VERY HOMOGENEOUS GENERALIZED n-GONS OF FINITE MORLEY RANK

K. TENT

Abstract

(Almost strongly minimal) generalized n-gons are constructed for all $n \geqslant 3$ for which the automorphism group acts transitively on the set of ordered ordinary $(n+1)$-gons contained in it, a new class of BN-pairs thus being obtained. Through the construction being modified slightly, $2^{\aleph_0}$ many non-isomorphic almost strongly minimal generalized n-gons are obtained for all $n \geqslant 3$, none of which interprets an infinite group. Furthermore, a characterization is given of all graphs whose simple cycles all have length $2n$ for some $n \geqslant 3$.

1. *Preliminaries*

A *generalized n-gon* is a bipartite graph (with respect to a predicate P and an incidence relation I where two vertices are incident if they have a common edge) such that the diameter of the graph is n and there are no simple cycles (that is, without repetitions) of length less than $2n$. Also, we require that the graph is *thick*, that is, that any element is incident with at least three other elements. By an *ordinary n-gon* we just mean a simple cycle of length $2n$, that is, we drop the assumption of thickness. If we fix labels $x_0, x_1, \ldots, x_{2n-1}, x_{2n} = x_0$ and $P(x_0)$ for an ordinary n-gon, we call this tuple an *ordered* ordinary n-gon. Note that a projective plane is nothing but a generalized 3-gon.

The construction proceeds via amalgamation and free extensions of bipartite graphs. The properties can then be easily checked using the existence of countable models satisfying rather strong universality and homogeneity conditions.

A similar approach was chosen in [**10**] for projective planes which are transitive on ordinary quadrangles. However, the generalization of this construction to generalized n-gons for arbitrary n in [**5**] is not correct, as was observed by Wassermann [**19**], and until now there has been no general class of examples for n-gons that are $(n+1)$-gon transitive.

For finite n-gons this is a rather restrictive condition; a complete classification of finite n-gons with this property was obtained by Thas and Van Maldeghem [**14**, **17**]. Joswig classified compact connected Moufang n-gons with this property [**7**, **8**]. In particular, in these classes the value of n is restricted to $n = 3$, 4 or 6 (see also [**18**, 6.8.9, 9.6.5]).

For generalized n-gons whose automorphism group acts transitively on ordered ordinary n-gons, a free construction was described in [**16**], and it was shown in [**15**, 3.2.6, 3.11] that this property is equivalent to the automorphism group having a *BN*-pair. See also the survey article by Funk and Strambach [**4**].

Our construction is inspired by that of Hrushovski [**9**], which has been adapted by Baldwin [**1**] for the construction of projective planes of finite Morley rank. There, a

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non-bipartite graph was constructed; this was then duplicated to present both the set of lines and the set of points of the projective plane. The same construction was used in [**3**] to obtain n-gons of finite Morley rank for *necessarily odd n* (which, in contrast to the polygons constructed in Section 4, are not even flag-transitive). We give here a uniform construction for all n, using a predicate to distinguish between points and lines instead of duplicating the graph.

To make the construction easier to understand for geometers, the condition needed to ensure finite Morley rank is introduced only in Section 4. By a small change in the requirements, we obtain $2^{\aleph_0}$ many non-isomorphic almost strongly minimal n-gons which do not interpret an infinite group.

REMARK 1.1. In contrast to this, it was shown in [**13**] that if $\mathfrak{B}$ is a generalized n-gon with strongly minimal point rows and line pencils, and if $G \leqslant \mathrm{Aut}(\mathfrak{B})$ is a group of finite Morley rank acting transitively on the set of ordered ordinary n-gons in $\mathfrak{B}$, then G is definably isomorphic to $\mathrm{PSL}_3(K)$, $\mathrm{PSp}_4(K)$ or $\mathrm{G}_2(K)$ for some algebraically closed field K, and the corresponding polygon is either the projective plane, the symplectic quadrangle, or the split Cayley hexagon over K.

2. *The set-up*

We start with some graph-theoretic notions.

DEFINITION 2.1. (i) For $a, b \in A$, the *distance* $d(a,b)$ between a and b is the smallest number m for which there exists a path $a = a_0, a_1, \ldots, a_m = b$ with $a_i \in A$ where a_i and a_{i-1} are incident for $i = 1, \ldots, m$.

