ALMOST STRONGLY MINIMAL

GENERALIZED n-GONS

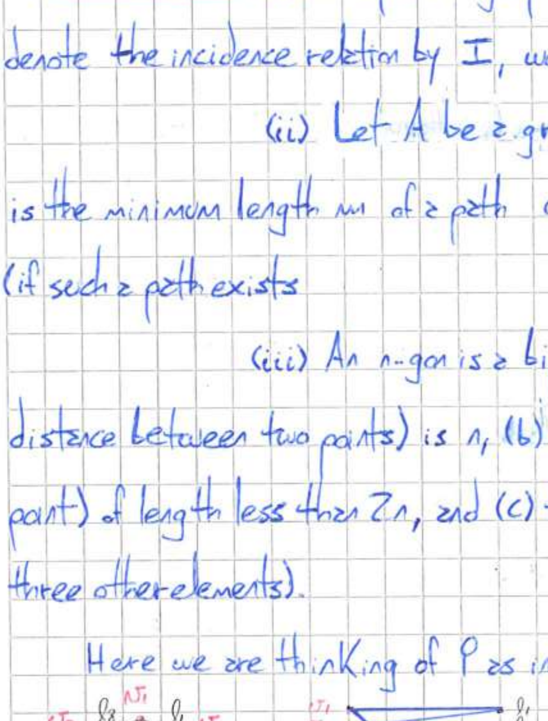
1. Introduction: Bipartite Graphs, n-Gons

Definition 1.1: (i) A bipartite graph is a graph equiped with a predicate P in which, if we denote the incidence relation by I, we add an axiom $\forall x \forall y \; I(x,y) \Rightarrow ([P(x) \wedge \neg P(y)] \vee [\neg P(x) \wedge P(y)])$.

(ii) Let A be a graph (bipartite or not), the distance between two points $a, b \in A$ is the minimum length m of a path $a = a_0, a_1, \ldots, a_m = b$ such that $I(a_i, a_{i+1})$ $i = 0, \ldots, m-1$ (if such a path exists

(iii) An n-gon is a bipartite graph such that (a) the diameter of the graph (i.e., the maximum distance between two points) is n, (b) there are no simple cycles (i.e., cycles without repetition of a point) of length less than $2n$, and (c) the graph is thick (i.e., any element is incident with at least three other elements).

Here we are thinking of P as inducing a colouring on the graph in such a way that the elements in P are the points of the polygon and the elements not in P are the lines.



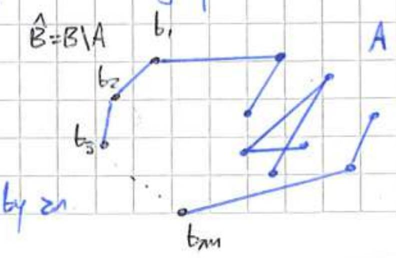
$P(x)$ iff $\bullet$; $I(x,y)$ iff $\overset{x}{\bullet}\!\!-\!\!\overset{y}{\bullet}$

Remark: From now on we fix $n \geq 3$.

Definition 1.2: (i) For any finite graph A we define the weighted Euler characteristic as $\delta(A) = (n-1)|A| - (n-2)e(A)$, where $|A|$ denotes the number of elements in A and $e(A)$ the number of edges.

(ii) For A, B finite disjoint subgraphs of a graph M we define $r(A,B)$ as the number of edges between A and B

(iii) For A, B subgraphs of a graph M we denote by AB the subgraph of M with vertices in $A \cup B$ and incidence relation induced from M



Definition 1.3: If B is obtained from A by attaching a string of m elements from one element of A to another then B is an extension by an

arc of length m.

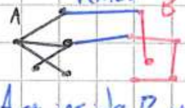
Definition 1.4: Let A, B be bipartite graphs with respect to P, $A \subseteq B$, then A is strong in B, written $A \leq B$ if $P(A) = P(B) \cap A$, A finite, and for every finite A' such that $A \subseteq A' \subseteq B$ we have $\delta(A) \leq \delta(A')$.

Remark: '$\leq$' is a transitive relation on finite subgraphs of a given graph.

Lemma 1.5: (i) If A, B are finite disjoint subgraphs of a graph M, then $\delta(AB) = \delta(A) + \delta(B) - (n-2) \cdot r(A,B)$

(ii) If B is obtained by attaching an arc of length m, then $\delta(B) = \delta(A) + m - n + 2$

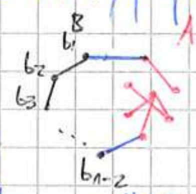
(iii) If $B_0 \leq B_1$ and $C \subseteq B_1$, then $C \cap B_0 \leq C$.

Proof: (i) 

It is clear that every point occurs either in A or in B, and the only edges not occurring inside A or inside B are the ones in $r(A,B)$.

(ii) It is a straight calculation from the fact that we are adding m elements and $m+1$ edges, $\delta(B) = \delta(A) + (n-1)m - (n-2)(m+1) = \delta(A) + (n-1)m - (n-2)m - n + 2 = \delta(A) + m - n + 2$

Definition 1.6: Let A and B be finite subgraphs of M, $i = 0, 1$, we say B is i-simple over A if A and B are disjoint, $\delta(AB) - \delta(A) = i$ and for every proper non-empty subset $C \subseteq B$, $\delta(AC) - \delta(A) > i$. We say that B is minimally 0-simple over A if it is 0-simple over A and it is not 0-simple over any proper non-empty subset of A.



Examples: . Expansion by an arc of length $n-2$ is a 0-simple extension (in fact, δ is defined in this way in order to allow this "free extensions" to be 0-simple). It is only minimally 0-simple if A consists of only the two points attached to B.

. Expansion by a single element set B attached to a single element in A is a 1-simple expansion.

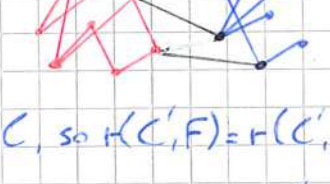
Lemma 1.7: (i) Let $B_0 \leq B_1$, $C \subseteq B_1$ 0-simple over $F \subseteq B_0$. Then, $C \subseteq B_0$ or $C \subseteq B_1 \setminus B_0$. In the latter case, C is 0-simple over B_0 and $r(C, B_0 \setminus F) = 0$. So, if C and $C' \subseteq B_1 \setminus B_0$ are isomorphic over F, they are isomorphic over B_0.

(ii) If C is 0-simple over B_0, then it is 0-simple over any $F \subseteq B_0$ with $r(C, B_0) = r(C, F)$. Thus there is a unique $F' \subseteq B_0$ with C minimally 0-simple over F', namely the set of elements of B_0 incident with elements of C.

(iii) If B is a connected graph with no cycles, $\delta(B) = (n-2) + |B|$

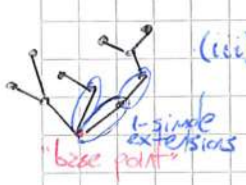
Proof: (i) See references in the paper.

(ii) Suppose C was not 0-simple over some $F \subseteq B_0$ with $r(C, B_0) = r(C, F)$. That could happen for two reasons. First, $\delta(CF) - \delta(F) \neq 0$, but that would contradict $\delta(CF) - \delta(F) =$ $= \delta(C) - (n-2)r(C,F) = \delta(C) - (n-2)r(C,B_0) = \delta(CB_0) - \delta(B_0) = 0$, this last equality given by the fact that C is 0-simple over B_0. The second reason could be that for $C' \subsetneq C$ a proper non-empty subset we had $\delta(C'F) - \delta(F) \leq 0$, but similarly that would contradict the 0-simplicity of C over B_0 since



$0 \geq \delta(C'F) - \delta(F) = \delta(C') - (n-2)r(C',F)$, and if F has the same number of edges with C than B_0, then this clearly holds also for any subset C' of C, so $r(C',F) = r(C',B_0)$, $0 \geq \delta(C') - (n-2)r(C',B_0) = \delta(C'B_0) - \delta(B_0)$, and we get the contradiction.

For the second part of the assertion, let F' be the set of elements of B_0 incident with elements of C. Clearly F' is 0-simple and it is minimal because for any $G' \subseteq B_0$ such that $F' \nsubseteq G'$, $r(G', C) < r(F', C)$, so $0 = \delta(C) - r(F', C) < \delta(C) - r(G', C) = \delta(G'C) - \delta(G')$, and that includes every proper non-empty subset of F'. So, the subsets of B_0 such that C is 0-simple over them is exactly the set of subsets that contain F', and no other of these subsets than F' can be minimally 0-simple since all of them contain F' that is 0-simple itself.



(iii) If B is a connected graph with no cycles, it can be seen as -1 1-simple extensions of a graph consisting of a single point (which would have δ-value of $n-1$), so $\delta(B) = (n-1) +$ $+ (|B| - 1) = n - 2 + |B|$

2. The Construction

This section will be devoted to the proof of the following theorem:

Theorem 2.1: For all $n \geq 3$ there exist generalized n-gons for which the automorphism group acts transitively on the set of ordinary $(n+1)$-gons contained in it.

The construction proceeds by free extensions and amalgamation of graphs from a class K of finite bipartite graphs partially ordered by the strong substructure relation '$\leq$'.

