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Universität Münster
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Geometric Group Theory I
Exercise Sheet 7

Exercise 1.
Let $G := H_1 * H_2 * \cdots * H_n$ with $H_i \cap H_j = \{1\}$ for all $i \neq j$. Define a normal form in G as a sequence $(g_i)_{i<\ell}$ of nontrivial elements in $\bigcup_{j \leq n} H_j$ such that for all i, there is no t_i with $\{g_i, g_{i+1}\} \subseteq H_{t_i}$.

a) Let X be the set of all normal forms, prove that $f : X \to G$ which maps $(g_i)_{i<\ell}$ to $\prod_{i \in I} g_i$ is a bijection.
Hint: You can prove this by induction on n and use the normal form for $G_1 * G_2$; or you can prove it directly as in the case of $G_1 * G_2$.

b) Show that if A is an abelian subgroup of G, then either A is cyclic (namely $A = \langle g \rangle$ for some $g \in G$) or $A \leq g^{-1} H_i g$ for some $g \in G$ and $1 \leq i \leq n$.

(4 Points)

c) *(Bonus exercise, 2 bonus points)* Suppose each H_i has no element of order 2 and $g = \prod_{j<\ell} g_j \in G$ with $(g_j)_{j<\ell}$ a normal form with $\ell > 1$ such that g_0 and $g_{\ell-1}$ are not both in some H_i. Let $A := \langle g \rangle$. Show that if $h^{-1}Ah = A$ (i.e. h normalises A), then there is $d \in G$ such that both h and g are in $\langle d \rangle$.

d) *(Bonus exercise, 2 bonus points)* Suppose each H_i is infinite cyclic, show that all solvable subgroups of G are abelian.[1]

Exercise 2. Let $D_4 = \langle\, d, i \mid d^4 = 1, i^2 = 1, idi = d^{-1} \,\rangle$ be the dihedral group of order 8. Let $H \leq D_4$ be the subgroup generated by d^2 and i.

a) Show that $H \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and the partial map $d^2 \mapsto i; i \mapsto d^2$ extends to an automorphism $\varphi : H \to H$.

b) Let $G := D_4 *_H D_4$ be the amalgamated free product with respect to the inclusion of H in D_4, namely $id : H \to D_4$. Let G' be the amalgamated free product of the following two embeddings: $id : H \to D_4$ and $\varphi : H \to D_4$. Show that G and G' are not isomorphic.
Hint: Use Exercise 4 in the previous exercise sheet to show that $\mathrm{Hom}(G, \mathbb{Z}/2\mathbb{Z})$ and $\mathrm{Hom}(G', \mathbb{Z}/2\mathbb{Z})$ are not isomorphic.

(4 Points)

[1] Note that the infinite dihedral group $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ is solvable.

Exercise 3.

Let H_i be a subgroup of G_i for $i \in \{1, 2\}$. Suppose A is a common subgroup of G_1 and G_2 such that $H_1 \cap A = H_2 \cap A =: B$.

a) Show that the homomorphism $\phi : H_1 *_B H_2 \to G_1 *_A G_2$ induced by the embeddings $H_i \hookrightarrow G_i$ is injective.

b) Deduce from a) that the subgroup generated by $H_1 \cup H_2$ in $G_1 *_A G_2$ is isomorphic to $H_1 *_B H_2$.

(4 Points)

Exercise 4.

a) Suppose F is a subgroup of $G := A * B$ such that $F \cap g^{-1}Ag = \{1\} = F \cap g^{-1}Bg$ for all $g \in G$. Show that F is a free group.
Hint: Consider a suitable action on a tree.

b) Show that the kernel of the canonical homomorphism $\varphi : A * B \to A \times B$ is a free group.

(4 Points)

*Submission by **Wednesday** morning 11:00, 30.11.2022, in Briefkasten 161.*
The exercise sheets should be solved and submitted in pairs.
Tutorial: Fridays 12:00-14:00, in room SR1d.
If you have questions about the problem sheet, please write to Tingxiang: tingxiangzou@gmail.com.