(ii) If B is obtained from A by attaching a string of m elements from one element of A to another element of A, we call this an extension by an *arc of length m*.

Hence an arc of length m yields a path of length $m+1$ between the elements of A to which it was attached, since we also have to count the end elements in the path.

Note that, from now on, we fix $n \geqslant 3$. The standard metric on graphs is given by the distance function defined above. For generalized n-gons we can also consider the *bounded metric* defined as follows.

DEFINITION 2.2. Let A be a bipartite graph with respect to a predicate P, and let $a, b \in A$. We define the *bounded distance* of a and b in A to be equal to the ordinary distance $d(a,b) = k$ if $k \leqslant n$. If there is no path of length less than n we set $d(a,b) = n$ if a, b have the same colour with respect to P and n is even, or if a, b are different in colour and n is odd, and $d(a,b) = n-1$ otherwise.

We think of P as inducing a colouring on the graph corresponding to the set of points in the polygon that we are going to construct. The elements not in P are considered as lines in the polygon.

For any finite graph A we define the weighted Euler characteristic as

$$y(A) = (n-1)\,|A| - (n-2)\,e(A)$$

where $|A|$ denotes the number of elements in A, and $e(A)$ denotes the number of edges in A. If A and B are disjoint subgraphs of a graph M, then $r(A, B)$ denotes the number of edges between elements of A and elements of B. If A and B are subgraphs of a graph M, we denote by AB the subgraph of M whose vertices are in $A \cup B$, with the incidence relation induced from M.

DEFINITION 2.3. If A and B are bipartite with respect to a predicate P and $A \subseteq B$ is finite, we say that A is *strong in* B and write $A \leqslant B$ if $P(A) = P(B) \cap A$, and if for any finite subgraph A' with $A \subseteq A' \subseteq B$ we have $y(A) \leqslant y(A')$.

Note that '$\leqslant$' is a transitive relation on finite subgraphs of a fixed graph. If $A \leqslant B$, then A is *isometrically embedded* in B, that is, for $a, b \in A$ the bounded distance with respect to A and the bounded distance with respect to B are the same. This is due to the fact that if $A \leqslant B$ and $\hat{B} = B \backslash A$ contains an arc of length k attached to two elements of A, then the function y forces $k \geqslant n-2$, that is, the string must have at least $n-2$ elements (not $n-1$ as claimed in [**3**]); see the following lemma.

LEMMA 2.4. (i) *If B is obtained from A by attaching an arc of length m, then $y(B) = y(A) + m - n + 2$.*

(ii) *If $B_0 \leqslant B_1$ and $C \subseteq B_1$, then $C \cap B_0 \leqslant C$* [**1**, **3**].

Proof. These are easy calculations. For (i) use the fact that if A and B are disjoint, we have $y(AB) = y(A) + y(B) - (n-2) \cdot r(A, B)$. □

DEFINITION 2.5. For A and B being finite subgraphs of M and $i = 0, 1$, we say that B is *i-simple* over A if A and B are disjoint, $y(AB) - y(A) = i$ and for every proper nonempty subset C of B, we have $y(AC) - y(A) > i$. A graph B is *minimally 0-simple* over A if it is 0-simple over A and if it is not 0-simple over any proper nonempty subset of A.

By Lemma 2.4(i), the typical example for a 0-simple extension is the expansion of a graph A by an arc of length $n-2$. This arc is *minimally* 0-simple if A consists only of the two elements to which the arc is attached. Clearly, an expansion of a graph A by a single element b incident with a unique element of A is 1-simple. In the class of graphs that we consider below, this will in fact be the only kind of 1-simple expansion.

The following lemma collects together some rather easy but useful facts.