Definition 2.2: Let $(K, \leq)$ be a collection of finite relational structures closed under substructures (i.e., under subsets with the restricted relations)

(i) We say that $(K, \leq)$ has the amalgamation property if for $A, B, C \in K$ and embeddings $f_0 : A \to B$, $g_0 : A \to C$ with $f_0(A) \leq B$, $g_0(A) \leq C$, there exists $D \in K$ and embeddings $f_1 : B \to D$, $g_1 : C \to D$ such that $f_1(B) \leq D$, $g_1(C) \leq D$ and $f_1 \circ f_0 = g_1 \circ g_0$.

(ii) We say that the class $(K, \leq)$ has the joint embedding property if for any $A, B \in K$ there exists some $C \in K$ and embeddings $f : A \to C$, $g : B \to C$ with $f(A) \leq C$, $g(B) \leq C$.

(iii) A countable structure M is called a $(K, \leq)$-homogeneous universal model if

(H_1) If $A \in K$ is finite, then there exists an embedding $f : A \to M$ such that $f(A) \leq M$.

(H_2) If $A \subseteq M$ is finite, $A \in K$

(H_3) If $A \leq M$, $B \leq M$ have an isomorphism $f : A \to B$ then there is an automorphism of M extending f.

Remark 2.3: If there is a structure in $(K, \leq)$, call it F, such that for any $A, B \in K$ there are embeddings $g_0 : F \to A$, $g_1 : F \to B$ with $g_0(F) \leq A$, $g_1(F) \leq B$, then the amalgamation property implies the joint embedding property. However, such a structure need not exist!

The Key to the construction is the following fact explored by Shelah:

Assertion 2.4: If a collection of finite relational structures $(K, \leq)$ partially ordered by '$\leq$' is closed under substructures and has both the amalgamation property and the joint embedding property, then there exists a countable $(K, \leq)$-homogeneous universal model.

We now define our class of structures

Definition 2.5: Let K be the collection of finite graphs A, bipartite with respect to P, with the following properties:

(K_1) A contains no ordinary k-gon for $k < n$.

(K_2) If $B \subseteq A$ contains an ordinary k-gon for $k > n$, then $\delta(B) \geq 2n+2$.

Remark 2.6: The purpose of condition $(K1)$ is clear: if our $(K, \leq)$-homogeneous universal structure is going to be a generalized n-gon, then it can not contain any ordinary k-gons for $k < n$.

• Condition (K2) has also a simple purpose, perhaps not as straightforward as (K1). First notice that any two ordered ordinary $(n+1)$-gons are isomorphic (   ). If we can be sure that any ordinary $(n+1)$-gon in the $(K,\leq)$-homogeneous universal model M is a strong substructure, then (H_3) ensures us that the automorphism group of the generalized n-gon acts transitively over the set of ordered ordinary $(n+1)$-gons. Since for an ordinary $(n+1)$-gon C we have $\delta(C) = (n-1)|C| - (n-2)e(C) = |C| = 2n+2$, this is precisely what (K2) accomplishes.

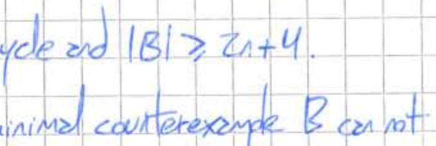
In addition, condition (K2) provides a characterization of 1-simple extensions

Lemma 2.7: If $A \subseteq B \in K$, then $\hat{B} = B \setminus A$ is 1-simple over A if and only if $B = Ab$, where b is incident with a unique element in A.

Proof: See the references in the paper.

Lemma 2.8: Let B be any bipartite graph satisfying condition (K1) with $|B| \geq n+1$. Then if $\delta(B) < 2n$ either B is a connected tree with $|B| = n+1$ or B contains a cycle and $|B| \geq 2n+4$.

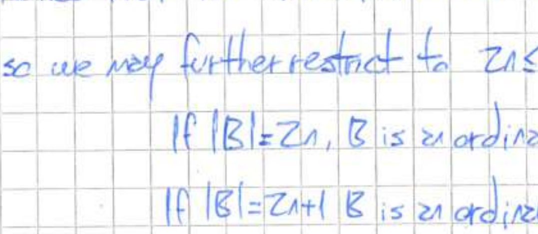
Proof: It can be seen by checking a few cases that a minimal counterexample B can not be disconnected. Moreover, if B is a connected tree, then by Lemma 1.7 (iii), if B has $n+2$ elements or more, $\delta(B) \geq 2n$. So we may take this counterexample as a connected graph containing a cycle. Notice that for $n+1 \leq |B| < 2n+4$ the only possibilities for cycles are ordinary n-gons and $(n+1)$-gons, so we may further restrict to $2n \leq |B| \leq 2n+3$.



If $|B| = 2n$, B is an ordinary n-gon and $\delta(B) = 2n$.

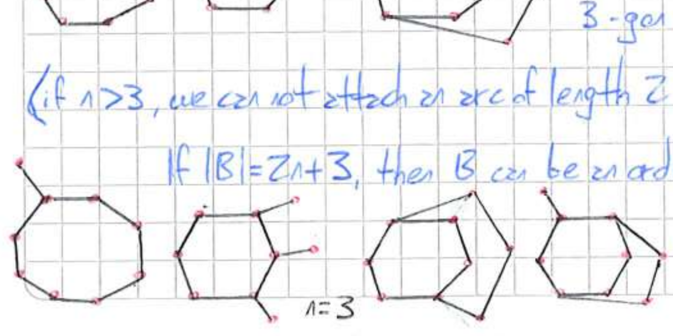
If $|B| = 2n+1$ B is an ordinary n-gon with a 1-simple expansion attached, and $\delta(B) = 2n+1$

If $|B| = 2n+2$, B can be an ordinary $(n+1)$-gon, an ordinary n-gon with two 1-simple expansi



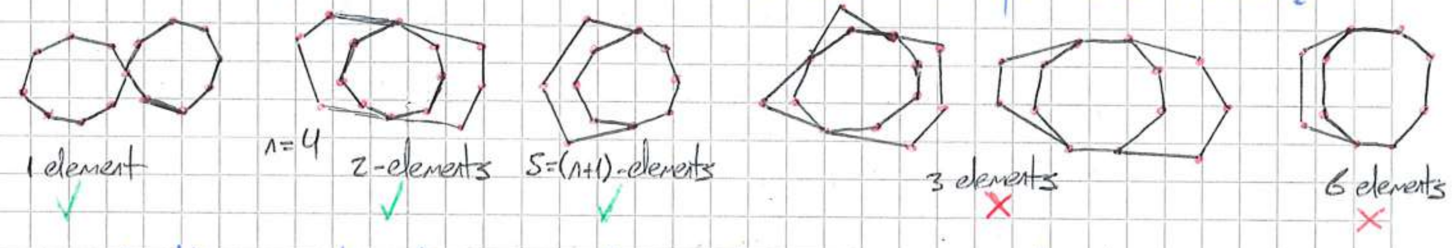
(in both cases $\delta(B) = 2n+2$), or, if $n=3$, it can be an ordinary 3-gon with an arc of length $2 = n-1$ attached, and $\delta(B) = 7 = 2n+1$ (if $n > 3$, we can not attach an arc of length 2 to an ordinary n-gon without violating (K1)).

If $|B| = 2n+3$, then B can be an ordinary $(n+1)$-gon or n-gon expanded by one or three 1-simple



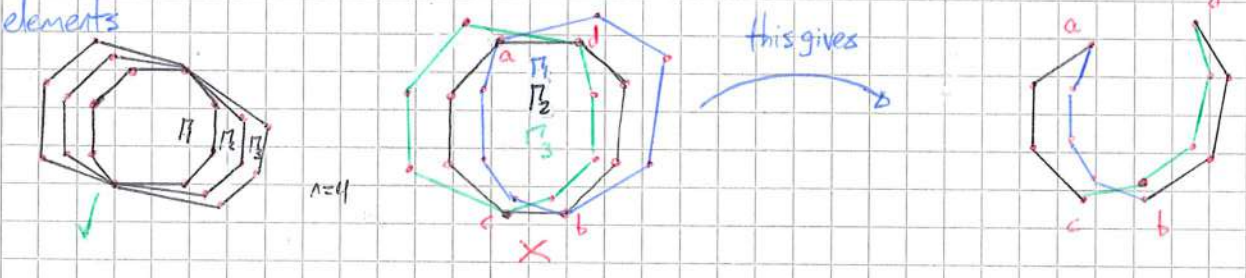
extensions (and $\delta(B) = 2n+3$). For $n \leq 4$, B can be an ordinary n-gon expanded by an arc of length 3, and $\delta(B) \geq n+1$, and for $n=3$ it can be an ordinary 3-gon expanded by an arc of length 2 and a 1-simple expansion, and $\delta(B) = 8 = 2n+2$.

Lemma 2.9: Suppose A is a graph that does not contain any simple K-cycles for $K \neq 2n$. If $\Gamma_1, \Gamma_2 \subseteq A$ are two distinct ordinary n-gons, then they must intersect in $0, 1, 2$ or $n+1$ elements. If $\Gamma_1 \cap \Gamma_2$ contains at least two elements, then it contains elements that have distance n with respect to both Γ_1 and Γ_2



Proof: No new simple cycles can appear from an intersection in 0 or 1 elements. If Γ_1 and Γ_2 intersect in two elements, they must have distance n for the new simple cycles to be $2n$-cycles. If $n>3$, then at least two of them must have distance less than n, and this generates a simple K-cycle for $K<2n$ unless Γ_1 and Γ_2 intersect on a string of $n+1$ elements. If $K > n+1$, then again new simple K-cycles for $K=2n$ appear unless $\Gamma_1 = \Gamma_2$.

