easy to prove.

Let us first demonstrate how to derive Theorem B from Theorem 4.1. Recall that if R is commutative, the determinant $\det : \mathrm{GL}_n(R) \to R^{\times}$ induces a homomorphism

$$\det : K_1(R) \to R^{\times}.$$

If R is a euclidean ring, det is an isomorphism. For example, it follows that $K_1(\mathbb{Z}) \to \{\pm 1\}$ is an isomorphism.

Lemma 4.3. *The units in the group ring* $\mathbb{Z}[\mathbb{Z}^n] = \mathbb{Z}[t_1^{\pm}, \ldots, t_n^{\pm}]$ *are precisely the elements of the form* $\pm t_1^{k_1} \cdots t_n^{t_k}$.

Proof. First consider the case $n = 1$. For $p = \sum_i a_i t^i \in \mathbb{Z}[t^{\pm}]$, $p \neq 0$, define

$$\deg(p) := \max\{j | a_j \neq 0\} - \min\{j | a_j \neq 0\} \in \mathbb{N}_0.$$

It is easily checked that $\deg(pq) = \deg(p) + \deg(q)$, so that for a unit p, we must have $\deg(p) = 0$. This means $p = at^k$ for some k, $a \in \mathbb{Z}$, and the only possibility for a is ± 1.

If $n \geq 2$ and $p = \sum_{g \in \mathbb{Z}^n} a_g g \in \mathbb{Z}[\mathbb{Z}^n]^{\times}$ is a unit, let $\mathrm{supp}(p) \subset \mathbb{Z}^n$ be the set of all g with $a_g \neq 0$. Under all coordinate projections $\mathbb{Z}^n \to \mathbb{Z}$, p maps to a unit. This shows that $\mathrm{supp}(p)$ maps to a singleton for each coordinate projection by the $n = 1$ case of the lemma. Hence $\mathrm{supp}(p)$ has a single element, so that $p = ag$ with $g \in \mathbb{Z}^n$ and $a \in \mathbb{Z}$; but $a = \pm 1$ is the only possibility. □

Proposition 4.4. *The homomorphism*

$$\det : K_1(\mathbb{Z}[\mathbb{Z}^n]) \to \mathbb{Z}[\mathbb{Z}^n]^{\times} \cong \pm \mathbb{Z}^n$$

is an isomorphism. The Whitehead group $\mathrm{Wh}(\mathbb{Z}^n)$ *is trivial.*

Proof. We prove the first claim by induction on n. The case $n = 0$ is clear since $\mathbb{Z}$ is euclidean. For the induction step, we write $\mathbb{Z}[\mathbb{Z}^n] = \mathbb{Z}[\mathbb{Z}^{n-1}][t_n^{\pm}]$ and let $j : \mathbb{Z}[\mathbb{Z}^{n-1}] \to \mathbb{Z}[\mathbb{Z}^n]$ be the inclusion. The composition

$$\mathbb{Z} = K_0(\mathbb{Z}) \to K_0(\mathbb{Z}[\mathbb{Z}^{n-1}]) \xrightarrow{\beta} K_1(\mathbb{Z}[\mathbb{Z}^n])$$

sends 1 to $t_n \in \mathrm{GL}_1(\mathbb{Z}[\mathbb{Z}^n]) \to K_1(\mathbb{Z}[\mathbb{Z}^n])$. Together with the map $K_1(\mathbb{Z}[\mathbb{Z}^{n-1}]) \to K_1(\mathbb{Z}[\mathbb{Z}^n])$ induced by the inclusion, it defines

$$\alpha : K_1(\mathbb{Z}[\mathbb{Z}^{n-1}]) \oplus \mathbb{Z} \to K_1(\mathbb{Z}[\mathbb{Z}^n]).$$

Theorems A and 4.1 together show that α is surjective. The diagram

$$\begin{tikzcd}
K_1(\mathbb{Z}[\mathbb{Z}^{n-1}]) \oplus \mathbb{Z} \arrow[r, "\alpha"] \arrow[d, "\det \oplus \mathrm{id}"] & K_1(\mathbb{Z}[\mathbb{Z}^n]) \arrow[d, "\det"] \\
(\pm \mathbb{Z}^{n-1}) \oplus \mathbb{Z} \arrow[r, "="] & \pm \mathbb{Z}^n
\end{tikzcd}$$

commutes by inspection. The left vertical map is an isomorphism, by induction. By a diagram chase, α is injective, and so α is an isomorphism. Therefore, all maps in the diagram are isomorphisms. Furthermore, the composition

$$\pm \mathbb{Z}^n \xrightarrow{i} K_1(\mathbb{Z}[\mathbb{Z}^n]) \xrightarrow{\det} \pm 1 \mathbb{Z}^n$$

is the identity. Since det is an isomorphism, i is surjective, and this shows that $\mathrm{Wh}(\mathbb{Z}^n) = \mathrm{coker}(i) = 0$. □

