Paul Broussous
Université de Poitiers, France
broussou@math.univ-poitiers.fr

The Special Linear Group over a field with discrete valuation

In one of the previous lectures, one constructed a natural BN-pair for the group $G = \mathrm{SL}(n, K)$, where K is a (commutative) field and $n \geqslant 2$ is an integer. In this talk we show that when the field K is equipped with a *discrete valuation*, there is a second natural BN-pair. For this new BN-pair the group "N" is going to be the same, but the group "B" is going to be what one calls an *Iwahori subgroup* of G. Its Weyl group is infinite of type $\tilde{A}_{n-1}$; it is an euclidean reflexion group. In particular the associated building is *affine* : its apartments are affine euclidean spaces.

We shall use the following notation. We denote by B the Borel subgroup of upper triangular matrices and by $T \subset B$ the subgroup of diagonal matrices. The normalizer N of T in G is the group of monomial matrices. The Weyl group $W = N/T$ is isomorphic to the permutation group S_n on n letters.

1. *Review on discrete valuations*

We assume that K is endowed with a discrete valuation, i.e. a surjective group homomorphism $v : K^\times \longrightarrow \mathbb{Z}$, satisfying

$$v(x+y) \geq \min(v(x), v(y)) \ , \ x, \ y \in K^\times \text{ s.t. } x+y \neq 0 \ .$$

One extends v to K by setting $v(0) = +\infty$. One easily sees that $v(-1) = 0$, $v(-x) = v(x)$, for all x in K, and that (exercise !) the set

$$\mathcal{O} := \{x \in K \ ; \ v(x) \geqslant 0\}$$

is a subring of K, called the *valuation ring* associated to (K, v).

For instance, fix a prime number p and consider the field $K = \mathbb{Q}$ equipped with the p-adic valuation $v = v_p$:

$$v(\frac{r}{s}) = v_p(r) - v_p(s) \ ,$$

where for any non-zero integer n, $v_p(n)$ is the unique integer satisfying

$$p^{v_p(n)} \mid n \text{ and } p^{v_p(n)+1} \not| n \ .$$

Then v is indeed a discrete valuation and the valuation ring is the ring of p-adic integers in $\mathbb{Q}$, that is the set of rational numbers of the form n/p^k, $n \in \mathbb{Z}$, $k \in \mathbb{N}$. One can get another example by completing $\mathbb{Q}$ w.r.t. this valuation. Indeed fix any real number $a \in (0,1)$. Then the map $x \mapsto |x|_p := a^{v_p(x)}$ is a norm on $\mathbb{Q}$ (with the convention that $a^{+\infty} = 0$). You may take the standard completion of $\mathbb{Q}$ w.r.t. this norm to get the field $\mathbb{Q}_p$ of p-adic numbers. Recall that $\mathbb{Q}_p$ is defined as the quotient of the ring of Cauchy sequences in $\mathbb{Q}$ w.r.t. $|-|_p$ by the too-sided ideal formed of Cauchy sequences converging to 0 (students wanting to learn more about this may read Cassels' book *Local Fields*). The valuation v_p extends naturally to $\mathbb{Q}_p$ and one gets a new pair $(\mathbb{Q}_p, v_p)$.

One easily checks the following points (exercises !):

• The group $\mathcal{O}^\times$ of units in $\mathcal{O}$ is the kernel $v^{-1}(0)$ of v.

• Let π any element of K such that $v(\pi) = 1$. Then any non-zero x in K may be written in a unique way as $\pi^n u$, where $n = v(x)$ and $u \in \mathcal{O}^\times$. As a consequence K is the field of fractions of $\mathcal{O}$.

• The ring $\mathcal{O}$ is principal. Its ideals are $\{0\}$ and ideals of the form $(\pi^k) = \pi^k\mathcal{O}$, for some $k \in \mathbb{Z}$.

• The unique maximal ideal of $\mathcal{O}$ is $\mathfrak{P} = \pi\mathcal{O}$.

The element π is called an *uniformizer* of $\mathcal{O}$. One fixes it once for all for the rest of the lecture. The quotient $k := \mathcal{O}/\mathfrak{P}$ is a field called the *residue field* of (K, v).