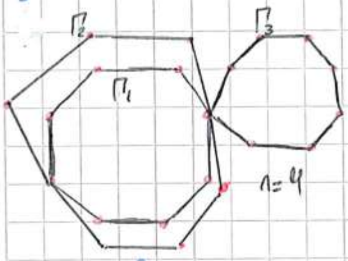
Lemma 2.10: Suppose A is a graph that does not contain simple K-cycles for $K \neq 2n$. If $\Gamma_1, \Gamma_2, \Gamma_3 \subseteq A$ are distinct n-gons such that $\Gamma_1 \cap \Gamma_2$ and $\Gamma_2 \cap \Gamma_3$ both contain at least two elements, then also $\bigcap_{i=1,2,3} \Gamma_i$ contains at least two elements



Proof: Let Γ_1, Γ_2 intersect in a and b (where they have distance n); Γ_2 and Γ_3 intersect in c and d, again at distance n. Then we obtain a K-gon for $K > n$ by moving from a to b inside Γ_1, then from b to d inside Γ_2, then from d to c in Γ_3, and finally back to a inside Γ_2. This path has no repetitions unless $\{c,d\} = \{a,b\}$

Remark 2.11: Since having two or more elements in the intersection is, by virtue of Lemma 2.10, a transitive relation between ordinary n-gons of a graph A not containing simple cycles for $K \neq 2n$, it is an equivalence relation '$\sim$'. We call such an equivalence class a stack of n-gons, and the two elements on the intersection of the stack glueing points. Any two elements on a stack have distance at most n and thus are contained in some ordinary n-gon belonging to the stack.

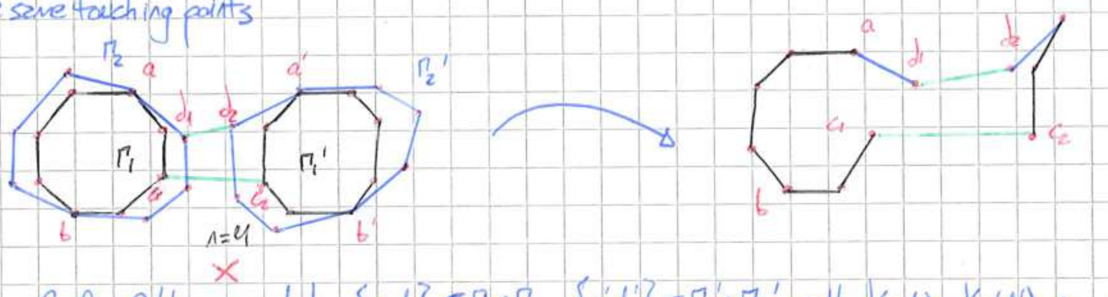
Lemma 2.12: Suppose that A is a graph that does not contain any simple K-cycle for $K \neq 2n$. Let $\Gamma_1, \Gamma_2 \subseteq A$ be distinct n-gons in the same stack, and suppose $\Gamma_3 \subseteq A$ intersects both Γ_1 and Γ_3. Then either Γ_3 is in the same stack as Γ_1 and Γ_2 or it intersects Γ_1 and Γ_2 in a common point



Proof: Suppose $\Gamma_3 \cap \Gamma_2 = \{a\}$, $\Gamma_1 \cap \Gamma_3 = \{b\}$, $a \neq b$, then by Remark 2.11 there is an ordinary n-gon Γ' in the same stack as Γ_1 and Γ_2 containing $\{a, b\}$, so $\Gamma_3 \sim \Gamma'$ and Γ_3 is in the stack.

Definition 2.13: We call two n-gons $\Gamma_1, \Gamma_2 \subseteq A$ (with A as in the previous Lemma) neighbours if they intersect in at most one element and if there is an element in Γ_1 incident with an element of Γ_2. These elements are called touching points.

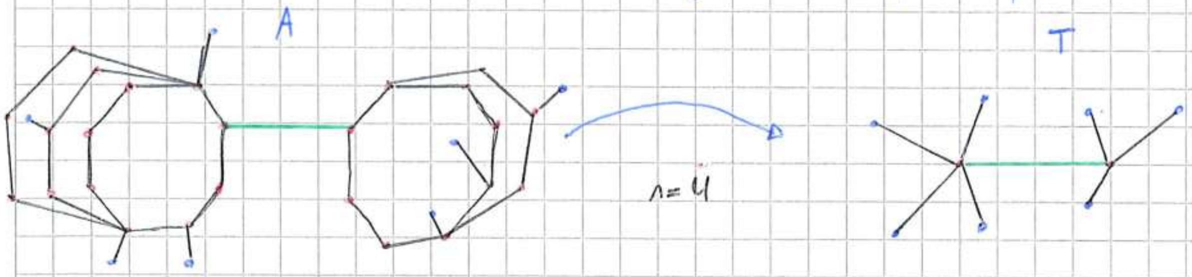
Lemma 2.14: Let A be a graph that does not contain any simple K-cycles for $K \neq 2n$. Suppose that, for ordinary n-gons $\Gamma_1, \Gamma_2, \Gamma_1', \Gamma_2' \subseteq A$ we have $\Gamma_1 \sim \Gamma_2$, $\Gamma_1' \sim \Gamma_2'$. If Γ_1 and Γ_1' are neighbours, and Γ_2 and Γ_2' too, then they have the same touching points



Proof: Otherwise, let $\{a, b\} \subseteq \Gamma_1 \cap \Gamma_2$, $\{a', b'\} \subseteq \Gamma_1' \cap \Gamma_2'$ with $d(a,b) = d(a', b') = n$, $c_1 \in \Gamma_1$, $c_2 \in \Gamma_1'$, $d_1 \in \Gamma_2$, $d_2 \in \Gamma_2'$ be the touching points contradicting the Lemma. Then we obtain a simple cycle for $K > 2n$ by going from b to a in Γ_1, from a to d_1 in Γ_2, then to d_2, then from d_2 to a' in Γ_2', from a' to c_2 in Γ_1', from c_2 to c_1 and then back to b in Γ_1.

Remark 2.15: In virtue of Lemma 2.14, if two stacks of n-gons contain neighbouring n-gons, then there is a unique touching point. We call the stacks neighbours in that case.

Theorem 2.16: Suppose that A is a graph, connected, and it does not contain any simple K-cycles for $K \neq 2n$. Then A can be described as a tree in which some single nodes are replaced by stacks of n-gons.

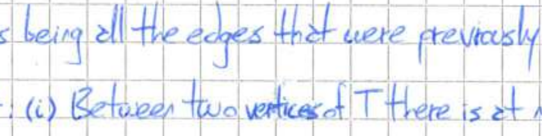


Proof: Let A be such a graph. If A contains no $2n$-cycles, then A is a tree and we are done.

If A contains $2n$-cycles, we form a new graph T by replacing each stack of A by a single node, with edges being all the edges that were previously connected to the stack. To show that T is a tree we must show that: (i) Between two vertices of T there is at most one edge. (ii) T does not contain any simple cycles.

To prove (i), let a and b be vertices of T. There are three cases to consider. If both a and b were vertices in A, then there is trivially at most one edge between them. If a comes from a stack $\tilde{a}$ in A and b from a single vertex, then b is incident with a unique element of $\tilde{a}$, this is because any two elements of $\tilde{a}$ are at distance at most n and thus b being incident with more than one of them would yield a simple K-cycle in A with $K \leq n+2 < 2n$, so there is also a unique edge between a and b in T. If both a and b come from stacks $\hat{a}$ and $\tilde{b}$, then by Lemma 2.14 there is a unique edge between them and thus between a and b.

To prove (ii), let's suppose that T contains a simple cycle γ, then there must be a simple



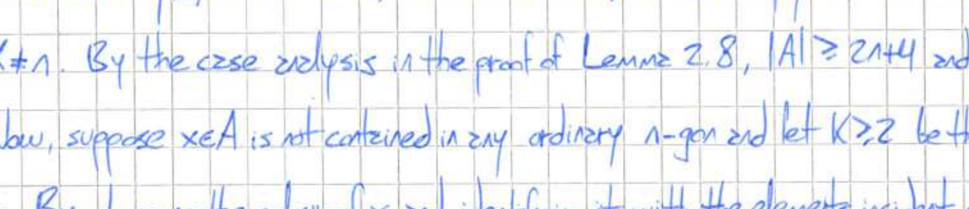
cycle Γ in A yielding γ after the stacks in A are replaced by vertices in T. However, by assumption, Γ is a cycle of length $2n$ and was replaced by a single node in T, proving that there are no simple cycles in T and thus it is a tree.

Remark 2.17: A stack of n-gons can be obtained by attaching succesively arcs of length $n-1$ to an ordinary n-gon, and thus has δ value of at least $2n$.

Lemma 2.18: Let $A \in H$, $|A| \geq n+2$. Then $\delta(A) \geq 2n$. Moreover, $\delta(A) \geq 2n+2$ unless A has at most $n+3$ elements, or A is an ordinary n-gon with either a single arc of length $n-1$ or a single element possibly attached.

