Dr. M. Bays
Dr. T. Zou

Universität Münster
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Geometric Group Theory I
Exercise Sheet 9

Exercise 1.

Let $\phi_1 : \mathbb{Z} \to \mathbb{Z}$ be the homomorphism mapping z to $3z$, and let ϕ_2 be the identity map on $\mathbb{Z}$. Describe the HNN-extension $\mathrm{HNN}(\mathbb{Z}, \mathbb{Z}, \phi_1, \phi_2)$ as a semi-direct product $A \rtimes \mathbb{Z}$ where $A \leq \mathbb{Q}$.
Hint: Use the presentation of HNN extensions and the presentation of a semi-direct product we saw on Exercise Sheet 4, Exercise 4.

(4 Points)

Exercise 2.

Let $A \leq G$, $B \leq H$ and $\phi \colon A \to B$ be an isomorphism. Show that the homomorphism from $G *_{A=B} H$ to the HNN-extension $\langle G * H, t \mid t^{-1}at = \phi(a), a \in A\rangle$ induced by $g \mapsto t^{-1}gt$, $h \to h$, $g \in G$, $h \in H$, is an embedding.

(4 Points)

Exercise 3.

a) Let $(\mathcal{G}, T)$ be a graph of groups, where T is a tree. Let $\mathcal{D}(\mathcal{G}, T)$ be the diagram of groups consisting groups $\{G_x : x \in T^0\} \cup \{G_e : e \in T^+\}$ and embeddings $\phi_e : G_e \to G_{\omega(e)}$ and $\phi_{\bar{e}} : G_e \to G_{\alpha(e)}$, for $e \in T^+$. Show that $\pi_1(\mathcal{G}, T, T)$ is isomorphic to the colimit of $\mathcal{D}(\mathcal{G}, T)$.
Hint: Use the presentations.

b) Let $(\mathcal{G}, Y)$ be a graph of groups and T be a maximal subtree of Y. Let $(\mathcal{G}|_T, T)$ be the restriction of $(\mathcal{G}, Y)$ on T. Let $G' := \pi_1(\mathcal{G}|_T, T, T)$. For any edge $e \in Y^1 \setminus T^1$, denote $\phi'_e : G_e \to G'$ the embedding $\eta_{G_{\omega(e)}} \circ \phi_e$ where $\phi_e : G_e \to G_{\omega(e)}$ is the embedding given by $(\mathcal{G}, T)$ and $\eta_{G_{\omega(e)}} : G_{\omega(e)} \to G'$ is the natural embedding. Recall that Y/T is the graph obtained by contracting T in Y. Let $(\mathcal{G}', Y/T)$ be the graph of groups consisting G' as the vertex group (note that Y/T has only one vertex $x := T/_\sim$), and for each $e \in Y^1 \setminus T^1$, the embedding $\phi'_e : G_e \to G'$. Prove that

$$\pi_1(\mathcal{G}, Y, T) \cong \pi_1(\mathcal{G}', Y/T, x).[1]$$

(4 Points)

[1] Combining a) and b) we see that the fundamental group of any graph of groups can be seen as an iterated HNN-extensions of some colimits of groups.

Exercise 4.

Let X be a connected non-empty graph and $x \in X^0$, and let $q : \widehat{X} \to X$ be a universal cover.

a) Show that there is an action of $\pi_1(X, x)$ on $\widehat{X}$ which respects q, i.e. $q(g * \hat{x}) = q(\hat{x})$ for all $\hat{x} \in \widehat{X}^0$ and $q(g * \hat{e}) = q(\hat{e})$ for all $\hat{e} \in \widehat{X}^1$.
Hint: Use the construction of the universal cover.

b) Let $H \leq \pi_1(X, x)$ be a subgroup, then H acts on $\widehat{X}$ by item a). Let X_H be the quotient graph ${}_H\backslash^{\widehat{X}}$ and $x_H := H\hat{x} \in (X_H)^0$, where $q(\hat{x}) = x$.

Show that q induces a natural morphism of graphs $q_H : X_H \to X$ such that $q_H(x_H) = x$. Let $(q_H)_* : \pi_1(X_H, x_H) \to \pi_1(X, x)$ be the pushforward of q_H as defined in Exercise 4 in Exercise Sheet 5. Show that $(q_H)_*(\pi_1(X_H, x_H)) = H$. Conclude that $\pi_1(X_H, x_H) \cong H$.
Hint: Use Exercise 4 b) in Exercise Sheet 5.

Submission by ***Wednesday*** *morning 11:00, 14.12.2022, in Briefkasten 161.*
The exercise sheets should be solved and submitted in pairs.
Tutorial: Fridays 12:00-14:00, in room SR1d.
If you have questions about the problem sheet, please write to Tingxiang: tingxiangzou@gmail.com.