In the case of $(\mathbb{Q}, v_p)$ or $(\mathbb{Q}_p, v_p)$, the prime $p \in K$ is a uniformizer and the residue field is isomorphic to $\mathbb{F}_p$, the field with p elements (exercise !). Note that there are pairs (K, v) for which k is infinite (consider a field af rational functions with coefficients in an infinite field).

2. *The Cartan decomposition for* $\mathrm{SL}(n, K)$ *and lattices in* K^n

Let $\tilde{T}$ be the subgroup of diagonal matrices in $\tilde{G} = \mathrm{GL}(n, K)$. Let $\mathcal{K} = \mathrm{SL}(n, \mathcal{O})$ (resp. $\tilde{\mathcal{K}} = \mathrm{GL}(n, \mathcal{O})$) be the subgroup of G formed of matrices with entries in $\mathcal{O}$ and determinant 1 (resp. matrices with entries in $\mathcal{O}$ and determinant a unit of $\mathcal{O}$).

Theorem 1 (*Cartan decomposition*). *We have* $G = \mathcal{K}T\mathcal{K}$ *and* $\tilde{G} = \tilde{\mathcal{K}}\tilde{T}\tilde{\mathcal{K}}$.

Sketch of proof (see Brown's book, page 129). Let $g \in \tilde{G}$. Using elementary operations on lines and columns (right and left multiplications by transvections with coefficients in $\mathcal{O}$), one can write: $g = k_1 m k_2$, where k_1, $k_2 \in \mathcal{K}$ and where m is a monomial matrix. Such a monomial matrix may be written $t\sigma$ where t is a diagonal matrix of the form $\mathrm{diag}(\pi^{k_1}, \pi^{k_2}, ..., \pi^{k_n})$ and where σ is a monomial matrix with coefficients units of $\mathcal{O}$. This matrix σ lies in $\tilde{\mathcal{K}}$ and this gives the Cartan decomposition for $\tilde{G}$. If g is in G, then m must be in G. We get

$$1 = \det(m) = \pi^{\sum_{i=1}^n k_i} \det(\sigma)$$

with $\det(\sigma) \in \mathcal{O}^\times$. So by a unicity argument, we must have $\det(\sigma) = 1$ and $\sum k_i = 0$. It follows that $\sigma \in \mathcal{K}$ and $\mathrm{diag}(\pi^{k_1}, \pi^{k_2}, ..., \pi^{k_n}) \in T$, and we get the Cartan decomposition for G.

Let V be the vector space K^n. A lattice in V is a sub-$\mathcal{O}$-module of V of the form $L = \mathcal{O}e_1 \oplus \cdots \oplus \mathcal{O}e_n$, for some basis $(e_1, ..., e_n)$ of V. It is clear that the group $\tilde{G}$ acts transitively on the set of lattices in V. The stabilizer of the standard lattice (take $(e_1, \ldots, e_n)$ to be the standard basis of V) in $\tilde{G}$ is $\tilde{\mathcal{K}}$; its stabilizer in G is $\mathcal{K}$ (exercise !). The Cartan decomposition implies the following property of the space of lattices.

Theorem 2 *Let L and L' be lattices in V. Then one can choose a basis $(e_1, \ldots, e_n)$ of V, non-zero scalars t_1, ..., t_n, so that*

$$L = \bigoplus_{i=1}^{n} \mathcal{O}e_i \text{ and } L' = \bigoplus_{i=1}^{n} \mathcal{O}t_i e_i \ .$$

Proof. Let $(v_1, ..., v_n)$ and $(w_1, ..., w_n)$ be $\mathcal{O}$-bases of L and L' respectively. Expressing the elements of $(w_1, ..., w_n)$ as linear combinations of those of $(v_1, ..., v_n)$ yields a matrix g of $\tilde{G}$. The Cartan decomposition gives $g = k_1 t k_2$, with k_1, $k_2 \in \tilde{\mathcal{K}}$ and $t \in T$. Note that linear combinations whose corresponding matrix lies in $\tilde{\mathcal{K}}$ transforms an $\mathcal{O}$-basis of a lattice in another $\mathcal{O}$-basis. So we get bases $(v'_1, ..., v'_n)$ and $(w'_1, ..., w'_n)$ of L and L' respectively, related by $w'_n = t_i v'_n$, where $t = \mathrm{diag}(t_1, \ldots, t_n)$, as required.