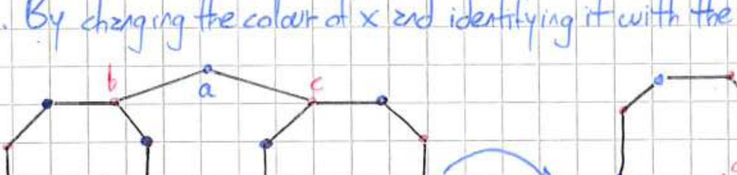
Proof: By Lemma 2.8, if $\delta(A) < 2n$, $|A| \geq 2n+4$, we can consider the case in which A is connected (since otherwise we can further reduce the δ value by connecting every connected component); if A contains no cycles, by Lemma 1.7 (iii) $\delta(A) = n-2+|A| \geq 3n+2$; so we can further concentrate in the case where A contains a cycle. If A is a connected graph with cycles, $A \in K$, then the cycles must have length $\geq 2n$. If A also contains no ordinary k-gons for $k > n$, then we are in the situation of Theorem 2.16, and Remark 2.17 together with Theorem 2.16 yield $\delta(A) \geq 2n$. If A contains an ordinary k-gon for $k > n$, then condition (K2) yields $\delta(A) \geq 2n+2$, so we are done with the first part.

For the second part, we may assume that any element in A (with A a minimal counterexample) is incident with at least two other elements (otherwise we remove this element and get an even smaller counterexample). Also, since A is a counterexample, $\delta(A) \leq 2n+1$, so conditions (K1) and (K2) imply that it can not contain any ordinary k-gon for $k \neq n$. By the case analysis in the proof of Lemma 2.8, $|A| \geq 2n+4$ and that it is of course connected. Now, suppose $x \in A$ is not contained in any ordinary n-gon and let $k \geq 2$ be the number of elements incident with x. By changing the colour of x and identifying it with the elements incident with x, we remove k elements



and k instances of incidence, thus also reducing δ and getting a smaller counterexample. So we can assume that every element of A lies in an ordinary n-gon.

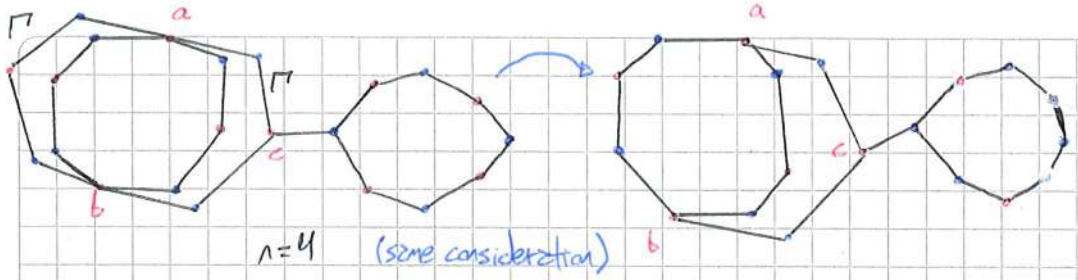
By Theorem 2.16 A can be described as a tree of stacks, and since A is finite there are finitely many stacks. At least one of them has only one neighbour, since otherwise we would have a cycle in T. Let $S \subseteq A$ be a stack with only one neighbour. If the stack is a sole ordinary n-gon, by removing it and leaving only the touching point we reduce the δ value in $(n-1)(2n-1)-(n-2)2n = 2n^2-3n+1-2n^2+2n = n+1$ (since we



(just exemplifying the procedure, not the case of the proof)

remove $2n-1$ vertices and $2n$ edges), so the remaining graph has δ value at most n, it can not have thus any cycles, and thus it must consist of at most $n+1$ elements. Therefore, A must have been an n-gon with an arc of length $n-1$ attached.

Finally, if S contains at least two n-gons, let a and b be the glueing points and c be the touching point, $\Gamma \subseteq S$ an ordinary n-gon containing c. Then, one path from a to b in Γ does not contain c, and we can get a smaller counterexample by removing this path (since δ again decreases). Similarly, again A must be



Corollary 2.19: If $A \in K$, then for any non-empty graph $B \subseteq A$ we have $\delta(B) \geq n-1$, and if $|B| \geq n+2$, then $\delta(B) \geq 2n$

Proof: Since K is closed under substructures, $B \in K$, so the second part follows directly from Lemma 2.18. For the first part, if B contained a cycle, by (K1) $|B| \geq 2n$ so we are back in the second part. If B was disconnected, we could further reduce the δ value by connecting the connected components, so we may as well assume B is a connected graph with no cycles, so we are in condition to apply Lemma 1.7(iii), $\delta(B) = n-2+|B| \geq n-1$.

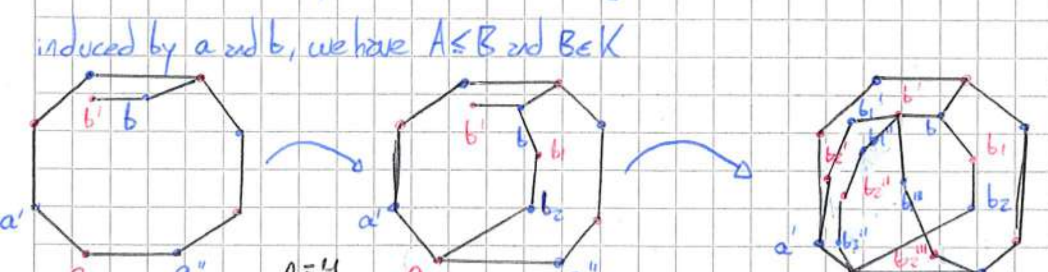
Remark 2.20: (i) Since for a single point a we have $\delta(\{a\}) = n-1$, then by the first part of Corollary 2.19 singletons are strong in every $A \in K$ non-empty.

(ii) Since an ordinary n-gon Γ has $\delta(\Gamma) = 2n$, $|\Gamma| = 2n > n+2$, ordinary n-gons are strong in every $A \in K$ containing one (also by Corollary 2.19)

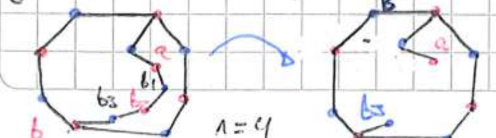
(iii) Since $\delta(\emptyset) = 0$, the empty set is strong in every $A \in K$. We thus satisfy the conditions of Remark 2.3 and we only need to verify the amalgamation property to get the joint embedding property.

We need the following Lemma to see that K is closed under certain important free extensions.

Lemma 2.21: Let $A \in K$, $a, b \in A$ be points that are not connected by a path of length at most n. Consider the graph B obtained from A by adding an arc of length K between a and b, that is, vertices $\{b_1, \dots, b_K\}$ and edges $(a, b_1), (b_1, b_2), \dots, (b_K, b)$ where $K = n-1$ if a and b have the same colouring with respect to P and n is even or if they have different colouring and n is odd, $K = n-2$ otherwise. Then with the colouring of $\{b_1, \dots, b_K\}$ induced by a and b, we have $A \leq B$ and $B \in K$



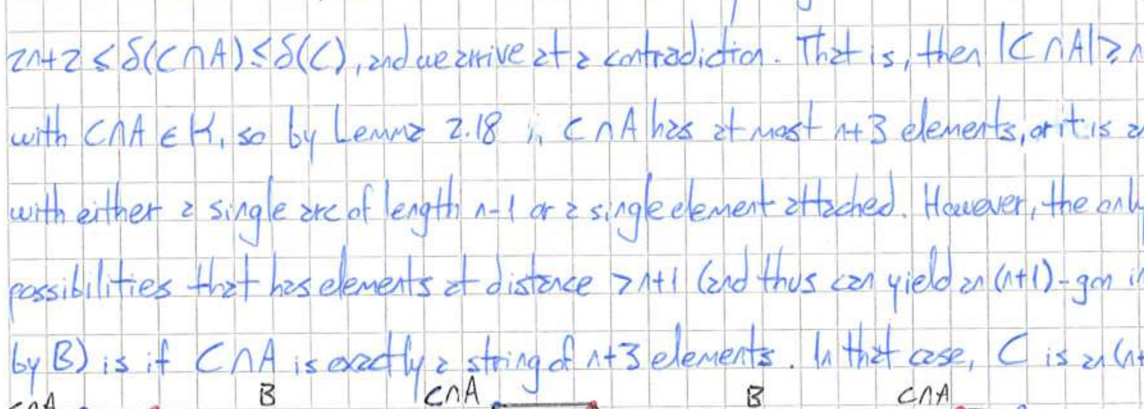
Proof: In the case $K = n-1$ we have $\delta(B) = \delta(A)+1$ and for $K = n-2$ $\delta(B) = \delta(A)$ (both by Lemma 1.5(ii)). Also, for any B' such that $A \subseteq B' \subsetneq B$, we must obtain B' from B by removing j vertices and at least $j+1$ edges, so $\delta(B') = \delta(B) + (j+1)(n-2) - j(n-1)$



$\delta(B') = \delta(A)+1+\cancel{n_j}-2j+n-2-\cancel{n_j}+j = \delta(A)+(n-1)-j$, in any case, $n-1 \geq k > j$, so $\delta(B') > \delta(A)$, $A \leq B$.