LEMMA 2.6. (i) *If $B_0 \leqslant B_1$ and $C \subseteq B_1$ is 0-simple over $F \subseteq B_0$, then $C \subseteq B_0$ or $C \subseteq B_1 \backslash B_0$* [**1**, **3**]. *In the latter case, C is 0-simple over B_0 and $r(C, B_0 \backslash F) = 0$. Hence if C and $C' \subseteq B_1 \backslash B_0$ are isomorphic over F, they are isomorphic over B_0.*

(ii) *If C is 0-simple over B_0, then it is 0-simple over any $F \subseteq B_0$ with $r(C, B_0) = r(C, F)$* [**1**, **3**]. *Thus there exists a unique $F \subseteq B_0$ with C minimally 0-simple over F, namely the set of elements of B_0 that are incident with elements of C.*

(iii) *If B is a connected graph which does not contain any cycles, then $y(B) = (n-2) + |B|$.*

Proof. The proofs in [**3**] also work in this context. The last assertion of (ii) follows from the equality $y(CB_0) = y(C) + y(B_0) - (n-2)\, r(C, B_0)$, so $y(C) = (n-2)\, r(C, B_0)$. Part (iii) follows easily from the observation that, starting with any $b \in B$, the whole graph B is obtained by successive 1-simple expansions by single elements. □

3. *The construction*

In this section we prove the following theorem.

THEOREM 3.1. *For all $n \geqslant 3$, there exist generalized n-gons for which the automorphism group acts transitively on the set of ordered ordinary $(n+1)$-gons contained in it.*

We will construct these generalized n-gons from a class $\mathscr{K}$ of finite bipartite graphs partially ordered by the strong substructure relation '$\leqslant$' by free extensions and amalgamation.

Throughout this section we fix $n \geqslant 3$.

DEFINITION 3.2. Let $(K, \leqslant)$ be a collection of finite relational structures closed under substructures.

(i) We say that $(K, \leqslant)$ has the *amalgamation property* if for $A, B, C \in K$ and embeddings $f_0: A \longrightarrow B$ and $g_0: A \longrightarrow C$ with $f_0(A) \leqslant B$ and $g_0(A) \leqslant C$ there exists some $D \in K$ and embeddings $f_1: B \longrightarrow D$ and $g_1: C \longrightarrow D$ such that $f_1(B), g_1(C) \leqslant D$ and $f_1 \circ f_0 = g_1 \circ g_0$.

(ii) The class $(K, \leqslant)$ is said to have the *joint embedding property* if for any $A, B \in K$ there exists some $C \in K$ and embeddings $f: A \longrightarrow C$ and $g: B \longrightarrow C$ with $f(A), g(B) \leqslant C$.

(iii) A countable structure M is called a $(K, \leqslant)$-homogeneous universal model if the following conditions are satisfied.

(H1) If $A \in K$ is finite, then there exists an embedding $f: A \longrightarrow M$ such that $f(A) \leqslant M$.

(H2) If $A \subseteq M$ is finite, then $A \in K$.

(H3) If $A, B \leqslant M$ are finite and there exists an isomorphism $f: A \longrightarrow B$, then there exists an automorphism of M extending f.

We will use the following fact explored by Shelah, generalizing a construction by Jónsson-Fraissé (see [**1**]).

ASSERTION 3.3. If $(K, \leqslant)$ has the amalgamation property and the joint embedding property, then there exists a countable $(K, \leqslant)$-homogeneous universal model.

In view of condition (H3), in order to obtain generalized n-gons for which the automorphism group acts transitively on ordinary $(n+1)$-gons, it will suffice to force any $(n+1)$-gon to be strong in M. Therefore we choose our class $\mathscr{K}$ in the following way.

DEFINITION 3.4. Let $\mathscr{K}$ be the collection of finite graphs A, bipartite with respect to P, with the following properties.

(K1) Graph A contains no ordinary k-gons for $k < n$.

(K2) If $B \subseteq A$ contains an ordinary k-gon for $k > n$, then $y(B) \geqslant 2n+2$.

Condition (K2) allows for an easy characterization of 1-simple extensions.

LEMMA 3.5 [**1**, **3**]. *If $A \subseteq B \in \mathscr{K}$, then $\hat{B} = B \backslash A$ is* 1*-simple over A, if and only if $B = Ab$ where b is incident with a unique element in A.*

We first need the following observation.

