Notes on the Classification of Spherical and Affine Buildings

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These notes contain a brief overview of the main classification results in the theory of spherical and affine buildings. The results we describe are all due to Jacques Tits (with some assistance in the case of affine buildings from François Bruhat). All the proofs can be found in the books and articles listed in the bibliography (but have not tried to give exact citations).

1. Buildings

Let Π be a Coxeter diagram with vertex set I and let W be the corresponding Coxeter group. Thus W is a group generated by the set I subject to relations which can be read off from the labels on the edges of Π. Let Δ be a building of type Π. For these notes, it suffices to consider Δ as a graph whose vertices are the chambers, where two chambers are joined by an edge whenever they lie in a common panel. If the type of a panel P (viewed as a simplex) is $I\backslash\{i\}$, we give each edge joining two chambers of P the "color" i.

The pair (W, I) is called a *Coxeter system.* Let Σ_Π be the corresponding Cayley graph. Thus Σ_Π is the graph with vertex set W, where two vertices $x, y \in W$ are joined by an edge whenever $y^{-1}x \in I$. We endow Σ_Π with its natural edge-coloring whereby an edge $\{x, y\}$ has "color" $i \in I$ if and only if $x = yi$ in W. The *apartments* of Δ are subgraphs isomorphic to this edge-colored Cayley graph.

For each edge $\{x, y\}$ of Σ_Π, the set of vertices nearer to x than to y and the set of vertices nearer to y than to x form a partition of the vertex set of Σ_Π (i.e. of W). Sets of this form are called *roots.* A root of an apartment Σ of Δ is the image in Σ of a root under some isomorphism from Σ_Π to Σ. A root of Δ is a root of one of its apartments.

For each subset J of I, let Π_J denote the subgraph of Π spanned by J. The J-*residues* of Δ (for some $J \subset I$) are the connected components of the graph obtained from Δ by deleting all the edges of Δ whose color is not in J (but without deleting any chambers). For a given J-residue R, the set J is precisely the set of colors appearing on the edges of R; this set is called the *type* of R and its cardinality is called the *rank* of R. A *residue* is a J-residue for some $J \subset I$. The *panels* of Δ are the residues of rank one. Every J-residue is a building of type Π_J. In particular, the panels are complete graphs. A J-residue is *irreducible* if the diagram Π_J is connected.

2. Spherical Buildings

The classification of spherical buildings (in rank at least three) was first given in [**3**]. In this section we follow more closely the revised proof given in Chapter 40 of [**5**].

From now on, we suppose that Δ is *spherical* (i.e. that its apartments are finite), *thick* (i.e. that every panel contains more than two chambers) and *irreducible* (i.e. that the Coxeter diagram Π is connected).

2.1 Definition. For each root α of Δ, let U_α denote the intersection of the stabilizers in the group $\mathrm{Aut}(\Delta)$ of all the chambers of Δ that are contained in a panel containing two chambers of α. The group U_α is called the *root group* belonging to α. It acts trivially on α and faithfully on every panel in the wall of α (i.e. on every panel containing exactly one chamber of α).

2.2 Definition. The building Δ is *Moufang* if for every root α, the root group U_α acts transitively on the set of apartments containing α. If Δ is Moufang and P is a panel in the wall of a root α, then the root group acts sharply transitive on the set $P\backslash\alpha$.

Now let C be a chamber of Δ and let $E_2(C)$ be the subgraph of Δ spanned by all the irreducible rank two residues of Δ that contain C. (There is exactly one such residue for each edge of Π and the intersection of two of these residues is a panel containing C if the corresponding edges are adjacent, $\{C\}$ otherwise.) The subgraph $E_2(C)$ is called the *foundation* of Δ.

2.3 Theorem. *The building Δ is uniquely determined by $E_2(C)$.*

In other words, if Δ' is a second building of type Π and π is an isomorphism from $E_2(C)$ to $E_2(C')$, where C' is a chamber of Δ', then there is an isomorphism from Δ to Δ' that extends π. A map π from $E_2(C)$ to $E_2(C')$ sending C to C' is an isomorphism if and only if for each edge $J = \{i, j\}$ of Π, the restriction of π to the J-residue of Δ containing C is an isomorphism from this residue to the J-residue of Δ' containing C'.

Applying 2.3 with $\Delta' = \Delta$, it is possible to deduce that the automorphism group of Δ is large. More precisely, we have the following:

2.4 Theorem. *If the rank of Δ (i.e. the cardinality of I) is at least three, then Δ is Moufang and all the irreducible residues of rank at least two of Δ are also Moufang.*

2.5 Definition. Suppose that Δ is Moufang. Let Σ be an apartment of Δ and let α and $-\alpha$ be a pair of opposite roots in Σ. Then for each $u \in U_\alpha^*$, there exist unique elements $y, z \in U_{-\alpha}^*$ such that the product

$$m_\Sigma(u) = yuz$$

interchanges α and $-\alpha$ and maps each panel in the wall of α to itself. (Thus $m_\Sigma(u)$ induces a reflection on Σ, but it is not necessarily an element of order two.) In particular, the product

$$m_\Sigma(u)m_\Sigma(v)$$

acts trivially on the apartment Σ for all $u, v \in U_\alpha^*$. Let

$$T_\alpha = \langle m_\Sigma(u)m_\Sigma(v) \mid u, v \in U_\alpha^* \rangle.$$

Note that the group T_α normalizes U_β for all roots β of Σ.