Now, for condition (K1), if n is even and a, b have the same colour and $d(a,b) > n$, we generate a different path of length $\geq n+1$ between a and b, and we can not be closing any simple cycle of length $< 2n$. If n is even and a, b have different colour with $d(a,b) > n$, then $d(a,b) \geq n+1$, and again, adding $n-2$ elements between a and b can not close a simple cycle of length $< 2n$. The case with n odd can be treated similarly, so B satisfies condition (K1) in any case

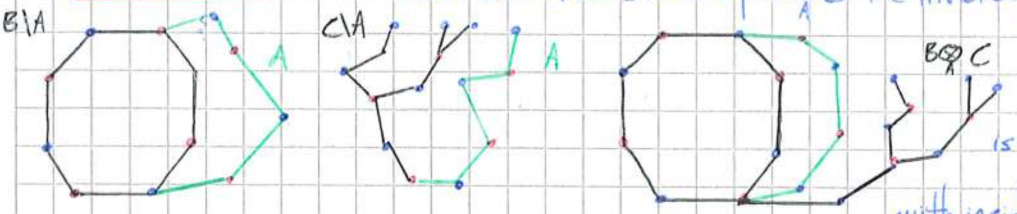
In order to check condition (K2), let $C \subseteq B$ contain an ordinary k-gon with $k > n$ and $\delta(C) < 2n+2$. Then, C must have at least $2n+3$ elements (otherwise it would just be an ordinary $(n+1)$-gon with $\delta(C) = 2n+2$). Since $|B \setminus A| \leq n-1$, $|C \cap A| \geq n+2$. Since K is closed under finite substructures, $C \cap A \in K$ and thus by Lemma 2.18, $\delta(C \cap A) \geq 2n$. Also, by Lemma 1.5 (iii) we have $C \cap A \leq C$. So, if $C \cap A$ contains an ordinary k-gon with $k > n$, we have by virtue of condition (K2) $2n+2 \leq \delta(C \cap A) \leq \delta(C)$, and we arrive at a contradiction. That is, then $|C \cap A| \geq n+2$ and $\delta(C \cap A) < 2n+2$ with $C \cap A \in K$, so by Lemma 2.18 , $C \cap A$ has at most $n+3$ elements, or it is an ordinary n-gon with either a single arc of length $n-1$ or a single element attached. However, the only of these three possibilities that has elements at distance $> n+1$ (and thus can yield an $(n+1)$-gon in C when extended by B) is if $C \cap A$ is exactly a string of $n+3$ elements. In that case, C is an $(n+1)$-gon, $\delta(C) = 2n+2$



and we arrive at a contradiction again. So B satisfies (K2) and $B \in K$.

In addition to the free extensions within K described by the previous Lemma, we need the following case of an amalgamation to prove that K in fact satisfies the amalgamation property

Definition 2.22: If $A \le B$ and $B \le C$, we denote by $B \otimes_A C$ the trivial amalgamation of B and C over A, obtained

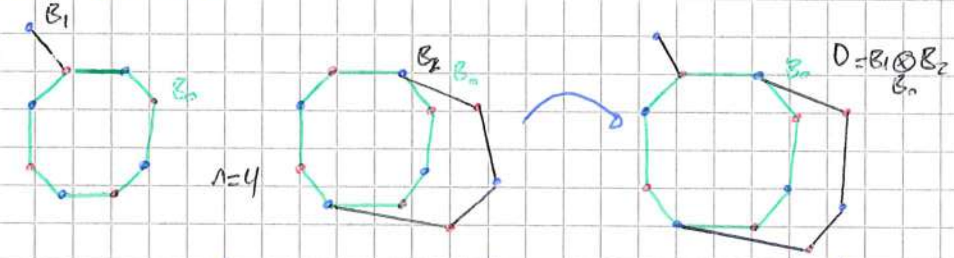


as the graph whose set of vertices is the disjoint union $(B\setminus A)\sqcup(C\setminus A)\sqcup A$ with incidence relation and colouring induced by A.

Remark 2.23: If A is non-empty, the colouring of B and C will be consistent with that of A, and thus $B\otimes_A C$ will be bipartite with respect to P. If A is empty, $B\otimes_A C$ is a disjoint union and thus again bipartite. So, colouring is never a problem for a trivial amalgamation.

In order to prove that there is a $(K,\le)$-homogeneous universal model with respect to the partial order relation of being a strong substructure, we need the following two Lemmas.

Lemma 2.24: If $B_1, B_2 \in K$, $B_0 \le B_1, B_2$ and $\hat{B}_1 = B_1 \setminus B_0$ is 1-simple over B_0, then $D = B_1 \otimes_{B_0} B_2 \in K$, with $B_1, B_2 \le D$.

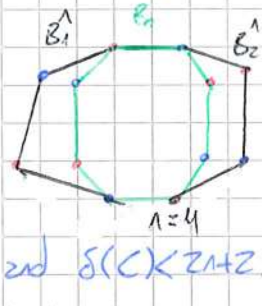


Proof: By Lemma 2.7, we obtain B_1 from B_0 by attaching a single new element b to some $a \in B_0$. So also D is obtain from B_2 by attaching b to $a \in B_0$. So it is immediate that $B_2 \le D$. Also, since $B_0 \le B_2$, for any B' such that $B_0 \subseteq B' \subseteq B_2$ we have $\delta(B_0) \le \delta(B')$, and since any B'' such that $B_1 \subseteq B'' \subseteq D$ is a 1-simple extension of such a B', we have $\delta(B'') = 1+\delta(B') \ge 1+\delta(B_0) = \delta(B_1)$, we get $B_1 \le D$. Finally, D can not contain any K-gon that is not already contained in B_2, so D satisfies conditions (K1) and (K2), $D \in K$.

Lemma 2.25: If $B_0, B_1, B_2 \in K$ with $B_0 \le B_1, B_2$ and $\hat{B}_1 = B_1 \setminus B_0$ is 0-simple over B_0, then either $D = B_1 \otimes_{B_0} B_2 \in K$ and $B_1, B_2 \le D$, or there is an isomorphic copy of B_1 over B_0 inside B_2, and it is strong in B_2.

Proof: Suppose that D contains an ordinary k-gon with $k<n$, it can not lie entirely inside either B_1 or B_2, and thus there must be paths γ_1 and γ_2 inside $\hat{B}_1$ and $\hat{B}_2$ respectively connecting elements of B_0. Since $B_0 \le B_1$, $\hat{B}_1$ can not contain any arc of length less than $n-2$ attaching elements of B_0 (since otherwise by Lemma 1.5 (ii) we would obtain B' with $B_0 \subseteq B' \subseteq B_1$ and $\delta(B') < \delta(B_0)$), so $|\gamma_1| \ge n-2$. Similarly, $|\gamma_2| \ge n-2$, and moreover, we must have $|\gamma_1| = n-2 = |\gamma_2|$, since if any of them was bigger, then both arcs would connect a single element in B_0 (since $|\gamma_1|+|\gamma_2|+j = 2n-3+j \le 2n-2$, where j is the number of elements in B_0 closing the k-gon), and we do not obtain a simple cycle. Also, since $\hat{B}_1$ is 0-simple over B_0, $\hat{B}_1$ must be exactly γ_1, since otherwise $\delta(\hat{B}_1 B_0)-\delta(B_0) = 0 = \delta(\gamma_1 B_0)-\delta(B_0)$ and $\gamma_1 \subsetneq \hat{B}_1$, contradicting 0-simplicity. So, in this case B_2 contains an isomorphic copy of B_1 over B_0, namely $B_0\gamma_2$. Thus, either we can amalgamate B_1 and B_2 into B_2, or $D = B_1 \otimes_{B_0} B_2$ satisfies (K1).

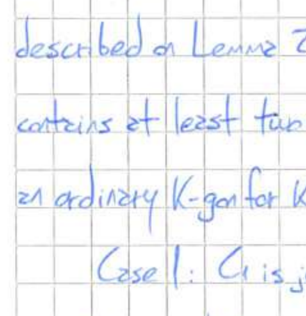
Now, suppose we had $B_1 \not\le D$. This means that for some B' with $B_1 \subseteq B' \subseteq D$ we have $\delta(B')<\delta(B_1)$. Since every relation between elements of B_1 and B_2 occurs in B_0, this would yield a set $B'' = B_2 \cap B'$ with



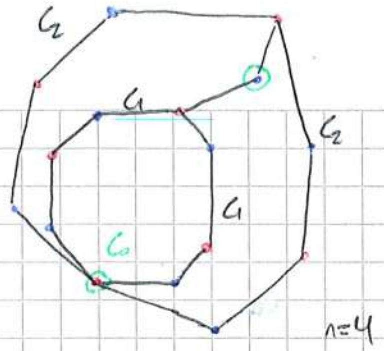
$B_0 \subseteq B''$ and $\delta(B' \cap B_2) = \delta(B'') < \delta(B_1 \cap B_2) = \delta(B_0)$, contradicting $B_0 \le B_2$. Thus, $B_1 \le D$ and in a completely analogous way we get $B_2 \le D$.