If for some lattice L and some basis $(e_1, ..., e_n)$ of V, we have $L = \mathcal{O}t_1e_1 \oplus \cdots \oplus \mathcal{O}t_ne_n$, where t_1, ..., t_n are non-zero scalars, one says that the basis *splits* the lattice, or that we have a *splitting basis* of L. We shall see that the building attached to the affine BN-pair of G is a simplicial complex whose set of vertices may be identified with the G-set of homothety classes of lattices in V, and where vertices of an apartment correspond to lattices that are split by a fixed basis.

3. *The affine BN-pair of* $\mathrm{Sl}(n, K)$

Reduction mod $\mathfrak{P}$ gives a group homomorphism $\mathcal{K} = \mathrm{SL}(n, \mathcal{O}) \to \mathrm{SL}(n, k)$. Using the fact that $\mathrm{SL}(n)$ of a field is generated by transvection matrices, one sees that this homomorphism is onto.We have the diagram of groups :

$$\begin{array}{ccc} \mathrm{SL}(n, \mathcal{O}) & \hookrightarrow & \mathrm{SL}(n, K) \\ \downarrow & & \\ \mathrm{SL}(n, k) & & \end{array}$$

As we said in the introduction the group "N" is going to be the same, that is the normalizer N of T in G. As group "B" we take the inverse image I by the reduction map of the standard Borel subgroup $\bar{B}$ of $\mathrm{SL}(n, k)$. This subgroup of $\mathcal{K}$ is called the *standard Iwahori subgroup* of G. So we have

$$I = \begin{pmatrix} \mathcal{O} & \cdots & \cdots & \mathcal{O} \\ \mathfrak{P} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \mathfrak{P} & \cdots & \mathfrak{P} & \mathcal{O} \end{pmatrix}$$

that is

$$I = \{a = a_{ij} \in \mathcal{K} \ ; \ a_{ij} \in \mathcal{O} \text{ if } j \geqslant i, \ a_{ij} \in \mathfrak{P} \text{ if } i > j\} \ .$$

Theorem 3. *The pair (I, N) is a BN-pair for G.*

We must prove the following facts:

(a) I and N generate G

(b) The intersection $T^0 := I \cap N$ is normal in N.

(c) The quotient $W^a := N/T^0$ admits a set of generators S such that for all $s \in S$ and $w \in W^a$, the two conditions hold:

(BN1) $$C(s)C(w) \subset C(w) \cup C(sw).$$

(BN2) $$sBs^{-1} \not\subset B$$

Proof of (a). The group $\mathrm{SL}(n,k)$ admits a spherical BN-pair, where group "B" is the subgroup $\bar{B}$ of upper triangular matrices and group "N" is the subgroup $\bar{N}$ of monomial matrices. The Bruhat decomposition $\mathrm{SL}(n,k) = \bar{B}\bar{N}\bar{B}$ according to this BN-pair lifts to give the decomposition $\mathcal{K} = IN^0I$, where $N^0 = \mathcal{K} \cap N$ is the set of monomial matrices in $\mathcal{K}$. Using the Cartan decomposition, we get $G = IN^0ITIN^0I$. This proves (a).

Proof of (b). We first note that a matrix in $I \cap N$ must be diagonal. Indeed if it is not diagonal, one of its entries must lie in $\mathfrak{P}$ and its determinant can not be 1. So $T^0 = I \cap N = I \cap T$, that is the set of diagonal matrices whose entries are units in $\mathcal{O}$. Now we see that T^0 is normal by direct computation.

Proof of (c). We first have to describe W^a. We have a natural projection $W^a = N/T^0 \longrightarrow W = N/T$, whose kernel is T/T^0, that is a short exact sequence :

$$1 \longrightarrow T/T^0 \longrightarrow W^a \longrightarrow W \longrightarrow 1 \ .$$

We may lift W in W^a as the subgroup N^0/T^0 and this sequence splits. Moreover the valuation map induces a group isomorphism:

$$T/T^0 \longrightarrow F \subset \mathbb{Z}^n$$

$$(t_1, ..., t_n) \text{ mod } T^0 \mapsto (v(t_1), ..., v(t_n))$$

where F is the free $\mathbb{Z}$-module of rank $n-1$: $\{(k_1, k_2, ..., k_n) \in \mathbb{Z}^n \ ; \ k_1 + \cdots + k_n = 0\}$. Indeed we have $W^a = F \rtimes W$, where the action of W on F is by permutation of coordinates.