2.6 Notation. Now suppose that Δ is Moufang and of rank two (so $|I| = 2$), let n be the label on the unique edge of Π, let C be a chamber of Δ, let P_1 and P_n be the two panels containing C and let Σ be an apartment of Δ containing C. Thus Σ is a circuit of length $2n$. We label the root $\alpha_1, \ldots, \alpha_n$ of Σ containing C so that $|\alpha_1 \cap P_1| = 1 = |\alpha_n \cap P_n|$ and $|\alpha_{i-1} \cap \alpha_i| = n - 1$ for all $i \in [2, n]$. Let $U_i = U_{\alpha_i}$ for all $i \in [1, n]$ and let

$$U_+ = \langle U_1, U_2, \ldots, U_n \rangle.$$

The sequence

$$\Omega := (U_+, U_1, U_2, \ldots, U_n)$$

is called the *root group sequence* of Δ. It depends on the choice of the pair (C, Σ), but $\mathrm{Aut}(\Delta)$ acts transitively on the set of such pairs, so the Ω is unique up to conjugation. Well, not really; it is the pair $(\Omega, \Omega^{\mathrm{op}})$ which is unique, where

$$\Omega^{\mathrm{op}} = (U_+, U_n, U_{n-1}, \ldots, U_1).$$

(Ω^{op} is called the *opposite root group sequence*.)

A building of the kind described in 2.6 is essentially the same thing as a *Moufang polygon*; see 7.11–7.17 in [**6**] for details.

2.7 Theorem. *If Δ is Moufang and has rank two, then it is uniquely determined by its root group sequence.*

Moufang buildings of rank two were classified in terms of their root group sequences in [**5**]. In [**5**], they are divided into nine families. We describe three of them:

2.8 Example. Let K be either a field, a skew field or an octonion division algebra. (The definition of an octonion division algebra can be found, for example, in Chapter 9 of [**5**], where they are called *Cayley-Dickson division algebras.*) For each $i \in [1,3]$, let x_i be an isomorphism from the additive group of K to a group U_i. Let U_+ be the group generated by the groups U_1, U_2, U_3 subject to the commutator relations $[U_1, U_2] = [U_2, U_3] = 1$ and

$$[x_1(s), x_3(t)] = x_2(st)$$

for all $s, t \in K$. There is a unique building whose root group sequence is

$$(U_+, U_1, U_2, U_3).$$

2.9 Example. Let (K, L, q) be an *anisotropic quadratic space.* In other words, K is a field, L is a vector space over K and q is a map from L to K such that $q(ta) = t^2 q(a)$ for all $a \in L$ and all $t \in K$, the map f given by

$$f(a, b) := q(a + b) - q(a) - q(b)$$

is a bilinear form on L and $q(a) = 0$ if and only if $a = 0$. For each odd (respectively, even) $i \in [1,4]$, let x_i be an isomorphism from the additive group of K (respectively, L) to a group U_i. Let U_+ be the group generated by the groups U_1, U_2, U_3, U_4 subject to the commutator relations $[U_1, U_2] = [U_2, U_3] = [U_3, U_4] = 1$, $[U_1, U_3] = 1$,

$$[x_2(a), x_4(b)^{-1}] = x_3(f(a, b))$$

for all $a, b \in L$ and

$$[x_1(t), x_4(a)^{-1}] = x_2(ta)x_3(tq(a))$$

for all $t \in K$ and all $a \in L$. There is a unique building whose root group sequence is $(U_+, U_1, U_2, U_3, U_4)$.

2.10 Example. Let (K, K_0, σ) be an *involutory set.* By this we mean that K is a skew field (possibly commutative), σ is an involution of K (i.e. an anti-automorphism whose square is the identity), K_0 is an additive subgroup of K containing 1, the set

$$K_\sigma := \{a + a^\sigma \mid a \in K\}$$

(the set of *traces* of σ) is contained in K_0, K_0 is contained in the set K^σ of fixed points of σ and $a^\sigma K_0 a \subset K_0$ for all $a \in K$. (If the characteristic of K is not two, then $K_\sigma = K_0 = K^\sigma$, but if the characteristic of K is not two, neither equality need hold.) For each odd (respectively, even) $i \in [1, 4]$, let x_i be an isomorphism from K_0 (respectively, the additive group of K) to a group U_i. Let U_+ be the group generated by the groups U_1, U_2, U_3, U_4 subject to the commutator relations $[U_1, U_2] = [U_2, U_3] = [U_3, U_4] = 1$, $[U_1, U_3] = 1$,

$$[x_2(a), x_4(b)^{-1}] = x_3(a^\sigma b + b^\sigma a)$$

for all $a, b \in K$ and

$$[x_1(t), x_4(a)^{-1}] = x_2(ta)x_3(a^\sigma ta)$$

for all $t \in K_0$ and all $a \in K$. There is a unique building whose root group sequence is $(U_+, U_1, U_2, U_3, U_4)$.