Let us now turn to the proof of Theorem 4.1. To that end, we write $[n, B] \in K_1(R)$ for the class represented by $\mathrm{GL}_n(R)$.

Proposition 4.5. *Each element $\mathbf{x} \in K_1(R[t^{\pm}])$ can be written as the sum of elements in the image of (ι, β) and elements of the form $[P_0, 1+tN_0]+[P_1, 1+t^{-1}N_1]$ with P_i projective and N_i nilpotent.*

Proof. Start with an arbitrary $\mathbf{x} = [n, B] \in K_1(R[t^{\pm}])$. We will write $\mathbf{x} \equiv \mathbf{y}$ if $\mathbf{x} - \mathbf{y} \in \mathrm{Im}(\iota, \beta)$. There is k so that $t^k B \in \mathrm{GL}_n(R[t^{\pm}]) \cap \mathrm{Mat}_{n,n}(R[t])$ (which is *not* the same as $\mathrm{GL}_n(R[t])$. Then

$$[n, B] = [n, t^{-k}] + [n, t^k B] = -nk\beta(1) + [n, t^k B] \equiv [n, t^k B].$$

In other words, we may henceforth assume $B \in \mathrm{GL}_n(R[t^{\pm}]) \cap \mathrm{Mat}_{n,n}(R[t])$. Write

$$B = B_0 + B_1 t + \ldots + B_m t^m$$

with $B_m \in \mathrm{Mat}_{n,n}(R)$. Now we use the *Higman trick*:

$$\begin{aligned}[n, B] = \left[2n, \begin{pmatrix} B & \\ & 1 \end{pmatrix}\right] &= \left[2n, \begin{pmatrix} B_0 + \ldots + B_m t^m & -t^{m-1} B_m \\ & 1 \end{pmatrix}\right] = \\ &= \left[2n, \begin{pmatrix} B_0 + \ldots + B_{m-1} t^{m-1} & -t^{m-1} B_m \\ t & 1 \end{pmatrix}\right]\end{aligned}$$

by elementary row and column operations. The latter matrix is polynomial of degree $\leq \max(1, m-1)$. Hence, as long as $m \geq 2$, we can reduce the degree by 1, at the price of enlarging the size of the matrices. Therefore

$$\mathbf{x} \equiv [n, B_0 + tB_1]$$

for $B_0, B_1 \in \mathrm{GL}_n(R)$ and some n. But setting $t = 1$ shows that $B_0 + B_1$ is invertible and so

$$\begin{aligned}[n, B_0 + tB_1] &= [n, B_0 + B_1] + [n, (B_0 + B_1)^{-1}(B_0 + tB_1)] \equiv \\ &\equiv [n, (B_0 + B_1)^{-1}(B_0 + B_1 + (t-1)B_1)] = \\ &= [n, 1 + (t-1)(B_0 + B_1)^{-1} B_1] =: [n, 1 + (t-1)A].\end{aligned}$$

Claim 4.6. If $A \in \mathrm{Mat}_{n,n}(R)$ is such that $1+(t-1)A \in \mathrm{GL}_n(R[t^{\pm}])$, then $(1-A)A$ is nilpotent.

To prove the claim, write $(1 + (t-1)A)^{-1} = C_{-m} t^{-m} + \ldots + C_m t^m$ with $C_i \in \mathrm{Mat}_{n,n}(R)$. The equation $(1 + (t-1)A)(1 + (t-1)A)^{-1} = 1$ can be written as

$$\begin{aligned}(1-A)C_{-m} &= 0 \\ (1-A)C_{1-m} + AC_{-m} &= 0 \\ \ldots &= 0 \\ (1-A)C_{-1} + AC_{-2} &= 0 \\ (1-A)C_0 + AC_{-1} &= 1 \\ (1-A)C_1 + AC_0 &= 0 \\ \ldots &= 0 \\ (1-A)C_m + AC_{m-1} &= 0 \\ AC_m &= 0.\end{aligned}$$