We do not have enough time to work out a detailled proof of (BN1) and (BN2). So we restrict to the case $n = 2$ which is already interesting in itself. Here W^a is the infinite dihedral group $\mathbb{Z} \rtimes S_2$. F is generated by an element whose representative in T can be taken to be

$$d = \begin{pmatrix} \pi & 0 \\ 0 & \pi^{-1} \end{pmatrix}$$

and W is generated by an element s_1 whose representative is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

So W^a is generated by $S = \{s_1, s_2\}$, where s_2 is represented by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \pi & 0 \\ 0 & \pi^{-1} \end{pmatrix} = \begin{pmatrix} 0 & \pi^{-1} \\ \pi & 0 \end{pmatrix} .$$

It turns out that this choice of S works. The proof of (BN1) and (BN2) is not difficult: it works by case by case computations. The reader may refer to pages 131 and 132 of Brown's book.

4. *The Building of the BN-pair* (I, N)

It is easy to see that the Coxeter complex $\mathcal{A}$ of (W^a, S) is described as follows. It is isomorphic to the 1-dimensional simplicial complex whose set of vertices is $\mathbb{Z}$ and set of chambers (i.e. edges in this case) is the set of pairs $\{k, k+1\}$, $k \in \mathbb{Z}$. Its geometric realization is the real line $\mathbb{R}$. One may fix the notation so that s_1 (resp. s_2) acts as the reflection with respect to the hyperplan $x = 0$ (resp. with respect to the hyperplane $x = 1$). Then the element $\delta = s_2 s_1$ of W^a represented by

$$\begin{pmatrix} \pi^{-1} & 0 \\ 0 & \pi \end{pmatrix}$$

acts as the translation $x \mapsto x + 2$. The geometric realization of the fundamental chamber is the closed interval $[0, 1]$.

As a consequence, the building $\Delta(G, I)$ of the BN-pair (I, N) is a 1-dimensional building whose apartements are real lines. From this one easily proves the existence of a unique geodesic between two points of the geometric realization and that this realization is contractible. So $\Delta(G, I)$ is a *tree without terminal points*.

Let us describe the special subgroups. Of course I is the special subgroup corresponding to the subset $\emptyset \subset S$, and according to the Bruhat decomposition $G = IW^aI$, G is the special subgroup corresponding to the whole set S. The Bruhat decomposition at the residual level $\mathrm{SL}(n, k) = \bar{B}\langle s_1\rangle\bar{B}$ lifts to the decomposition $\mathcal{K} = I\langle s_1\rangle I$. Therefore $\mathcal{K}$ is the special subgroup corresponding to $\{s_1\} \subset S$. To find the last one, i.e. the special subgroup corresponding to s_2, we use the fact that s_1 and s_2 are conjugate in $\mathrm{GL}(2, K)$:

$$s_2 = \begin{pmatrix} 0 & -\pi^{-1} \\ \pi & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\pi & 0 \end{pmatrix} s_1 \begin{pmatrix} 0 & -\pi^{-1} \\ 1 & 0 \end{pmatrix} .$$

Moreover the matrix $\Pi := \begin{pmatrix} 0 & 1 \\ -\pi & 0 \end{pmatrix}$ normalizes I (exercise!). So the special subgroup corresponding to $\{s_2\} \subset \mathcal{K}$ is

$$\mathcal{K}' = \Pi\mathcal{K}\Pi^{-1} = \left\{ \begin{pmatrix} a & \pi^{-1}b \\ \pi c & d \end{pmatrix} \; ; \; a, b, c, d \in \mathcal{O} \; , \; ad - bc = 1 \right\} .$$

5. *The lattice model of* $\Delta(G, I)$

We are going to give a model of $\Delta(G, I)$ in terms of $\mathcal{O}$-lattices in $V = K^n$. For simplicity sake we restrict to the case $n = 2$.