2.11 Remark. Let K be an octonion division algebra with center F and let σ be its standard involution and q its reduced norm (as defined, for example, in 9.10 of [**5**]). Then (K, F, σ) is not an involutory set (since K is not a skew field), but the recipe in 2.9 nevertheless yields the root group sequence of a building. We therefore call such a triple (K, F, σ) an *honorary* involutory set. (The root group sequence we obtain in this fashion is not new, however; it is precisely the root group sequence we obtain by applying 2.9 to the anisotropic quadratic space (F, K, q).)

In 2.8, $n = 3$ (where n is the label on the unique edge of the Coxeter diagram Π). In 2.9 and 2.10, $n = 4$. There is a third family with $n = 4$ where the defining algebraic structure is an *anisotropic pseudo-quadratic space*. This is something like an anisotropic quadratic space defined over a skew field with involution. In this family, not all the root groups are abelian. There are five remaining families, three with $n = 4$, one with $n = 6$ and one with $n = 8$. See Chapter 17 of [**5**] for details. These five families do not appear as rank two residues of spherical buildings in higher rank as we will see below.

We note explicitly that the buildings described in 2.8 are the only Moufang buildings whose Coxeter diagram is A_2. These buildings are especially important since in higher rank, almost every irreducible rank two residue is a Moufang building of type A_2. Here is one basic fact about these buildings:

2.12 Proposition. *Let K and $\Omega := (U_+, U_1, U_2, U_3)$ be as in 2.8. We denote the corresponding building by $A_2(K)$. Let $\Delta = A_2(K)$ and let Σ be an apartment of Δ whose roots have been numbered so that Ω is the corresponding root group sequence. Let m_Σ be the map from U_i^* to $\mathrm{Aut}(\Delta)$ (for $i = 1$, 2 or 3) described 2.5 and let*

$$h_1 = m_\Sigma(x_1(-1))m_\Sigma(x_1(s^{-1})) \quad \textit{and} \quad h_3 = m_\Sigma(x_3(-1))m_\Sigma(x_3(t^{-1}))$$

for some $s \in U_1^*$ *and some* $t \in U_3^*$. *Then*

$$x_1(u)^{h_1} = x_1(s^{-1}us^{-1}), \quad x_2(u)^{h_1} = x_2(s^{-1}u) \quad \textit{and} \quad x_3(u)^{h_1} = x_3(su)$$

and

$$x_1(u)^{h_3} = x_1(ut), \quad x_2(u)^{h_3} = x_2(ut^{-1}) \quad \textit{and} \quad x_3(u)^{h_3} = x_3(t^{-1}ut^{-1})$$

for all $u \in K$.

Similar formulas for all the other families of Moufang buildings of rank two can be found in 33.11–33.17 of [**5**].

We turn now to spherical buildings of rank greater than two. To begin, we let Π be the Coxeter diagram A_3 and we assume that Δ is a building of type Π. We can suppose that $I = \{1, 2, 3\}$, where $e := \{1, 2\}$ and $f := \{2, 3\}$ are the two edges of Π. Let Σ be an apartment of Δ, let C be a chamber of Σ and for each $i \in I$, let α_i be the unique root of Σ containing C but not the unique chamber of Σ that is i-adjacent to C (i.e. joined to C by an edge of color i), let U_{α_i} be the corresponding root group and let T_{α_i} be as in 2.5. Let Δ_e and Δ_f be the unique e- and f-residues of Δ containing the chamber C. Thus $\Delta_e \cap \Delta_f = P$, where P is the unique 2-panel (i.e. $\{2\}$-residue) containing the chamber C. The root group U_{α_2} acts sharply on the set $P \backslash \{C\}$.

By 2.4, Δ_e and Δ_f are both Moufang. Thus there exist fields (or skew fields or octonion division algebras) K and K' such that $\Delta_e \cong A_2(K)$ and $\Delta_f \cong A_2(K')$, where $A_2(K)$ and $A_2(K')$ are as defined in 2.12. Furthermore:

(i) $\alpha_2 \cap \Delta_e$ and $\alpha_2 \cap \Delta_f$ are roots of Δ_e and Δ_f and U_{α_2} induces the corresponding root group on each of these two residues. Both of these root groups act faithfully on the panel P.

(ii) By (i), the root group U_{α_2} is isomorphic both to the additive group of K and the additive group of K'. We can thus identify these two additive groups with each other and with U_{α_2}. The multiplication in K we denote by $\cdot$ and the multiplication in K' (which we now call K) we denote by $*$.

(iii) The isomorphism from Δ_e to $A_2(K)$ can be chosen so that $(K, \cdot)$ and $(K, *)$ have the same multiplicative identity.

(iv) Since the vertices 1 and 3 are not adjacent in Π, it can be shown that the groups T_{α_1} and T_{α_3} commute elementwise. (This is, in fact, a consequence of 3.2 below.) By 2.12, T_{α_1} induces the maps $u \mapsto t \cdot u$ for all $t \in K^*$ on K (which we have identified with U_{α_2}) and T_{α_3} induces the map $u \mapsto u * s$ for all $s \in K^*$ on K. Since $[T_{\alpha_1}, T_{\alpha_3}] = 1$, it follows that

$$(t \cdot u) * s = t \cdot (u * s)$$

for all $s, u, t \in K^*$.

(v) By (iii), $t \cdot 1 = t$ and $1 * s = s$ for all $t, s \in K$. Thus setting $u = 1$ in the identity in (iv), we deduce that the multiplications $\cdot$ and $*$ are the same. Thus not only is $K \cong K'$, but K is, again by the identity in (iv), associative, i.e. not octonion.