LEMMA 3.6. *Let B be any bipartite graph satisfying condition* (K1) *with $|B| \geqslant n+1$. If $y(B) < 2n$, then $|B| \geqslant 2n+4$.*

Proof. Suppose that this is not true, and let B be a minimal counterexample. By Lemma 2.6(iii), we may assume that B is connected and contains a cycle, so $|B| \geqslant 2n$.

If $|B| = 2n$, then B is an ordinary n-gon and $y(B) = 2n$.

For $|B| = 2n+1$, B must be a 1-simple expansion of an ordinary n-gon since it is not possible to attach an arc of length 1 to an ordinary n-gon without contradicting condition (K1). Therefore $y(B) = 2n+1$.

If $|B| = 2n+2$, B can be either an ordinary $(n+1)$-gon or an n-gon expanded by two 1-simple expansions, and in either case we have $y(B) = 2n+2$. If $n = 3$, B can also be an ordinary 3-gon with one arc of length 2 attached, in which case $y(B) = 2n+1$. Note that for $n \geqslant 4$ it is not possible to attach an arc of length 2 to an ordinary n-gon without contradicting condition (K1).

If $|B| = 2n+3$, B can be either an $(n+1)$-gon or an n-gon expanded by one or three 1-simple expansions, respectively. In either case $y(B) = 2n+3$. For $n \leqslant 4$, B can also be an expansion by an arc of length 3, in which case $y(B) \geqslant 2n+1$, and for $n = 3$ it is finally also possible that B is an expansion of an ordinary triangle by an arc of length 2 and a 1-simple expansion, but then also $y(B) = 2n+2$. For $n \geqslant 5$ it is again impossible to attach an arc of length 3 without contradicting condition (K1).

This finishes the proof of the lemma. □

We need the following lemmas.

LEMMA 3.7. *Suppose that A is a graph that does not contain any simple k-cycles for $k \neq 2n$. If $\Gamma_1, \Gamma_2 \subseteq A$ are two distinct ordinary n-gons, then they intersect in* 0, 1, 2 *or $n+1$ elements. If $\Gamma_1 \cap \Gamma_2$ contains at least two elements, then it contains elements that have distance n with respect to both Γ_1 and Γ_2.*

Proof. It is clear that Γ_1, Γ_2 can intersect in zero or one elements. If they intersect in exactly two elements, it can easily be seen that they must have distance n. If they intersect in at least three elements, then at least two of them must have distance less than n, and we must find some k-gon for $k \neq n$, unless Γ_1, Γ_2 agree on a string of $n+1$ elements. If they agree on more than $n+1$ elements, we obtain a k-gon with $k < n$ unless $\Gamma_1 = \Gamma_2$. □

LEMMA 3.8. *Suppose that A is a graph that does not contain any simple k-cycles for $k \neq 2n$. If $\Gamma_i \subseteq A$, where $i = 1, 2, 3$, are distinct n-gons such that $\Gamma_1 \cap \Gamma_2$ and $\Gamma_2 \cap \Gamma_3$ both contain at least two elements, then also $\bigcap_{i=1,2,3} \Gamma_i$ contains at least two elements.*

Proof. Suppose that Γ_1 and Γ_2 intersect in a and b, where a and b have distance n, and that Γ_2 and Γ_3 intersect in c, d at distance n. Then we obtain a k-gon for $k > n$ by moving from a to b inside Γ_1, then inside Γ_2 from b to d, then in Γ_3 from d to c, and finally inside Γ_2 back to a. This path has no repetitions unless $\{c, d\} = \{a, b\}$. □

For graphs A not containing any simple k-cycles for $k \neq 2n$, it follows from Lemma 3.8 that we have an equivalence relation '$\sim$' on the set of ordinary n-gons contained in A defined by $\Gamma_1 \sim \Gamma_2$ if and only if $\Gamma_1 \cap \Gamma_2$ contains at least two elements.

By a *stack of n-gons* we denote the subgraph of A consisting of one equivalence class of ordinary n-gons with respect to this equivalence relation. The intersection of all n-gons in the same stack contains at least two elements at distance n, and we call these elements the *glueing points* of the stack.

Note that a stack has the property that any two elements have distance at most n inside this subgraph. Hence any two elements of the same stack are contained in some ordinary n-gon belonging to the stack.