From these equations, we read off that

$$-C_m = AC_{m-1} \Rightarrow A^2 C_{m-1} = 0, \ \ldots, \ A^{m+1} C_0 = 0.$$

Working downwards and switching the roles of A and $1-A$, we obtain similarly
$$(1-A)^m C_{-1} = 0.$$
Using $(1-A)C_0 + AC_{-1} = 1$, we get
$$A^{m+1}(1-A)^{m+1} = A^{m+1}(1-A)^{m+1}(1-A)C_0 + A^{m+1}(1-A)^{m+1}AC_{-1} =$$
$$= (1-A)^{m+2}A^{m+1}C_0 + A^{m+2}(1-A)^{m+1}C_{-1} = 0,$$
showing the claim.

Claim 4.7. If $A \in \mathrm{Mat}_{n,n}(R)$ is such that $1+(t-1)A \in \mathrm{GL}_n(R[t^{\pm}])$, then $A = N+P$, where P is an idempotent, N is nilpotent, and P and N commute.

By Claim 4.6, there is r with $A^r(1-A)^r = 0$. Write
$$1 = (x+(1-x))^{2r} = p(x)x^r + q(x)(1-x)^r$$
for some polynomials $p, q \in \mathbb{Z}[x]$. Put
$$f(x) := x - x^r p(x).$$
Because
$$f(x) = x(1-x^{r-1}p(x)) = (x-1)(1-(1-x)^{r-1}q(x)),$$
x and $(x-1)$ divide $f(x)$ and so we can write
$$f(x) = x(x-1)g(x)$$
for some $g \in \mathbb{Q}[x]$. A quick calculation shows that $g \in \mathbb{Z}[x]$. Now define
$$P := p(A)A^r.$$
Note that $1-P = q(A)(1-A)^r$. Because
$$P - P^2 = P(1-P) = p(A)q(A)A^r(1-A)^r = 0,$$
P is an idempotent matrix. Furthermore
$$N := A - P$$
is nilpotent since
$$N = f(A) = A(1-A)g(A),$$
showing Claim 4.7. So far, we have seen that each element in $K_1(R[t^{\pm}])$ is the sum of an element in the image of (ι, β) and an element of the form
$$[n, 1+(t-1)(N+P)]$$
where $P \in \mathrm{Mat}_{n,n}(R)$ is an idempotent, $N \in \mathrm{Mat}_{n,n}(R)$ is nilpotent and $NP = PN$. Since $P^2 = P$, both $\mathrm{Im}(P)$ and $\mathrm{Im}(1-P) = \ker(P)$ are projective and $\mathrm{Im}(P) \oplus \mathrm{Im}(1-P) = R^n$. Thus
$$[n, 1+(t-1)(N+P)] = [\mathrm{Im}(P), 1+(t-1)(1+N)] + [\ker(P), 1+(t-1)N].$$
Now calculate
$$[\ker(P), 1+(t-1)N] = [\ker(P), (1-N)+tN] =$$
$$= [\ker(P), 1-N] + [\ker(P), 1+t(1-N)^{-1}N] =$$
$$\equiv [\ker(P), 1+t(1-N)^{-1}N]$$

and

$$\begin{aligned}[\mathrm{Im}(P), 1 + (t-1)(N+1)] &= [\mathrm{Im}(P), t(N+1) - N] = \\ &= [t] + [1+N] + [\mathrm{Im}(P), 1 - t^{-1}(N+1)^{-1}N] = \\ &\equiv [\mathrm{Im}(P) 1 - t^{-1}(N+1)^{-1}N].\end{aligned}$$

This concludes the proof. □

The proof of Theorem 4.1 is completed by the following result:

Proposition 4.8. *Let R be a regular noetherian ring and let $N : P \to P$ be nilpotent endomorphism of a finitely generated projective R-module. Then*

$$[P, 1 + N] = 0 \in K_1(R).$$

Example 4.9. The following example shows that the regularity hypothesis in Proposition 4.8 cannot be dropped. Let $R = \mathbb{Z}/4$. The element $2 \in R$ is nilpotent. Then $\mathbf{x} = [1 + 2] = [-1] \in K_1(R)$. But under the determinant homomorphism, $\mathbf{x}$ is mapped to the nontrivial element of $R^\times = \{1, -1\}$.