From (i)–(v), we deduce that $E_2(C)$ is isomorphic to the graph obtained by taking a field or a skew field K, then taking two copies of $A_2(K)$, then choosing panels P and P' and chambers $C \in P$ and $C' \in P'$ in both copies, then identifying both $P\backslash\{C\}$ and $P'\backslash\{C'\}$ with K via the action of root groups. The resulting graph is (up to isomorphism) independent of the choice of the various choices. It is the foundation of a building which we denote by $A_3(K)$.

If $\Delta = A_3(K)$ for some field or skew field K, then every irreducible rank two residue of Δ is isomorphic to $A_2(K)$.

2.13 The Simply Laced Case. Now suppose that $\Pi = X_n$ is simply laced for some $n \geq 4$ and that Δ is a building of type Π. Thus (by the classification of finite Coxeter groups), $\Pi = A_n$ or D_n for $n \geq 4$ or $\Pi = E_6$, E_7 or E_8. If $\Pi_J \cong A_3$ for some $J \subset I$ and Δ_J is the unique J-residue containing a fixed chamber C, then $\Delta_J \cong A_3(K_J)$ for some field or skew field K_J (as we have just seen above). Suppose that Π_J and $\Pi_{J'}$ are two subgraphs of Π having exactly one edge e in common and both isomorphic to A_3. Then the unique e-residue containing C is a residue of both Δ_J and $\Delta_{J'}$. Therefore $A_2(K_J) \cong A_2(K_{J'})$. From this it follows that $K_J \cong K_{J'}$. We conclude that $K := K_J$ is independent of the choice of J. Therefore every irreducible residue of rank two is isomorphic to $A_2(K)$ and if e and f are two edges of Π containing a vertex i in common, then the e- and f-residues containing C are glued together along a panel exactly as they are in $A_3(K)$. Thus $E_2(C)$ is uniquely determined by K alone. Conversely, the graph we obtain by such glueings is always the foundation of a building subject only to the condition that K is commutative if Π contains a subdiagram isomorphic to D_4.

Suppose, namely, that Π contains a subdiagram isomorphic to D_4. In this case, we can choose *three* groups of the form T_α (corresponding to the unique triple of nodes of Π having a common neighbor) that commute pairwise and that all induce the maps $u \mapsto tu$ on K for all $t \in K^*$. From this it follows that K must, in fact, be commutative.

The remaining spherical buildings of rank n at least three can be described in terms of their foundations as follows:

2.14 The Quadratic Form Case. Let $\Pi = B_n$ and let (K, L, q) be an anisotropic quadratic space. Take one copy of the building of rank two described in 2.9 and let C and P_1 be as in 2.6. The root group U_1 acts sharply transitively on $P_1\backslash\{C\}$. Using this action, we can identify $P_1\backslash\{C\}$ with K. Now take a copy of $A_2(K)$ (i.e. of the building defined in 2.8) and let C' and P_1' be as in 2.6 (with primes on all the notation this time). Again the action of U_1' yields an identification of $P_1'\backslash\{C'\}$ with K. We thus have an identification of P_1 with P_1' (which matches up C with C'). We then glue to this copy of $A_2(K)$ a second copy of $A_2(K)$, identifying its panel P_1'' with the panel P_3'. We continue in this fashion until we have accounted for all the edges of Π. The resulting graph depends only on (K, L, q) and not on the various choices involved in its description. It is the foundation of a unique building of type Π depending only on (K, L, q).

2.15 The Involutory Set Case. Let $\Pi = B_n$ and let (K, K_0, σ) be an involutory set as described in 2.10 or an honorary involutory set (as defined in 2.11) if $n = 3$. Thus again U_1 is isomorphic to K. Take one copy of the corresponding building of rank two and glue on $n-2$ copies of $A_2(K)$ exactly as in the previous case. (This can only be done once if K is octonion since there are no buildings of type A_3 in this case.) Again the resulting graph depends only on (K, K_0, σ) and not on the various choices involved in its description. It is the foundation of a unique building of type Π depending only on (K, K_0, σ).

2.16 The Pseudo-Quadratic Form Case. Let $\Pi = B_n$, let

$$(K, K_0, \sigma, L, q)$$

be an anisotropic pseudo-quadratic space and let $(U_+, U_1, U_2, U_3, U_4)$ be the root group sequence described in 16.5 of [**5**]. Thus again U_1 is isomorphic to K. Take one copy of the corresponding building of rank two and glue on copies of $A_2(K)$ exactly as in cases (i) and (ii). Again the resulting graph depends only on (K, K_0, σ, L, q) and not on the various choices involved in its description. It is the foundation of a unique building of type Π depending only on (K, K_0, σ, L, q). (If L is trivial, we obtain the buildings in 2.15.)