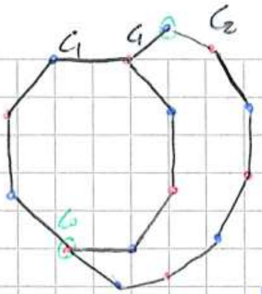
Assume now that there is a set $C \subseteq D$ that contains a generalized k-gon for $k>n$ and $\delta(C) < 2n+2$. Set $C_i = C \cap B_i$ $(i=0,1,2)$ and by Lemma 1.5 (iii) we get $C_i \le C$ $i=1,2$. If we had $\delta(C_i) \ge 2n+2$ for $i=1,2$, then we would arrive at a contradiction; so both C_1 and C_2 are of the form described on Lemma 2.18. So, we can check the different possibilities, always taking into account that C_0 contains at least two elements (the ones where the arcs in C_1 and C_2 forming the k-gon) and that C contains an ordinary k-gon for $k>n$.

Case 1: C_1 is just an ordinary n-gon, possibly with an arc of length $n-1$ attached. Then any two elements in C_1 have distance at most n. Then the only possibility for C_2 in order to form a k-gon for $k>n$ is to be string of $n+3$ elements, and thus C_0 contains exactly two elements, $\delta(C) = \delta(C_1)+2 = 2n+3$, we get a contradiction



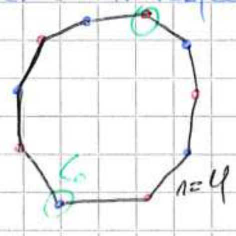
Case 2: C_1 is an ordinary n-gon with a single element attached to it. Then any two elements in C_1 have distance at most $n+1$. Then, since C contains an ordinary k-gon with $k>n$, C_2 must contain elements at distance at least $n+1$. One possibility is an ordinary n-gon with a single element attached, in which case we have that C_2 contains exactly two elements at distance $n+1$, and those must be the elements of C_0, and





we obtain C from C_1 by attaching an arc of length n and one arc of length $n-1$, so $\delta(C)=\delta(C_1)+3=2n+4$. The other possibility is C_2 being a string of $n+2$ elements, and thus $\delta(C)=\delta(C_1)+1=2n+2$; in both cases we get a contradiction.

Case 3: C_1 is a string of either $n+2$ or $n+3$ elements. Then one possibility for C_2 is to be a string of the same length, and thus C is an ordinary $(n+1)$-gon (in which case $\delta(C)=2n+2$) or $(n+2)$-gon (and $\delta(C)=2n+4$) respectively. Otherwise C_2 must be an ordinary n-gon with or without an extra point or arc of length $n-1$, and the situation reduces to the cases before with the roles of C_1 and C_2 exchanged. In any case, we get a contradiction.



Now, we can finally prove amalgamation:

Theorem 2.26: $(H, \leq)$ satisfies the amalgamation property

Proof: Let $B_0 \leq B_1, B_2$, $\hat{B}_i = B_i \setminus B_0$ $i = 1, 2$. We prove the theorem by induction on $|\hat{B}_1| + |\hat{B}_2|$. There are four cases to consider:

Case 1: There is a proper subset X of $\hat{B}_1$ with $\delta(XB_0) = \delta(B_0)$. In this case we have $B_0 \leq B_0X \leq B_1$, by the induction hypothesis, we can amalgamate B_0X with B_2 over B_0 to get D' and then D' with B_1 over B_0X.

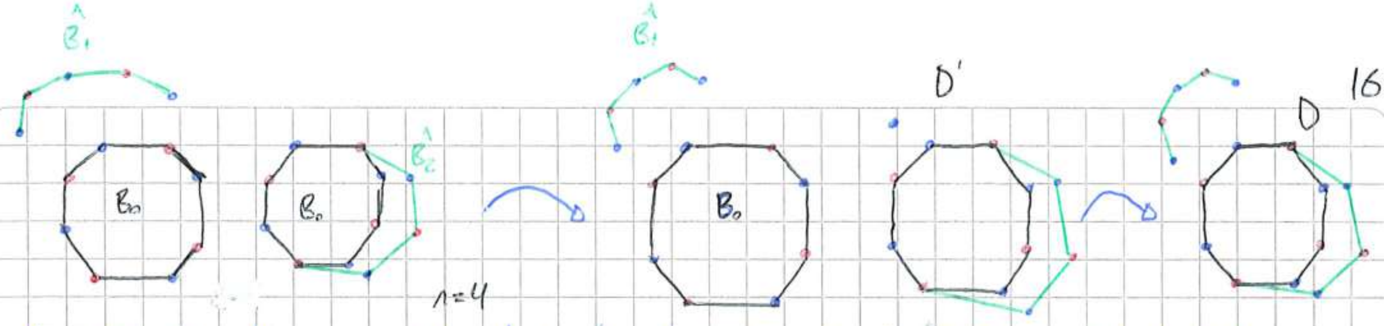
Case 2: Case 1 fails and $\delta(B_1) = \delta(B_0)$. Since Case 1 fails, there is no proper $X \subsetneq \hat{B}_1$ with $\delta(XB_0) = \delta(B_0)$, $\hat{B}_1$ is 0-simple over B_0 and we get the amalgamation through Lemma 2.25.

Case 3: Case 1 fails and $\delta(B_1) - \delta(B_0) = 1$. If there is a proper subset X of $\hat{B}_1$ with $\delta(XB_0) - \delta(B_0) = 1$, then failure of Case 1 gives $B_0 \leq B_0X \leq B_1$ and we amalgamate just as in Case 1. If there is not such a proper subset, $\hat{B}_1$ is 1-simple over B_0 and we get amalgamation through Lemma 2.24.

Case 4: Case 1 fails and $\delta(B_1) - \delta(B_0) > 1$. By failure of Case 1 we must have $\delta(bB_0) - \delta(B_0) \geq 1$ for all $b \in \hat{B}_1$. First, suppose there is $b \in \hat{B}_1$ with $\delta(bB_0) - \delta(B_0) = 1$. Failure of Case 1 implies $B_0 \leq bB_0 \leq B_1$, so we can use Case 3 to amalgamate bB_0 with B_2 with B_0 to get D', and by induction hypothesis we amalgamate D' with B_1 over bB_0 to get D.

If for all $b \in \hat{B}_1$ we have $\delta(bB_0) - \delta(B_0) > 1$, Lemma 1.5(i) gives $\delta(bB_0) - \delta(B_0) = n-1-(n-2)r(b,B$

so we must have $r(b, B_0) = 0$ for all $b \in \hat{B}_1$, $\delta(bB_0) - \delta(B_0) = n-1$. So, let $b \in \hat{B}_1$ be arbitrary. Clearly bB_0 and B_2 can be freely amalgamated into D' over B_0. Since $r(\hat{B}_1, B_0) = 0$, and $\delta(C) \geq n-1$ for any $C \subseteq \hat{B}_1$ (since thus $C \in H$),



we have $bB_0 \leq B_1$, and we can use the induction hypothesis to amalgamate D' and B_1 over B_0b to get D.

Proof of Theorem 2.1: We want to show that the $(K, \leq)$-homogeneous universal structure whose existence has just been proved satisfies the conditions of Theorem 2.1.

First, we have to show that this structure M is a generalized n-gon. It is immediate from conditions (H2) and (K1) that M does not contain ordinary k-gons for $k<n$ (and it is clearly a bipartite graph by condition (H2). Now, we need to see that any two elements $a, b \leq M$ are connected by a path of length at most n. Now, consider a subgraph A with $\{a,b\} \subseteq A \leq M$, clearly $A \in K$, and if a and b are connected by a path of length $\leq n$ in A, we are done. Otherwise, by Lemma 2.21, $B = A \cup \{b_1, \ldots, b_k\}$ with incidence $(a, b_1), \ldots, (b_k, b)$ (with k either $n-1$ or $n-2$) is a strong extension of A and also in K, so there is an embedding $f: B \to M$ with $f(B) \leq M$. By condition (H3), since $A \leq M$ and $A \leq B$, we also have $f|_A(A) \leq M$, and since A and $f|_A(A)$ are isomorphic, $f|_A$ extends to an automorphism of M. Since $f(a)$ and $f(b)$ are connected in M (in particular in $f(B)$) by a path of length $\leq n$, so are a and b.

To prove thickness, observe that for a with $P(a)$, the graph $\{a,b,c,d\}$ with incidence $\{(a,b),(a,c),(a,d)\}$  is in K, and thus it has an isomorphic copy inside M; that is, there is one point $x \in M$ incident with at least three elements. Now, for any $y \in M$, we have $\{x\}, \{y\} \leq M$ (since singletons are strong) and they are isomorphic, thus by condition (H3) this isomorphism extends to an automorphism of M and y must also be incident with at least three elements. For elements b with $\neg P(b)$ this is proved analogously, so M is a generalized n-gon.

That the automorphism group of M acts transitively on the set of generalized n-gons is an immediate consequence of the second part of Remark 2.6.

Remark 2.27: (i) By Remark 2.20 and a reasoning completely analogous to that of Remark 2.6, the automorphism group also acts transitively on the set of ordered ordinary n-gons. It also acts transitively on flags (i.e., pairs of incident elements), so it gives rise to a new class of BN-pairs.