Call two lattices L and L' of V *equivalent* if there exists a scalar t such that $L' = tL$. We denote by $[L]$ the equivalence class of L for this relation and by $\mathcal{L}$ the set of equivalence classes. Call two classes $[L]$ and $[L']$ *incident*, denoted $[L] \sim [L']$, if one can arrange the representatives so that

$$\pi L \subset L' \subset L \ ,$$

(the containments are strict). One easily sees that the relation $\sim$ is symmetric. Let $F(\mathcal{L})$ the be flag complex of this incidence geometry. It is naturally equipped with an action of $\tilde{G} = \mathrm{GL}(n, K)$ by simplicial automorphisms.

Lemma 1. *Any simplex in* $F(\mathcal{L})$ *has dimension less than or equal to* 1 *and any simplex is contained in an edge.*

Proof. Let $[L] = [\mathcal{O}e_1 \oplus \mathcal{O}e_2]$ be a vertex of $F(\mathcal{L})$, where (e_1, e_2) is a basis of V. Then $[L]$ is contained in the edge corresponding to the flag

$$([\mathcal{O}e_1 \oplus \mathcal{O}e_2] \sim [\mathcal{O}e_1 \oplus \mathcal{O}\pi e_2] \ .$$

This proves the second assertion. For the first one, assume for a contradiction that we have a two dimensional flag $([L], [L'], [L''])$. Then $[L]$, $[L']$ and $[L'']$ are pairwise incident and we must have arrange the representatives so that

$$\pi L \subset L' \subset L \text{ and } \pi L \subset L'' \subset L \ .$$

Reduction mod πL gives two lines $\bar{L}' = L'/\pi L$ and $\bar{L}'' = L''/\pi L$ in the 2-dimensional k-space $L/\pi L$. These two lines must be distinct otherwise we would have $L' = L''$. Pick e_1 and e_2 in L such that $\bar{L}' = k\bar{e}_1$ and $\bar{L}'' = k\bar{e}_2$, where $\bar{v}$ means reduction mod πL. Then $L = \mathcal{O}e_1 \oplus \mathcal{O}e_2$, $L' = \mathcal{O}e_1 + \pi L = \mathcal{O}e_1 + \mathcal{O}\pi e_2$ and $L'' = \mathcal{O}e_2 + \pi L = \mathcal{O}\pi e_1 + \mathcal{O}e_2$. But $[\mathcal{O}e_1 + \mathcal{O}\pi e_2]$ and $[\mathcal{O}\pi e_1 + \mathcal{O}e_2]$ cannot be incident.

Lemma 2. *The simplicial complex* $F(\mathcal{L})$ *is labellable and the action of* G *respects the labelling. Moreover* G *acts transitively on simplices of same type.*

Proof. Say that a lattice class $[L]$ is of type 0 (resp. 1) if L has a basis (e_1, e_2) so that $v(\det(e_1, e_2))$ is even (resp. odd). Here the determinant is relative to the standard basis of K^2. One easily sees that the type does not depend on the choice of the representative L nor on the choice of the basis (e_1, e_2) (this follows form the fact that $v(\det(t.\mathrm{id}_{K^2})) = 2v(t)$ is even and that $v(\det(k)) = 0$ for all $k \in \tilde{\mathcal{K}}$). If $[L]$ and $[L']$ are two vertices of an edge, then one can find a basis (e_1, e_2) and representatives so that $L = \mathcal{O}e_1 \oplus \mathcal{O}e_2$ and $L' = \mathcal{O}e_1 \oplus \mathcal{O}\pi e_2$. It follows that vertices in a common edge have different types and that we indeed have a labelling of $F(\mathcal{L})$. The action of G clearly preserves the labelling.

Let $[L] = [\mathcal{O}e_1 \oplus \mathcal{O}e_2]$ and $[L'] = [\mathcal{O}f_1 \oplus \mathcal{O}f_2]$ be two vertices of the same type and $g \in \tilde{G}$ such that $ge_i = f_i$, $i = 1, 2$. Since $\det(g) = \pi^{2r}u$, $u \in \mathcal{O}^\times$ and $r \in \mathbb{Z}$, one may write

$$g = g_1 \begin{pmatrix} \pi^r u & 0 \\ 0 & \pi^r \end{pmatrix} \text{ with } g_1 \in G \ .$$

But then $[L'] = g_1[\mathrm{diag}(\pi^r u, \pi^r)L]$, with $[\mathrm{diag}(\pi^r u, \pi^r)L] = [\mathcal{O}e_1 \oplus \mathcal{O}e_2]$. So the action of G is transitive on vertices of the same type. It is clearly transitive on edges and this proves the second assertion of the lemma.