2.17 The F_4-Case. Let $\Pi = F_4$. Suppose that either (a) L/K is an inseparable extension of fields of characteristic two such that $L^2 \subset K$, (b) L is a field equal to K, (c) L/K is a separable quadratic extension or (d) L is a quaternion or octonion division algebra with center K. Let σ be the identity and $q(x) = x^2$ for all $x \in L$ in cases (a) and (b). In case (c) let σ be the unique non-trivial element in the Galois group of L/K and let q be the norm of the extension L/K. In case (d) let σ be the standard involution of L and let q be the reduced norm of L. In each case (L, K, σ) is an involutory set (or an honorary involutory set) and (K, L, q) is an anisotropic quadratic space. Moreover, the root group sequence obtained by applying the recipe in 2.9 to (K, L, q) is the same as the root group sequence obtained by applying the recipe in 2.10 to (L, K, σ). Take a copy of this building of rank two and let C, P_1, P_4 be as in 2.6. We can thus identify $P_1\backslash\{C\}$ with K via the action of U_1 and $P_4\backslash\{C\}$ with L via the action of U_4. We glue a copy of $A_2(K)$ on by identifying one of its panels with P_1 and then glue on a copy of $A_2(L)$ by identifying one of its panels with P_4. The resulting graph depends only on the pair (K, L) and not on the various choices involved in its description. It is the foundation of a unique building of type Π depending only on the pair (K, L).

This completes the list of all (irreducible thick) spherical buildings of rank at least three. It can be shown that almost all of these buildings arise (via the notion of a BN-pair) from absolutely simple algebraic groups. The exceptions are those buildings whose defining algebraic data involves a skew field K of infinite dimension over its center; or a vector space L of infinite dimension over K; or an anisotropic quadratic form q whose defect (i.e. the radical of the corresponding bilinear form) is of dimension greater than one; or an anisotropic pseudo-quadratic form q whose defect is non-trivial; or a purely inseparable extension L/K of exponent one.

We call K the *defining field* of the corresponding building Δ (even though K is not always, in fact, a field). The defining field K is an invariant of Δ except in subcase (a) of case (iv), when only the pair (K, L) and not K itself is an invariant of Δ (so Δ really has a pair of defining fields in this case).

3. Root Systems

Let V be an n-dimensional Euclidean space and let Φ be an irreducible reduced root system spanning V (as defined in Chapter 6 of [**1**]; see especially §4). Thus $\Phi = X_n$ for some $X \in \{A, B, C, D, E, F, G\}$. For each $\alpha \in \Phi$, let

$$H_\alpha = \{v \in V \mid \alpha \cdot v = 0\}$$

and

$$s_\alpha(v) = v - 2(v \cdot \alpha)\alpha/(\alpha \cdot \alpha)$$

for all $v \in V$. A *Weyl chamber* of Φ is the closure of a connected component of the space V with all the hyperplanes of the form H_α for $\alpha \in \Phi$ removed. We define two Weyl chambers to be adjacent if their intersection spans a subspace of dimension $n - 1$. This defines a graph Θ on the Weyl chambers that is isomorphic to the

Cayley graph associated with the spherical Coxeter diagram called X_n. For each $\alpha \in \Phi$, the set of all Weyl chambers contained in

$$\{v \in V \mid \alpha \cdot v \geq 0\}$$

is a root of Θ (as defined in Section 1) and every root of Θ is of this form (for a unique $\alpha \in \Phi$). We can thus identify Φ with the set of roots of Θ.

3.1 Notation. Let α, β be two roots of Φ such that $\beta \neq \pm\alpha$. Let (α, β) denote the set of elements

$$\alpha_1, \dots, \alpha_s$$

of Φ of the form $p\alpha + q\beta$ for positive real numbers p and q. We order the vectors in this set so that the angle between α and α_i increases as i increases. This ordered set is called the *interval* from α to β.

3.2 Remark. Suppose that Δ is a building of type X_n with $n \geq 3$, so Δ is one of the spherical buildings described in the previous section. Let Σ be an apartment of Δ, let α, β be linearly independent elements of Φ and let $\alpha_1, \dots, \alpha_s$ be as in 3.1. If we identify Σ with the graph Θ and thus Φ with the set of roots of Σ, then

$$[U_\alpha, U_\beta] \subset \prod_{i=1}^{s} U_{\alpha_i}$$

unless the interval (α, β) is empty, in which case $[U_\alpha, U_\beta] = 1$. We have seen special cases of this in 2.8–2.10.

Let $\tilde{X}_n$ denote the corresponding extended Dynkin diagram with the arrows on the multiple bonds deleted. The diagrams $\tilde{X}_n$ which arise in this way are precisely the connected affine Coxeter diagrams; they can be viewed, for example, in Theorem 4 in Chapter VI, §4, of [**1**].

For each $\alpha \in \Phi$ and each integer k, let

$$H_{\alpha,k} = \{v \in V \mid \alpha \cdot v = k\}$$

and

$$s_{\alpha,k}(v) = s_\alpha(v) + 2k\alpha/(\alpha \cdot \alpha)$$

for all $v \in V$. The affine hyperplane $H_{\alpha,k}$ is thus the fixed point set of $s_{\alpha,k}$. An *alcove* of Φ is the closure of a connected component of the space V with all the affine hyperplanes of the form $H_{\alpha,k}$ removed. Let Γ be the graph whose vertices are the alcoves such that two alcoves are adjacent whenever their intersection is of dimension $n-1$ (i.e. the vectors $u-v$ for all u, v in the intersection span a subspace of dimension $n-1$). The graph Γ is isomorphic to the Cayley graph of the Coxeter system associated with the Coxeter diagram $\tilde{X}_n$, and the corresponding Coxeter group is isomorphic to the group generated by all the maps of the form $s_{\alpha,k}$.