(ii) The action of the automorphism group is not regular on the ordered ordinary n-gons. Consider one such n-gon $\Gamma = (x_0, \ldots, x_{2n-1})$ and k new elements $b_1, \ldots, b_k$ incident with x_0. This graph is in K, so we can identify it with some $A \leq M$, and since $\Gamma b_i \leq A$ for $i=1,\dots,K$; $\Gamma b_i \leq M$. Also, $\Gamma b_i \cong \Gamma b_j$ for $i,j=1,\dots,K$, by condition (H3) it can be extended to an automorphism of M, that is, there are automorphisms of M fixing Γ and taking b_1 to each one of the b_j, $j=1,\dots,K$. Since K is arbitrary, this implies that the stabilizer of each ordered ordinary $(n+1)$-gon is at least countable.

(iii) Finite generalized n-gons exist only for $n=3,4,6$ and 8. The generalized n-gons constructed here contain no finite generalized n-gons.

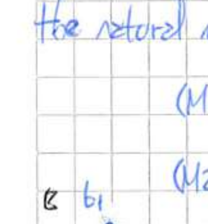
3. Almost Strong Minimality

Definition 3.1: A structure M is almost strongly minimal if it is in the definable closure of a strongly minimal subset plus a finite number of extra elements.

In order to get almost strongly minimal generalized n-gons, we need to have some control over how the algebraic closure behaves and to that purpose we need to add further conditions to the class K.

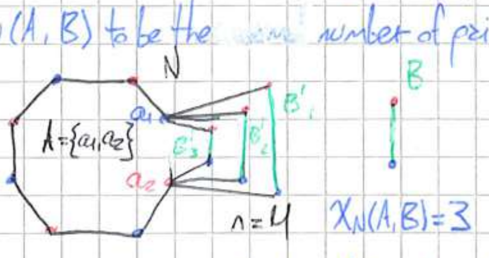
Definition 3.2: Fix a function μ from pairs (A,B) of finite graphs with B minimally 0-simple over A into the natural numbers which satisfies:

(M1) If (A,B), (A',B') are such that $A\cong A'$, $B\cong B'$, then $\mu(A',B')=\mu(A,B)$



(M2) If A and B are as in the first part of the Example after Definition 1.6, that is B is an extension of A by an arc of length $n-2$ and A is just the two elements attached to B, then $\mu(A,B)=1$. Otherwise $\mu(A,B)\geq \max\{\delta(A), n\}$.

Definition 3.3: For a finite graph N and a pair (A,B) with B minimally 0-simple over A, $A\subseteq N$, we define $\chi_N(A,B)$ to be the number of pairwise disjoint graphs $B'\subseteq N$ such that $B'\cong_A B$ (B' is isomorphic to B over A).



Definition 3.4: Let K^{μ} be the class of finite graphs bipartite with respect to P, satisfying conditions (K1), (K2) and the following extra condition:

(K3) For every pair of subgraphs (A,B) of $N\in K^{\mu}$, B minimally 0-simple over A, $\chi_N(A,B)\leq \mu(A,B$

Remark 3.5: It is immediate that K^4 is closed under substructures, and that the empty set and singletons (as well as ordinary $(n+1)$-gons and n-gons) are strong substructures in any graph containing them (inherited straight from the fact that this also happens in K). We now only remains to prove that condition (K3) survives amalgamation to prove that a $(K^4, \leq)$-homogeneous universal model exists and that it survives extensions by an arc of length $n-1$ or $n-2$ as in Lemma 2.21 so all the properties of Section 2 remain (being a generalized n-gon).

Lemma 3.6: Let $A \in K^4$ and let B be an extension of A by an arc of length $n-1$ or $n-2$ as in Lemma 2.21. Then $A \leq B$ and $B \in K^4$.

Proof: See the references in the Paper.

Theorem 3.7: The class $(K^4, \leq)$ has the amalgamation property.

Proof: The proof of Theorem 2.26 can be replicated completely if we prove that Lemmas 2.24 and 2.25 still work for the class K^4.

So, suppose that $B_0 \leq B_1, B_2$, $\hat{B_1} = B_1 \setminus B_0$ is 1-simple over B_0, $\hat{B_1} = \{b\}$. Note that no 0-simple extension inside $D = B_1 \otimes_{B_0} B_2$ of B_0 can contain b: otherwise, we remove b from this extension, and we obtain an extension of B_0 contained in the former one, and (since b is not incident with any element of $\hat{B_2}$) it will have less δ value than the former one, contradicting 0-simplicity. Also, if $C \subseteq D$ is minimally 0-simple over $F \subseteq D$ with $b \in F$, $\chi_D(F, C)$ represent the number of isomorphic copies of F over C (pairwise disjoint), but since b is incident with a unique element of B_0, then $1 = \chi_D(F, C) \leq \mu(F, C)$.

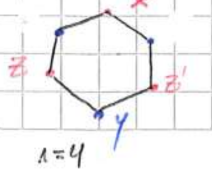
If $\hat{B_1} = B_1 \setminus B_0$ is 0-simple over B_0, it was shown in Lemma 2.25 that $D \in K$ unless there is an isomorphic copy of $\hat{B_1}$ inside B_2 (and we get the amalgam in B_2, we do not have to worry about (K3)). If $D \in K$, either D satisfies (K3) or there is again an isomorphic copy of $\hat{B_1}$ over B_0 in B_2 (see references in the paper).

Thus, as it was stated in Remark 3.5, it is proved exactly as in Section 2 that the $(K^4, \leq)$-homogeneous universal model is a generalized n-gon.

Definition 3.8: (i) For $x \in M^4$, we define $D_k(x) = \{y \in M^4 / d(x, y) = k\}$ (for $k = 1, \dots, n$)

(ii) If $x \in P(M)$ $D_1(x)$ is called a line pencil, if $x \notin P(M)$ $D_1(x)$ is a point row.

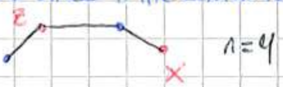
Remark 3.9: (i) Since for every pair $x, y \in M^4$ the property $d(x,y) = k$ is first order, $D_k(x)$ is definable.

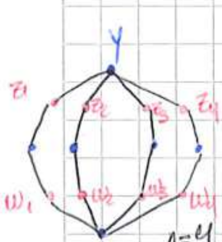


$n = 4$

(ii) If $1 < d(x,y) < n$, then there is a unique $z \in D_1(y)$ with $d(x,z) = d(x,y) - 1$ (since otherwise a simple cycle of length $< 2n$ would exist). We denote it by $z = proj_y(x)$, and thus $proj_y$

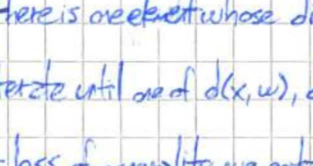
is a function from $D_2(y)\cup\ldots\cup D_{n-1}(y)$ into $D_1(y)$, and since the domain is definable and $d(x,z)=d(x,y)-1$ is a first order property, $proj_y$ is also definable.





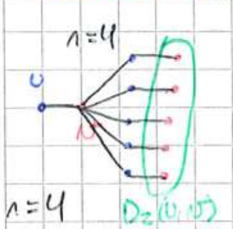
(iii) If $d(x,y)=n$, then each point at $D_1(x)$ has distance $n-1$ to y (the other option, $n+1$, would not be possible since the diameter of the graph is n). So, there is a function (definable by analogous reasons to the ones before) $[y;x]: D_1(x)\to D_1(y)$ with $z=proj_y(w)$ for $w\in D_1(x)$. Moreover, this function is a bijection: clearly $proj_x$ is the inverse of $proj_y$. A concatenation of this definable bijections is again definable and it is called a projectivity, we write $[x_3;x_2]\circ[x_2,x_1]=[x_3;x_2;x_1]: D_1(x_1)\to D_1(x_3)$.

(iv) Thickness implies that every element x has an element y at distance n: take y' such that $1\leq d(x,y')=k<n$, then y' is incident with three elements at least, and only one of them can have distance to x less than k (otherwise we would get a simple cycle of length $\leq 2k<2n$), so distance to x must increase for the rest of them. Moreover, it can be proved that if $x,y\in P(M)$ (and similarly if $x,y\notin P(M)$), then there is $z\in D_n(x)\cap D_n(y)$: if both elements have the same colouring, they can not have distance 1 (this would mean they are incident). This means that there is $w\in M^4$ such that $d(x,w),d(y,w)<d(x,y)$. From this point, $d(x,w),d(y,w)<n$, then every w' incident with w except one is at a greater distance from x than w (otherwise we would be closing a simple cycle of length less than $2n$), and similarly for y. Thus, thickness implies there is one element whose distance to both x and y increases, and we take this w' for our next step. At this point we iterate until one of $d(x,w)$, $d(y,w)$ reaches n. If both did at the same time, we are done. Otherwise, assume without loss of generality we got $d(x,w')=n$, $d(y,w')\neq n$. We can not have $d(y,w')=n-1$ since that would be inconsistent with $x,y\in P(M)$. So, at least one of the points incident with w' increases its distance to y (and of course has distance $n-1$ to x). Thus, we are in the situation to increase both distances again. We alternate both steps until we get $d(x,w''')=d(y,w''')=n$. This point w''' will yield a definable bijection $[y;w;x]: D_1(x)\to D_1(y)$.



That is, there is a definable bijection between any two point rows or line pencils. If n is odd, part (iii) of this Remark yields also a definable bijection between a point row and a line pencil at distance n of the line defining this point row, and thus between any point row and any line pencil.

Definition 3.10: Let u, v be incident elements of M^4. We define $D_k(u,v) = D_{k+1}(u) \cap D_k(v)$, $0 \le k \le n-1$, this sets are called Schubert cells.