LEMMA 3.9. *Suppose that A is a graph that does not contain any simple k-cycles for $k \neq 2n$. Let $\Gamma_1, \Gamma_2 \subseteq A$ be distinct n-gons in the same stack, and suppose that $\Gamma_3 \subseteq A$ is another n-gon intersecting both Γ_1 and Γ_2. Then either Γ_3 is in the same stack as Γ_1, Γ_2 or it intersects Γ_1 and Γ_2 in a common point.*

Proof. Suppose that $\Gamma_3 \cap \Gamma_1 = \{a\}$ and $\Gamma_3 \cap \Gamma_2 = \{b\}$. If $a \neq b$, then there is an ordinary n-gon Γ' in the same stack as Γ_1, Γ_2 containing a and b. However, now Γ_3 intersects Γ' in two elements, and therefore belongs to the same stack. □

We call two n-gons $\Gamma_1, \Gamma_2 \subseteq A$ *neighbours* if they intersect in at most one element and if there is an element in Γ_1 which is incident with an element of Γ_2. These elements are called the *touching points*.

LEMMA 3.10. *Let A be a graph that does not contain any simple k-cycles for $k \neq 2n$. Suppose that $\Gamma_1 \sim \Gamma_2$ and $\Gamma_1' \sim \Gamma_2'$ in A. If Γ_1 and Γ_1' are neighbours and Γ_2 and Γ_2' are neighbours, then they have the same touching points.*

Proof. If not, we can again use the touching points to find a k-gon for $k > n$. □

By Lemma 3.10, if two stacks contain neighbouring n-gons, there is a unique touching point. In this case we will say that the stacks are *neighbours*.

The following theorem is crucial for the construction, and might be interesting in its own right.

THEOREM 3.11. *Suppose that A is a connected graph that does not contain any simple k-cycles for $k \neq 2n$. Then A can be described as a tree in which single nodes are replaced by stacks of n-gons.*

Proof. Let A be a connected graph, all cycles of which have length $2n$. First assume that A does not contain any cycles at all. Then, clearly, A is a tree and we are done.

Now suppose that A does contain $2n$-cycles. Then we form a new graph T by replacing each stack of A by a single node, the edges of which are all the edges that were connected to the stack. To show that T is a tree, we have to show that between two nodes of T there is at most one edge, and that T does not contain any simple cycles. For the first part, let a, b be two nodes of T. There are three cases to consider, depending on whether a and b came from nodes in A or from stacks in A. If both a and b were nodes in A, then there is only a single edge between them in A, and hence in T. If a comes from a stack $\tilde{a}$ in A, and b from a single node, then in A, the node b is incident with a unique element of $\tilde{a}$ (as all nodes in one stack have at most distance n from each other); thus there is only a single edge between a and b in T.

If both a and b come from stacks $\tilde{a}$ and $\tilde{b}$ in A, respectively, then by Lemma 3.10, there is a unique touching point. Thus there is a unique edge between a and b in T.

Now suppose that T contains a simple cycle γ. Then, clearly, there must be a simple cycle Γ in A which yields $\gamma \subseteq T$ after the stacks of A are replaced by nodes in T. However, by assumption, Γ is a cycle of length $2n$ and was thus replaced by a single node in T. Thus there are no simple cycles in T, which shows that T is indeed a tree. This proves the theorem. □

Since a stack can be obtained from an ordinary n-gon by successively attaching arcs of length $n-1$, the y-value of any stack is at least $2n$.

LEMMA 3.12. *Suppose that $A \in \mathscr{K}$ with $|A| \geqslant n+2$. Then $y(A) \geqslant 2n$. Moreover, we have in fact $y(A) \geqslant 2n+2$, unless A has at most $n+3$ elements, or A is an ordinary n-gon with either a single arc of length $n-1$ or a single element attached.*

Proof. By Lemma 3.6 it suffices to prove the second part of the lemma. Assume, moving towards a contradiction, that A is a minimal counterexample, so A is not of the form described in the statement and $y(A) \leqslant 2n+1$. By definition of $\mathscr{K}$, A cannot contain any ordinary k-gons for $k \neq n$. By the proof of Lemma 3.6, we must have $|A| \geqslant 2n+4$, and we may assume that A is a connected graph (as otherwise some of the components would form a smaller counterexample).