For each $\alpha \in \Phi$ and each integer k, let $K_{\alpha,k}$ denote the set of alcoves in the set

$$\{v \in V \mid v \cdot \alpha \geq k\}.$$

Each $K_{\alpha,k}$ is a root of Γ and every root of Γ is of this form (for a unique pair α, k). We can think of the affine hyperplane $H_{\alpha,k}$ as the wall of $K_{\alpha,k}$. Two walls $H_{\alpha,k}$ and $H_{\beta,l}$ of Γ are *parallel* if $\alpha = \beta$ and *adjacent* if, in addition, $|k-l| = 1$.

A point v in V is called *special* if $v \cdot \alpha$ is an integer for all $\alpha \in \Phi$. A *sector* is a translation of a Weyl chamber by a special point, i.e. a set of the form $C + v$, where C is a Weyl chamber and v is a special point. We think of a sector S as the subgraph of Γ spanned by all the alcoves in S. Two sectors S_1 and S_2 are *adjacent* if there are two Weyl chambers C_1 and C_2 whose intersection spans a subspace of dimension $n-1$ and two special points v_1 and v_2 such that $S_1 = C_1 + v_1$ and $S_2 = C_2 + v_2$.

4. Affine Buildings

The classification of affine buildings (in dimension at least three) was first given in [**2**] and [**4**]. Proofs for all the assertions in this section can be found in [**7**].

We continue with all the notation in the previous section. Suppose from now on that Ξ is a thick building of type $\tilde{X}_n$. The apartments of Ξ are isomorphic to the graph Γ. A sector of Ξ is the image of a sector under an isomorphism from Γ to an apartment of Ξ. Moreover, two sectors in an apartment A are adjacent (respectively, two walls in A are parallel or adjacent) if they are the image under an isomorphism from Γ to A of two sectors that are adjacent (respectively, two walls that are parallel or adjacent).

The first main theorem in the theory of affine buildings is the following:

4.1 Theorem. *Let S and S_1 be two sectors. Then there exists an apartment of Ξ containing subsectors of both S and S_1.*

Two sectors are said to be *parallel* if their intersection is a subsector. This is an equivalence relation on the set of sectors; the parallel class containing a given sector S is denoted by S^∞. By 4.1, given two parallel classes of sectors there is always an apartment containing representatives of both. We declare two parallel classes of sectors to be adjacent if two representatives in a single apartment are adjacent.

This gives the set of parallel classes structure of a thick building of type X_n. This building is called the *building at infinity* of Ξ and is denoted by Ξ^∞. The building at infinity is spherical.

If A is an apartment of Ξ, then the set of chambers S^∞ of Ξ^∞ for all sectors S contained in A forms an apartment of Ξ^∞ which we denote by A^∞. The map $A \mapsto A^\infty$ is a bijection from the set of apartments of Ξ to the set of apartments of Ξ^∞. For each root α of Ξ, the set of chambers S^∞ of X^∞ for all sectors S contained in α forms a root of Ξ^∞ (which we denote by α^∞) and every root of Ξ^∞ is of this form.

We declare two walls of Ξ to be parallel (respectively, adjacent) if there is an apartment containing them both where they are parallel (respectively, adjacent). For each parallel class m of walls, let Λ_m be the graph whose vertices are the walls in m, where two walls in m are joined by an edge whenever they are adjacent. Then for each parallel class m of walls of Ξ, the graph Λ_m is a thick tree (i.e. a tree such that every vertex has at least three neighbors), and the ends of this tree are in one-to-one correspondence with the roots of Ξ^∞ of the form α^∞ for some root α of Ξ whose wall is in m. Furthermore, two roots α and β of Ξ have parallel walls if and only if the two roots α^∞ and β^∞ of Ξ^∞ have the same wall. The map sending the wall of a root α of Ξ to the wall of the root α^∞ thus yields a canonical bijection from the set of parallel classes of walls of Ξ to the set of walls of Ξ^∞.

Let Ξ_1 be a second building of type $\tilde{X}_n$ and let π be an isomorphism from Ξ^∞ to Ξ_1^∞. The map π is called *ecological* (or *tree-preserving*) if for every wall m of Ξ^∞ (i.e. every parallel class m of walls of Ξ), the map induced by π from the set of roots of Ξ^∞ with wall m to the set of roots of Ξ_1^∞ with wall $\pi(m)$ (i.e. from the set of ends of the tree Λ_m to the set of ends of the tree $\Lambda_{\pi(m)}$) is induced by an isomorphism from Λ_m to $\Lambda_{\pi(m)}$.

Here is the second main theorem in the theory of affine buildings:

4.2 Theorem. *If π is an ecological isomorphism from Ξ^∞ to Ξ_1^∞, then there is a unique isomorphism from Ξ to Ξ_1 that induces π.*

For the next result, we fix an apartment A of Ξ and choose an isomorphism $\tilde{\pi}$ from A to the graph Γ on the alcoves of the root system Φ described in Section 3. The isomorphism $\tilde{\pi}$ induces an isomorphism from the apartment $\Sigma := A^\infty$ of Ξ^∞ to the graph Θ on the Weyl chambers of Φ defined in Section 3, and this isomorphism yields, in turn, an identification of the root system Φ with the set of roots of Σ.