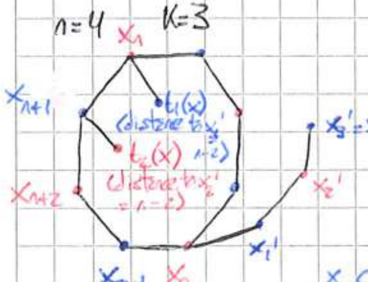
Remark 3.11: (i) We have also $D_k(u,v) = D_k(v) \setminus D_{k-1}(u)$, $D_n(u) = D_{n-1}(u,v)$, $D_k(v) = D_k(u,v) \cup D_{k-1}(v,u)$

(ii) Let u, v be incident with $u \in P(M)$ by the third equality in part (i) we have that $P = D_0(v,u) \cup D_1(u,v) \cup D_2(v,u) \cup \ldots$, where the last term is $D_{n-1}(u,v)$ or $D_{n-1}(v,u)$ if n is even or odd respectively. A similar construction can be done for $M^4 \setminus P(M^4)$

(iii) Since a Schubert cell is the intersection of two definable sets it is, of course, also definable.

Lemma 3.12: M^4 is in the definable closure of a line pencil, a point row and a finite number of extra elements.

Proof: Fix an ordinary n-gon $\{x_0, x_1, \ldots, x_{2n-1}\}$ in M^4, and take $x \in D_k(x_{2n-1}, x_0)$, let $(x_{2n-1}, x_0, x_1', x_2', \ldots, x_k' = x)$ denote the chain of incidences. Note that $d(x_i', x_{n+i}) = n$ for $i = 1, \ldots, k$; so $d(x_i', x_{n+i-1}) < n$ and we can



define functions on x t_i, $i = 1, \ldots, k$ such that $t_i(x) = \mathrm{proj}_{x_{n+i+1}}(x_i') \in T_i = D_1(x_{n+i}, x_{n+i-1})$. So we have attached coordinates $(t_1(x), t_2(x), \ldots, t_k(x))$ to the element x (that of course depend on the ordinary n-gon chosen as reference). Through this coordinates, x can be defined as the element such that $x_1' = \mathrm{proj}_{x_0}(t_1(x))$, $x_j' = \mathrm{proj}_{x_{j-1}'}(t_j(x))$ for $j = 2, \ldots, k$. (to see that this works, notice that $t_1(x)$ is at distance $n-1$ from x_0, and at distance $n-2$ from $t_1(x)$), so the coordinates reconstruct the path from any element to the chosen ordered pair of incident elements (x_{2n-1}, x_0). Thus, x is in the definable closure of $\{x_0, \ldots, x_{2n-1}\} \cup \{D_1(x_{n+i}, x_{n+i-1}) / i = 1, \ldots, n\}$. So, since x is arbitrary as long as its colouring is consistent with $d(x, x_{2n-1}) > d(x, x_0)$, but for the other colouring, a completely analogous process can be done swapping the roles of x_0 and x_{2n-1} and the elements defining the Schubert cells involved. Since there is also a bijection between any two point rows or line pencils, the definable closure of $\{x_0, \ldots, x_{2n-1}\} \cup D_1(x_0) \cup D_1(x_1)$ is the whole structure.

Remark 3.13: All the Definitions, Remarks, Lemmas from Definition 3.8 until Lemma 3.12 are valid for any generalized n-gon, not just for M^4.

Theorem 3.14: Let M^4 be the $(K^4, \leq)$-homogeneous universal model. Let $b \in M^4$, then $D_1(b)$ is strongly minimal.

Proof: See references in the paper, it uses the fact that M^4 is countably saturated.

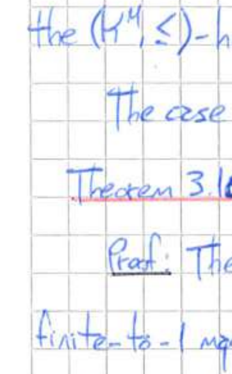
Remark 3.15: Theorem 3.14, Lemma 3.13 and Remark 3.9(iv) (the last paragraph) already imply that the $(K^4, \leq)$-homogeneous universal structure is almost strongly minimal.

The case n even is treated in the following Theorem.

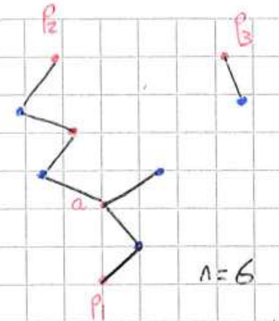
Theorem 3.16: Let M^4 be the $(K^4; \leq)$-homogeneous universal structure. Then M^4 is strongly minimal.

Proof: The case n odd is already covered, for n even, we will show there is a definable surjective finite-to-1 map from a line pencil to a point row, showing that the line pencil is in the algebraic closure of the point row, and thus proving the Theorem.

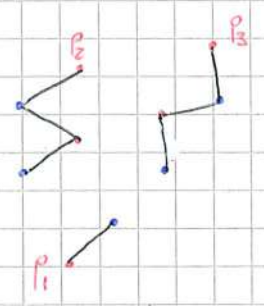
We first notice that there is always the following special case of a minimally 0-simple pair $(A, \hat{B})$: we choose $p_1, p_2, p_3 \in P(M^4)$ with $d(p_i, p_j) = n$ for $1 \leq i \neq j \leq 3$ (such three points always exist by Remark 3.9 (iv)). Let A be the graph consisting of these three points without any incidence, so $\delta(A) = 3n-3$. Now, extend



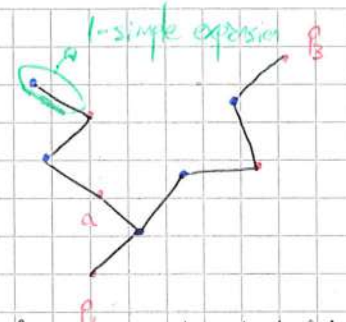
A by an element $a \in P(M^4)$ with $d(a, p_1) = 2$ and $d(a, p_2) = d(a, p_3) = n-2$, and let B be the graph containing Aa and the shortest paths between a and the p_i for $i = 1, 2, 3$ (such a subgraph of M^4 always exists since it is in K^4 and M^4 is countably saturated). To compute $\delta(B)$, notice that B can be described as a path of length n from p_1 through a to p_2 with a path of length $n-2$ from a to p_3 attached, so $\delta(B) = (n+1)(n-1) - (n-2)n + (n-2)(n-1) - (n-3)(n-2) - (n-2) = n^2 - 1 - n^2 + 2n + n^2 - 3n + 2 - n^2 + 5n - 6 - n + 2 = 3n-3$. Moreover, removing any elements from (excluding a) would imply at least removing m vertices and $m+1$ edges with $m \geq n-3$, so the change in the value of δ is $\geq (n-2)(n-2) - (n-1)(n-3) = 1$. If we removed a in addition, we would have a series of 1-simple extensions of A, thus, the δ-value would increase again and we would, and so $\hat{B} = B \setminus A$ is a 0-simple expansion of A. In addition, over any non-empty subset of A, an expansion by B differs from the one over A in that it is lacking both the element and the vertex removed, and it has 1-simple expansions that can be removed thus decreasing the δ value, and so it will not be 0-simple. Thus, $\hat{B}$ will even be minimally 0-simple over A. Also, since M^4 is saturated, there are exactly $l = \mu(A, \hat{B})$



we remove points and vertices such that δ increases (if we don't include a)



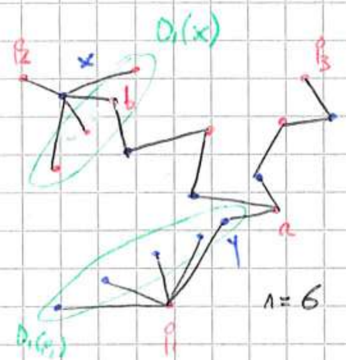
if we include a, then we get 1-simple expansions of A (δ increases again)



for any non-empty subset of A, expanding by $\hat{B}$ includes 1-simple expansions that can be remove decreasing δ

copies of $\hat{B}$ over A in M^H; that is, M^H contains exactly ℓ elements b lying in the same configuration with p_1, p_2, p_3 as a. Note also that ℓ does not depend on the choice of p_1, p_2, p_3 (as long as they have pairwise distance n) and a, since φ depends only on the configuration.

Now, let p_1, p_2, p_3 be as before, and fix x with $d(p_2, x) = 1$ (and thus $d(p_1, x) = d(p_3, x) = n-1$).



Now, consider the following map ϕ from $D_1(p_1)$ to $D_1(x)$: for any $y \in D_1(p_1)$, we have $d(y, p_3) = n-1$, and we can define $a = \mathrm{proj}_y(p_3)$. Similarly, since $a \in P(M^H)$ and $x \notin P(M^H)$, $d(x, a) < n$ and thus we can define $b = \mathrm{proj}_x(a)$. Put $\phi(y) = b$.

As we vary y, we vary a, and from the discussion in the paragraph before, it follows that ϕ is an ℓ-to-1 function (and of course definable, since it was constructed from proj). By the $(K^H, \leq)$-homogeneity of M^H, the image of ϕ must be all of $D_1(\ell)$, and therefore $D_1(p_1)$ is in the algebraic closure of $D_1(x)$.