We may also assume that any element in A is incident with at least two other elements (otherwise we could remove an element with only one incidence, yielding a smaller counterexample). Using similar considerations we conclude that any element of A must lie inside some ordinary n-gon. Suppose that $x \in A$ is not contained in any ordinary n-gon and let $k \geqslant 2$ be the number of elements incident with x. By changing the colour of x and identifying x with the elements incident with x, we are removing k elements and k instances of incidence, thereby obtaining a smaller counterexample.

By Theorem 3.11 we know that A can be described as a tree in which certain nodes were replaced by stacks. Since A is finite, there are only finitely many stacks. At least one of these stacks has at most one neighbour, as otherwise there must be a circle of at least three stacks, yielding a k-gon for $k > n$.

Now let $S \subseteq A$ be a stack with at most one neighbouring stack. If this stack consists of a single ordinary n-gon, then by removing S and leaving only the touching point we reduce the y-value of the graph by $n+1$. Thus the remaining graph has y-value at most n, and so consists of at most $n+1$ elements. Therefore A must have been an n-gon with an arc of length $n-1$ attached.

Finally, suppose that S consists of at least two n-gons with glueing points a, b and touching point c. Let $\Gamma \subseteq S$ be an n-gon containing c. Then one path from a to b inside Γ does not contain c, and no element on this path is incident with more than two elements. Therefore, by removing the $n-1$ elements of this path between a and b, we again obtain a graph of smaller y-value. Therefore, A must again have been an ordinary n-gon with an arc of length $n-1$ attached. □

COROLLARY 3.13. *If $A \in \mathscr{K}$, then for any non-empty graph $B \subseteq A$ we have $y(B) \geqslant n-1$, and if $|B| \geqslant n+2$, then $y(B) \geqslant 2n$.*

Proof. This follows from Lemmas 3.6, 3.12 and 2.6(iii). □

Thus the empty graph is strong in any element of $\mathscr{K}$. Note also that (by the previous corollary) singletons, ordinary n-gons and ordinary $(n+1)$-gons are strong in all graphs containing them with the same colouring.

The following lemma (adapted from [**1**, **3**]) shows that we can add certain free extensions inside $\mathscr{K}$.

Lemma 3.14. *Suppose that $A \in \mathscr{K}$ and $a, b \in A$ are not connected by a path of length at most n. Consider the graph B obtained from A by adding an* arc *of length k between a and b, that is, by adding new vertices $\{b_1, \ldots, b_k\}$ and new edges*

$$(a, b_1), (b_1, b_2), \ldots, (b_k, b)$$

where $k = n-1$ if a and b have the same colouring with respect to P and n is even, or if a and b have different colours and n is odd, and $k = n-2$ otherwise. Then with the colouring of $\{b_1, \ldots, b_k\}$ induced from a and b, we have $A \leqslant B$ and $B \in \mathscr{K}$.

Proof. It is clear from Lemma 2.4(i) that $A \leqslant B$, and obviously B satisfies condition (K1).

In order to check condition (K2), we assume now that $C \subseteq B$ contains an ordinary k-gon Γ with $k > n$ and $y(C) < 2n+2$. By the proof of Lemma 3.12 we must have $|C| \geqslant 2n+3$. Therefore $C \cap A$ contains at least $n+2$ elements, and thus $y(C \cap A) \geqslant 2n$. By Lemma 2.4(ii), we have $C \cap A \leqslant C$. Thus, if $C \cap A$ contains an ordinary k-gon for $k > n$ then $2n+2 \leqslant y(C \cap A) \leqslant y(C)$, yielding a contradiction. Hence we can apply Lemma 3.12 to see that $C \cap A$ must be a string of $n+3$ elements, and therefore C is an ordinary $(n+1)$-gon with $y(C) = 2n+2$. All other possibilities for $A \cap C$ described in Lemma 3.12 do not contain elements of distance greater than $n+1$, and therefore cannot be extended by γ in such a way as to contain an ordinary k-gon for $k > n$. □

Since the empty graph is strong in every graph of the class $\mathscr{K}$ defined in Section 2, and $\mathscr{K}$ is clearly closed under substructures, it suffices to verify the amalgamation property in order to obtain a $(\mathscr{K}, \leqslant)$-homogeneous universal model. Thus the next step will be to show that we can amalgamate inside the class $\mathscr{K}$ with respect to the strong substructure relation '$\leqslant$'.