4.3 Theorem. *Suppose that Ξ^∞ is Moufang (as defined in 2.2) and let $K_{\alpha,k}$ be as defined in Section 3. Then the following hold:*

(i) *For each* $\alpha \in \Phi$*, every element in the root group* U_α *is ecological. Thus by 4.2, each element* u *of the root group* U_α *has a canonical image in* $\mathrm{Aut}(\Xi)$ *which we denote by the same letter* u.

(ii) *For each* $\alpha \in \Phi$*, there exists a surjective map* φ_α *from* U_α^* *to the integers such that for each* $u \in U_\alpha^*$*, the fixed point set of* u *in* A *is the root* $K_{\alpha,-\varphi_\alpha(u)}$.

(iii) *For each* $\alpha \in \Phi$*, let*

$$d_\alpha(u,v) = 2^{-\varphi_\alpha(u-v)}$$

for all $u, v \in U_\alpha$ *(whose multiplication we write in additive notation even though it might not be abelian). Then* d_α *is a metric on* U_α*, and* U_α *is complete with respect to this metric.*

(iv) *The set* $\varphi := \{\varphi_\alpha \mid \alpha \in \Phi\}$ *satisfies the following three conditions:*

(V1) *For each* $\alpha \in \Phi$ *and each integer* k*, the subset*

$$U_{\alpha,k} := \{u \in U_\alpha \mid \varphi_\alpha(u) \geq k\}$$

is a subgroup of U_α *(where we define* φ_α *of the identity element of* U_α *to be* ∞*).*

(V2) *For each pair* $\alpha, \beta \in \Phi$ *such that* $\beta \neq \pm\alpha$ *and the interval* (α, β) *is non-empty,*

$$[U_{\alpha,k}, U_{\beta,l}] \subset \prod_{i=1}^{s} U_{\alpha_i, p_i\alpha + q_i\beta},$$

where $(\alpha_1, \ldots, \alpha_s)$ *is as in 3.1 and for each* $i \in [1, s]$*,* p_i *and* q_i *are the unique positive integers such that* $\alpha_i = p_i\alpha + q_i\beta$*. (Compare 3.2.)*

(V3) *For each pair* $\alpha, \beta \in \Phi$ *and each* $u \in U_\alpha^*$*, the value*

$$\varphi_{s_\alpha(\beta)}(x^{m_\Sigma(u)}) - \varphi_\beta(x)$$

is independent of the choice of $x \in U_\beta$ *and equals* $-2\varphi_\alpha(u)$ *if* $\alpha = \beta$*. Here* $m_\Sigma(u)$ *is as in 2.5; it induces the reflection* s_α *on* Φ*, i.e. on the set of roots of the apartment* Σ*.*

4.4 Definition. Let Δ be a Moufang building of type X_n and let Σ be an apartment of Δ. Let π be an isomorphism from the apartment Σ to the graph Θ on the Weyl chambers of Φ and let the root system Φ be identified with the set of roots of Σ via π. The set $\{U_\alpha \mid \alpha \in \Phi\}$ (together with the map $\alpha \mapsto U_\alpha$) is called the *root datum* of Δ (based at Σ). For each $\alpha \in \Phi$, let φ_α be a map from U_α^* onto the integers (sending the identity to ∞). Then the set

$$\varphi = \{\varphi_\alpha \mid \alpha \in \Phi\}$$

is called a *valuation of the root datum* of Δ (based at Σ) if the maps in φ satisfy the conditions (V1)–(V3) in 4.3(iv). If

$$\psi = \{\psi_\alpha \mid \alpha \in \Phi\}$$

is a second valuation of the root datum of Δ, then φ and ψ are called *equipollent* if for each $\alpha \in \Phi$, the difference

$$\psi_\alpha(u) - \varphi_\alpha(u)$$

is independent of the choice of the element $u \in U_\alpha^*$.

4.5 Proposition. *Suppose that* Ξ^∞ *is Moufang and let*

$$\varphi = \{\varphi_\alpha \mid \alpha \in \Phi\}$$

be as in 4.3. Then φ *is a valuation of the root datum of* Ξ^∞ *(with respect to the isomorphism* π *from* Σ *to* Θ *induced by the isomorphism* $\tilde{\pi}$ *from* A *to* Γ*). It depends on the choice of* $\tilde{\pi}$ *but only up to equipollence.*

Now suppose that Δ is an arbitrary thick building of type X_n with $n \geq 3$ and let Σ be an apartment of Δ. Since $n \geq 3$, Δ is one of the spherical buildings in 2.13–2.17. In particular, we can always choose roots $\beta_1, \beta_2, \beta_3$ of Σ and maps $x_{\beta_1}, x_{\beta_2}, x_{\beta_3}$ such that x_{β_i} is an isomorphism from the additive group of the defining field K to the root group U_{β_i} for each $i \in [1,3]$ and

$$[x_{\beta_1}(u), x_{\beta_3}(v)] = x_{\beta_2}(uv)$$

for all $u, v \in K$; compare 2.8.