Theorem 4. *The simplicial complexes $\Delta(G, I)$ and $F(\mathcal{L})$ are isomorphic as simplicial G-complexes.*

Proof. We call the flag $[\mathcal{O} \oplus \mathcal{O}] \sim [\mathcal{O} \oplus \pi\mathcal{O}]$ the *standard edge* C. The stabilizer of $s_0 := [\mathcal{O} \oplus \mathcal{O}]$ is $\mathcal{K}$. The vertex $s_1 := [\mathcal{O} \oplus \pi\mathcal{O}]$ may be written $s_1 = \Pi s_0$, so that the stabilizer of s_2 is $\mathcal{K}' = \Pi\mathcal{K}\Pi^{-1}$. Finally the stabilizer of C is $\mathcal{K} \cap \mathcal{K}' = I$. It follows that:

– The G-set of type 0 vertices is isomorphic to $G/\mathcal{K}$.
– The G-set of type 1 vertices is isomorphic to $G/\mathcal{K}'$.
– The G-set of egdes is isomorphic to G/I.

From this is follows that as a simplicial complex the poset $\Delta(G, I)$ of special cosets in G is isomorphic to the flag complex $F(\mathcal{L})$.

Finally let us say a few words on the apartments of $\Delta(G, I)$. The vertices of the fundamental apartment $\mathcal{A}$ are

$$\{ws_0 \ , \ ws_1 \ ; \ w \in W^a\}$$

Using the description of W^a, we see that the vertices of $\mathcal{A}$ correspond to lattice classes of the form $[\pi^l\mathcal{O} \oplus \pi^l\mathcal{O}]$, $l, m \in \mathbb{Z}$, i.e; to lattice classes whose lattices are split by the standard basis. The other apartments are the $g\mathcal{A}$, $g \in G$. They correspond to lattice classes whose lattices are split by a given basis of V. So apartments are in $1-1$ correspondence with the set of unordered pairs of lines $\{D_1, D_2\}$ in V such that $D_1 \oplus D_2 = V$.

6. *Remarks and further reading*

In *Trees*, Springer Verlag, 1980, Jean-Pierre Serre gives a construction of the tree $\Delta(G, I)$, in the case of $\mathrm{SL}(2, \mathbb{Q}_p)$ and proofs of its basic properties. His proofs are elementary and do not involve the theory of BN-pairs.

The book *Buildings and Classical Groups*, Chapman & Hall, 1997, by Paul Garrett, is similar to Brown's book. In is §19 he gives the lattice model of the building of $\mathrm{SL}(n, K)$ but its approach is different. He first proves that the flag geometry of lattice classes gives rise to a thick building. Then he uses the fact that the action of G on that building is strongly transitive and that the BN-pair one gets if formed of the Iwahori subgroup and the monomial subgroup. Garrett's book also deals with the affine buildings of some classical groups over a local field.

Historically the affine Building of $\mathrm{SL}(n, \mathbb{Q}_p)$ was first considered by Goldman and Iwahori in *The space of p-adic norms*, Acta. Math. 109, 1963, pp. 137-177. Actually they constructed a model of the geometric realization of that buildings in terms of *additive norms* on the vector space $V = \mathbb{Q}_p^n$. An additive norm is a map $\nu : V \mapsto \mathbb{R} \cup \infty$ such that

– $\nu(x+y) \geqslant \inf(\alpha(x), \alpha(y))$, $x, y \in V$.
– $\nu(tx) = v_p(t) + \nu(x)$, $t \in \mathbb{Q}_p$, $x \in V$.
– $\nu(x) = \infty$ if and only if $x = 0$.

Norms are related to flags of lattices in the following way. If ν is a norm and $r \in \mathbb{R}$, then the ball $L_r = \{x \in V \ ; \ \nu(x) \geqslant r\}$ is a lattice in V. The sequence of lattices L_r, $r \in \mathbb{R}$, only contains a finite number of lattices up to homothety and gives rise to a simplex of the building.

In their monographs (Publ. Math. IHES 41 and 60), Bruhat and Tits explain how to attach an affine building to any reductive group over a local field. They develop the axiomatic of BN-pairs and valuated root data. They then give lattice models of affine buildings of classical groups in their two papers in Bull. Soc. Math. France.