For the proof of Proposition 4.8, let us introduce the category $\mathbf{AutFin}(R)$ whose objects are R-modules M, together with automorphisms $f : M \to M$. A morphism $(M, f) \to (N, g)$ is a homomorphism $h : M \to N$ with $gh = hf$. Let $\mathbf{AutFin}(R) \subset \mathbf{AutMod}(R)$ be the full subcategory of all objects (M, f) with M finitely generated and let $\mathbf{AutProj}(R) \subset \mathbf{AutFin}(R)$ be the full subcategory of all objects (P, f) with p projective. Warning: an epimorphism $(M, f) \to (P, g)$ where P is projective does not necessarily split.

Proposition 4.10. *Let R be regular noetherian, let $(M, f) \in \mathbf{AutFin}(R)$ and let*

$$0 \to (P_n, g_n) \to \ldots \to (P_0, g_0) \to (M, f) \to 0$$

and

$$0 \to (Q_m, h_m) \to \ldots \to (Q_0, h_0) \to (M, f) \to 0$$

be two exact sequences with P_i, Q_i projective. Then

$$\sum_{i \geq 0} (-1)^i [P_i, g_i] = \sum_{i \geq 0} (-1)^i [Q_i, h_i] \in K_1(R).$$

Proof.

Claim 4.11. If

$$0 \to (P_n, g_n) \xrightarrow{d_n} \ldots \xrightarrow{d_1} (P_0, g_0) \to 0$$

is exact with P_i projective, then $\sum_i (-1)^i [P_i, g_i] = 0 \in K_1(R)$.

Let us first consider the case $n = 2$. A splitting s of d_1 yields an isomorphism

$$(P_1, d_1) \cong \left(P_2 \oplus P_0, \begin{pmatrix} g_2 & * \\ & g_0 \end{pmatrix} \right).$$

By elementary row operations, one sees that

$$[P_1, g_1] = \left[P_2 \oplus P_0, \begin{pmatrix} g_2 & * \\ & g_0 \end{pmatrix} \right] = \left[P_2 \oplus P_0, \begin{pmatrix} g_2 & \\ & g_0 \end{pmatrix} \right] = [P_2, g_2] + [P_0, g_0]$$

as claimed. The case $n \geq 3$ is settled by induction: given

$$0 \to (P_n, g_n) \xrightarrow{d_n} \ldots \xrightarrow{d_1} (P_0, g_0) \to 0$$

is exact with P_i projective, choose a splitting $s : P_0 \to P_1$ of d_1. Then (P_1, g_1) is isomorphic, in $\mathbf{AutProj}(R)$, to

$$(P_1' \oplus P_0, \begin{pmatrix} g_1' & * \\ & g_0 \end{pmatrix}),$$

with $g_1' := g_1|_{P_1'}$. Therefore

$$[P_1, g_1] = [P_1', g_1'] + [P_0, g_0] \in K_1(R).$$

Since P_1' is projective, we conclude $\sum_i (-1)^i [P_i, g_i] = 0$ by induction.

Claim 4.12. Suppose that

$$\begin{tikzcd} 0 \arrow[r] & (P_n, g_n) \arrow[r] \arrow[d, "k_n"] & \ldots \arrow[r] & (P_0, g_0) \arrow[r] \arrow[d, "k_0"] & (M, f) \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & (Q_n, h_n) \arrow[r] & \ldots \arrow[r] & (Q_0, h_0) \arrow[r] & (M, f) \arrow[r] & 0 \end{tikzcd}$$

is a commutative diagram in $\mathbf{AutFin}(R)$ with exact rows and P_i, Q_i projective, then

$$\sum_{j=0} (-1)^j [P_j, g_j] = \sum_{j=0} (-1)^j [Q_j, h_j] \in K_1(R).$$

To show this claim, consider the mapping cylinder (M_*, m_*) of the chain map $f_* : (P_*, g_*) \to (Q_*, h_*)$ of chain complexes in $\mathbf{AutProj}(()R)$. The jth term of the mapping cylinder is $(M_j, m_j) = (P_j, g_j) \oplus (Q_{j-1}, h_{j-1})$. Since f_* is a quasiisomorphism, the mapping cylinder is exact. By Claim 4.11, we get

$$0 = \sum_j (-1)^j [M_j, m_j] = \sum_j (-1)^j [P_j, g_j] - \sum_j (-1)^j [Q_j, h_j],$$

as claimed.