4.6 Theorem. *Let* Δ, Σ, $\beta_1, \beta_2, \beta_3$ *and* $x_{\beta_1}, x_{\beta_2}, x_{\beta_3}$ *be as above. Suppose that*

$$\varphi = \{\varphi_\alpha \mid \alpha \in \Phi\}$$

is a valuation of the root datum of Δ *based at* Σ *(with respect to some isomorphism* π *from* Σ *to the graph* Θ*) and that for each* $\alpha \in \Phi$*, the root group* U_α *is complete with respect to the metric* d_α *defined in 4.3(iii). Then the following hold:*

- *(i) There exists a unique affine building* Ξ *possessing an apartment* A *and a unique isomorphism* $\tilde{\pi}$ *from* A *to the graph* Γ *on the alcoves of* Φ *such that* $\Xi^\infty \cong \Delta$, $A^\infty = \Sigma$*, the isomorphism* π *from* Σ *to* Θ *is induced by* $\tilde{\pi}$ *and* φ *is as in 4.3(ii). Note that* B_n *and* C_n *are two names for the same Coxeter diagram. If* $X_n = B_n = C_n$*, then* Ξ *is of type* $\tilde{B}_n$ *or* $\tilde{C}_n$*; a simple calculation involving* φ *and* ν *reveals which one. In every other case, the Coxeter diagram of* Ξ *is* $\tilde{X}_n$*.*

(ii) *The building Ξ in (i) depends only on the equipollence class of φ (and not on the isomorphism π).*

(iii) *Let $k_i = \varphi_{\beta_i}(x_{\beta_i}(0))$ and let*

$$\nu(u) = \varphi_{\beta_i}(x_{\beta_i}(u)) - k_i$$

for all $u \in K$. Then ν is a discrete valuation of K which is independent of i, and K is complete with respect to this valuation.

4.7 Theorem. *Let Δ, Σ, $\beta_1, \beta_2, \beta_3$ and $x_{\beta_1}, x_{\beta_2}, x_{\beta_3}$ be as above. Suppose that K is complete with respect to a discrete valuation ν. Let k be an integer and let*

$$\varphi_{\beta_1}(x_{\beta_1}(u)) = \nu(u) + k$$

for all $u \in K$. Then the following hold:

(i) *The map φ_{β_1} can be extended to a valuation*

$$\varphi = \{\varphi_\alpha \mid \alpha \in \Phi\}$$

of the root datum of Δ based at Σ (with respect to some isomorphism π from Σ to Θ). Up to equipollence, the valuation φ is independent of the choice of the integer k, the roots $\beta_1, \beta_2, \beta_3$ and the maps $x_{\beta_1}, x_{\beta_2}, x_{\beta_3}$.

(iii) *There exists a unique building $\hat{\Delta}$ of type X_n containing Δ as a subbuilding (and hence Σ as an apartment) such that for each $\alpha \in \Phi$, the root group of $\hat{\Delta}$ corresponding to α is the closure of the root group U_α with respect to the metric d_α defined in 4.3(iii). The valuation φ has a canonical extension to a valuation $\hat{\varphi}$ of the root datum of $\hat{\Delta}$ based at Σ.*

(iv) *If the Coxeter diagram X_n of Δ is simply laced or if Δ is the building associated with an absolutely simple algebraic group, then $\hat{\Delta} = \Delta$.*

4.8 Definition. Let Δ be an arbitrary thick building of type X_n (with $n \geq 3$). We will say that Δ is *complete* if its defining field K is complete with respect to a discrete valuation and Δ equals the building $\hat{\Delta}$ introduced in 4.7(iii).

A field (or skew field or octonion division algebra) that is complete with respect to a discrete valuation has only one discrete valuation.

Together, the results 4.3, 4.6 and 4.7 yield the following conclusion:

4.9 Theorem. *Affine buildings of type $\tilde{X}_n$ for $n \geq 3$ are classified by complete buildings of type X_n (as defined in 4.8).*

Suppose, finally, that Ξ is a building of type $\tilde{X}_n$ for some $n \geq 3$ and that X_n is simply laced. Then every irreducible rank two residue is isomorphic to $A_2(\bar{K})$, where K the defining field of Ξ^∞ (which is complete with respect to a discrete valuation) and $\bar{K}$ is its residue field. (Something similar holds also in the non-simply laced cases.) In general there are many fields K with discrete valuation having the same residue field $\bar{K}$. We can deduce from this that affine buildings are *not* uniquely determined by their foundations, in contrast to the situation with spherical buildings.

References

1. N. Bourbaki, *Elements of Mathematics: Lie Groups and Lie Algebras*, Chapters 4–6, Springer, 1968.
2. F. Bruhat and J. Tits, Groupes réductifs sur un corps local, I: Données radicielles valuées, *Publ. Math. I.H.E.S.* **41**, 5-252.
3. J. Tits, *Buildings of Spherical Type and Finite BN-Pairs*, Lecture Notes in Mathematics **386**, Springer, 1974.
4. J. Tits, Immeubles de type affine, in: *Buildings and the Geometry of Diagrams* (Como 1984), pp. 159-190, Lecture Notes in Mathematics **1181**, Springer, 1986.
5. J. Tits and R. M. Weiss, *Moufang Polygons*, Springer, 2002.
6. R. M. Weiss, *The Structure of Spherical Buildings*, Princeton, 2003.
7. R. M. Weiss, *The Structure of Affine Buildings*, Annals of Mathematics Studies **168**, Princeton, to appear in 2008.