If $A \leqslant B$ and $A \leqslant C$, we denote by $B \otimes_A C$ the *trivial amalgamation* of B and C over A, obtained as the graph whose set of vertices is the disjoint union $(B \backslash A) \sqcup (C \backslash A) \sqcup A$ with incidence and colouring induced by B and C. Note that if A is non-empty, then the colouring of B and C will be consistent with that of A, and hence $B \otimes_A C$ will again be bipartite with respect to P. If A is empty, then $B \otimes C$ will be a disjoint union, and will hence again be bipartite. Thus the colouring never poses a problem in the amalgamation.

Theorem 3.15. *$(\mathscr{K}, \leqslant)$ satisfies the amalgamation property.*

Proof. Let $B_0 \leqslant B_1, B_2$, $\hat{B}_i = B_i \backslash B_0$. The proof is obtained by induction on $|\hat{B}_1| + |\hat{B}_2|$, and it follows the proof of [**1**, 3.2]. The corresponding proof in [**3**, Theorem 10] needs some amendments in step 4.

Case 1: There is a proper subset X of $\hat{B}_1$ with $y(XB_0) = y(B_0)$. Then $B_0 \leqslant B_0 X \leqslant B_1$, and we can use the induction hypothesis to amalgamate $B_0 X$ with B_2 over B_0 to get D', and then B_1 with D' to get D.

Case 2: Case 1 fails and $y(B_1/B_0) = 0$. Since Case 1 does not apply, we must have $\hat{B}_1$ 0-simple over B_0. Now the result follows from Lemma 3.17 below.

Case 3: Case 1 fails and $y(B_1/B_0) = 1$. If there is a proper subset X of B_1 with $y(X/B_0) = 1$, then the failure of Case 1 implies that $B_0 \leqslant B_0 X \leqslant B_1$ and we again use induction. If not, then B_1 is 1-simply algebraic over B_0, and we use Lemma 3.16 below.

Case 4: Assume that case 1 fails and $y(B_1/B_0) > 1$. By the failure of case 1 we must have $y(b/B_0) \geqslant 1$ for all $b \in \hat{B}_1$. First, suppose that there is some $b \in \hat{B}_1$ with $y(bB_0/B_0) = 1$. Therefore we can use case 3 to amalgamate b and B_2 over B_0 into some D'. Again, by the failure of case 1, it follows that bB_0 is strong in B_1 and we can use induction to amalgamate B_1 with D' over bB_0. (Note that in this case $b \in \hat{B}_1$ cannot be chosen arbitrarily because if $r(b, B_0) = 0$, then $y(bB_0/B_0) = n-1$, so for $n > 3$, the subgraph bB_0 need not be strong in B_1.)

If for all $b \in \hat{B}_1$ we have $y(bB_0/B_0) > 1$, we must have $r(b, B_0) = 0$ and $y(bB_0/B_0) = n-1$ for all $b \in \hat{B}_1$. Now let $b \in \hat{B}_1$ be arbitrary. Clearly, b and B_2 can be freely amalgamated over B_0 into some D'. Also, bB_0 is strong in B_1 because $r(\hat{B}_1, B_0) = 0$ and $y(C) \geqslant n-1$ for any $C \subset \hat{B}_1$. Using induction again, we amalgamate D' and $\hat{B}_1$ over bB_0 to obtain the required amalgam.

It is left to show the decisive lemmas.

LEMMA 3.16. *If $B_0 \leqslant B_1, B_2$ and $\hat{B}_1 = B_1 \backslash B_0$ is* 1-*simple over B_0, then $D = B_1 \otimes_{B_0} B_2 \in \mathscr{K}$ with $B_1, B_2 \leqslant D$.*

Proof. By Lemma 3.5, B_1 is obtained from B_0 by attaching a single new element b to some element a in B_0. Thus D is obtained from B_2 by attaching b at $a \in B_0$ as before. Therefore, clearly, $B_1, B_2 \leqslant D$. Also, it follows that D cannot contain any ordinary k-gons for any k that are not already contained in B_2, showing that D satisfies conditions (K1) and (K2). □

The proof of the following lemma is not correct in [**3**].