The problem with the general case is that if P is projective, then epimorphisms in $\mathbf{AutProj}(()R)$ onto (P, g) do not have to split. Therefore, we cannot invoke the fundamental lemma of homological algebra to constract a chain equivalence between the two complexes, and Claim 4.11 is not yet enough. But we are able to construct a further projective resolution (K_*, k_*) of (M, f) which maps to both, (P_*, g_*) and (Q_*, h_*). The existence of such a resolution implies the Proposition. Let us see how to construct (K_*, k_*).

Claim 4.13. Let $(M, f) \in \mathbf{AutFin}(R)$. Then there is a projective module Q, an epimorphism $q : Q \to M$ and an automorphism g of Q such that $qg = fq$.

First pick an epimorphism $p : P \to M$ from a projective P. We can lift the automorphism $f \oplus f^{-1}$ of $M \oplus M$ to an automorphism of $P \oplus P$. To see this, write

$$\begin{pmatrix} f & \\ & f^{-1} \end{pmatrix} = \begin{pmatrix} 1 & f \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ -f^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & f \\ & 1 \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}.$$

Since any *endomorphism* of M can be lifted to an endomorphism of P, it follows that $f \oplus f^{-1}$ can be lifted to an automorphism h of $P \oplus P$. Define $Q := P \oplus P$,

$q := p \circ \mathrm{pr}_1$ and $g := h$. By inspection, the diagram

$$\begin{tikzcd} P \oplus P \arrow[r, "p\circ\mathrm{pr}_1"] \arrow[d, "g"] & M \arrow[d, "f"] \\ P \oplus P \arrow[r, "p\circ\mathrm{pr}_1"] & M \end{tikzcd}$$

commutes, which shows the claim.

Claim 4.14. Let $b : (M, f) \to (N, g)$ be a morphism in $\mathbf{AutFin}(R)$ and let

$$0 \to (P_n, h_n) \to \ldots \to (P_0, h_0) \overset{a_0}{\to} (N, g) \to 0$$

be a finite resolution by objects in $\mathbf{AutProj}(R)$. Then there exists a finite resolution $(Q_*, k_*) \to (M, f)$ by objects in $\mathbf{AutProj}(R)$ and a chain map $(Q_*, k_*) \to (P_*, h_*)$ extending b.

Consider

$$(a_0, -b) : (P_0, h_0) \oplus (M, f) \to (N, g);$$

this is surjective, and let (B, r) be its kernel. Since R is noetherian, B is finitely generated and we can find a projective module Q_0 with an automorphism k_0 and an epimorphism $(Q_0, k_0) \to (B, r)$. This construction yields a commutative square

$$\begin{tikzcd} (Q_0, k_0) \arrow[r] \arrow[d] & (M, f) \arrow[d] \\ (P_0, h_0) \arrow[r] & (N, g). \end{tikzcd}$$

Proceeding in this manner, we get a partial projective resolution, which can be completed to a finite projective resolution because R is regular noetherian.

Claim 4.15. Under the assumptions of the proposition, there is a finite projective resolution (K_*, k_*) of (M, f) and two quasiisomorphisms $(K_*, k_*) \to (P_*, g_*)$ and $(K_*, k_*) \to (Q_*, h_*)$ covering the identity of (M, f).

To see this, we consider the diagonal $(M, f) \to (M \oplus M, f \oplus f)$ and apply the previous claim to the resolution $(P_* \oplus Q_*, g_* \oplus h_*) \to (M \oplus M, f \oplus f)$. The result is a resolution (K_*, k_*) of (M, f) with a map to $(P_* \oplus Q_*, g_* \oplus h_*)$ covering the diagonal. Composing with the two projections yields the desired map. □

Proof of 4.8. Let $f : P \to P$ be nilpotent of nilpotence index n (i.e. $f^n = 0$). We claim that $[P, 1 + f] = 0 \in K_1(R)$ and show this by induction on the nilpotence index. The induction beginning $n = 1$ is trivial. The following is a short exact sequence in $\mathbf{AutFin}(R)$:

$$0 \to (\mathrm{Im}(f), (1 + f)) \to (P, 1 + f) \to (P/\mathrm{Im}(f), 1 + f) \to 0.$$

Claim 4.16. Let M be a finitely generated R-module and let $f : M \to M$ be nilpotent of nilpotence index n. Then there is a projective Q, a nilpotent endomorphism g of Q of nilpotence index n and a surjective $q : Q \to M$ with $qg = fq$.

To see this, let $p : P \to M$ be an epimorphism from a finitely generate projective module P and put $Q := P^n$. Define

$$q : Q \to M;\ q = p \circ (1\, f\, f^2\, \ldots\, f^{n-1})$$

and

$$g = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \dots & \dots & \\ & & & 1 & 0 \end{pmatrix}.$$

Obviously $g^n = 0$ and $qg = fq$.

By Claim 4.16 and since R is regular noetherian, there is a finite projective resolution

$$0 \to (Q_m, 1 + f_m) \to \dots \to (Q_1, 1 + f_1) \to (\operatorname{Im}(f), 1 + f).$$

where each f_j is nilpotent of nilpotency index $\leq n-1$ (this is possible since $f|_{\operatorname{Im}(f)}$ has nilpotency index $\leq n-1$). This yields a finite projective resolution

$$0 \to (Q_m, 1 + f_m) \to \dots \to (Q_1, 1 + f_1) \to (P, 1 + f) \to (P/\operatorname{Im}(f), 1) \to 0$$

(since $f = 0$ on $P/\operatorname{Im}(f)$), and

$$0 \to (Q_m, 1) \to \dots \to (Q_1, 1) \to (P, 1) \to (P/\operatorname{Im}(f), 1) \to 0$$

is another finite projective resolution. It follows from Proposition 4.10 that

$$0 = [P, 1] + \sum_{j \geq 1} (-1)^j [Q_j, 1] = [P, 1 + f] + \sum_{j \geq 1} (-1)^j [Q_j, 1 + f_j].$$

By induction over the nilpotency index, $[Q_j, 1+f_j] = 0 \in K_1(R)$, and so $[P, 1+f] = 0$. □

References

[1] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[2] Michael Atiyah and Raoul Bott. On the periodicity theorem for complex vector bundles. *Acta Math.*, 112:229–247, 1964.

[3] H. Bass, A. Heller, and R. G. Swan. The Whitehead group of a polynomial extension. *Inst. Hautes Études Sci. Publ. Math.*, (22):61–79, 1964.

[4] Hyman Bass. *Algebraic K-theory*. W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[5] T. A. Chapman. Topological invariance of Whitehead torsion. *Amer. J. Math.*, 96:488–497, 1974.

[6] Steve Ferry. Geometric topology notes. Available at http://sites.math.rutgers.edu/ sferry/.

[7] Graham. Higman. The units of group-rings. *Proc. London Math. Soc. (2)*, 46:231–248, 1940.

[8] Michel A. Kervaire. Le théorème de Barden-Mazur-Stallings. *Comment. Math. Helv.*, 40:31–42, 1965.

[9] Robion C. Kirby and Laurence C. Siebenmann. *Foundational essays on topological manifolds, smoothings, and triangulations*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1977. With notes by John Milnor and Michael Atiyah, Annals of Mathematics Studies, No. 88.

[10] J. Milnor. Whitehead torsion. *Bull. Amer. Math. Soc.*, 72:358–426, 1966.

[11] John Milnor. *Introduction to algebraic K-theory*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1971. Annals of Mathematics Studies, No. 72.

[12] S. P. Novikov. Topological invariance of rational classes of Pontrjagin. *Dokl. Akad. Nauk SSSR*, 163:298–300, 1965.

[13] D. Quillen. Higher algebraic K-theory. I. pages 85–147. Lecture Notes in Math., Vol. 341, 1973.

[14] Jonathan Rosenberg. *Algebraic K-theory and its applications*, volume 147 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994.

[15] J.-P. Serre. Modules projectifs et espaces fibrés à fibre vectorielle. In *Séminaire P. Dubreil, M.-L. Dubreil-Jacotin et C. Pisot, 1957/58, Fasc. 2, Exposé 23*, page 18. Secrétariat mathématique, Paris, 1958.

[16] C. T. C. Wall. Finiteness conditions for CW-complexes. *Ann. of Math. (2)*, 81:56–69, 1965.

[17] James E. West. Mapping Hilbert cube manifolds to ANR's: a solution of a conjecture of Borsuk. *Ann. of Math. (2)*, 106(1):1–18, 1977.

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