LEMMA 3.17. *If $B_0, B_1, B_2 \in \mathscr{K}$ with $B_0 \leqslant B_1, B_2$ and $\hat{B}_1 = B_1 \backslash B_0$ is* 0-*simple over B_0, then either $D = B_1 \otimes_{B_0} B_2 \in \mathscr{K}$ with $B_1, B_2 \leqslant D$, or there is an isomorphic copy of B_1 over B_0 inside B_2.*

Proof. Here, the proof of condition (K1) in [**3**, Lemma 13] is wrong. Suppose that D contains an ordinary k-gon for $k < n$. Then this $2k$-cycle cannot lie entirely inside either B_1 or B_2, and hence there must be paths γ_1, γ_2 inside $\hat{B}_1, \hat{B}_2$, respectively, connecting elements of B_0. Since $\hat{B}_1$ is 0-simple over B_0, γ_1 must consist of $n-2$ elements and must be all of $\hat{B}_1$. However, B_0 is strong in B_2; hence B_2 cannot contain arcs of lengths less than $n-2$, so γ_2 must also consist of $n-2$ elements, and γ_1 and γ_2 must meet in some elements $a_1, a_2 \in B_0$. However this implies that B_0 extended by γ_2 is an isomorphic copy of B_1 over B_0 inside B_2, and we have achieved the amalgamation.

Therefore we may assume now that D does not contain an ordinary k-gon for $k < n$. Using Lemma 2.4(ii), it is immediate that B_1 and B_2 are strong in D.

Assume now that there is a set $C \subseteq A$ that contains an ordinary k-gon Γ with $k > n$ and $y(C) < 2n+2$. Set $C_i = C \cap B_i$ and note that $C_i \leqslant C$, for $i = 1, 2$. If either $y(C_1) \geqslant 2n+2$ or $C_2 \geqslant 2n+2$, then $y(C) \geqslant 2n+2$, and we have a contradiction. Therefore, both C_1 and C_2 are of the form described in Lemma 3.12. Since C_0 contains at least two elements and C contains an ordinary k-gon with $k > n$, we can check the different possibilities for C_1 and C_2.

Case 1: Suppose that C_1 is just an ordinary n-gon, or an ordinary n-gon with an arc of length $n-1$ attached. Then any two elements of C_1 have distance at most n. If C_2 is either an ordinary n-gon with or without an arc or a single point attached, the distance of elements of C_2 is at most $n+1$, and hence C would not contain any k-gon for $k > n$. Therefore C_2 must be a string of $n+3$ elements. Then C_0 contains exactly two elements and $y(C) = y(C_1)+2 = 2n+2$, a contradiction.

Case 2: Suppose now that C_1 is an ordinary n-gon with a single element attached to it. Then any two elements of C_2 have distance at most $n+1$, and there are exactly two elements that have distance $n+1$. Since C contains a k-gon for $k > n$, C_2 must also contain elements of distance at least $n+1$. Hence C_2 is also an ordinary n-gon with a single extra element attached, in which case also C_2 contains exactly two elements at distance $n+1$ and these must be the elements of C_0. Then we obtain C from C_1 by attaching one arc of length n and one arc of length $n-1$, and therefore $y(C) = y(C_1)+3 = 2n+4$. Alternatively, C_2 is a string of at least $n+2$ elements, and hence $y(C) \geqslant y(C_1)+1 = 2n+2$, again yielding a contradiction.

Case 3: Finally, suppose that C_1 is a string of either $n+2$ or $n+3$ elements. Then either C_2 is also a string of the same length, in which case C is just an ordinary $(n+1)$- or $(n+2)$-gon and $y(C) \geqslant 2n+2$, or C_2 is an ordinary n-gon with or without an extra arc or extra point, and we can apply one of the previous cases with the roles of C_1 and C_2 exchanged.

This finishes the proof of Lemma 3.17 and Theorem 3.15. □

It now follows that there is a countable $(\mathscr{K}, \leqslant)$-homogeneous universal structure M, and it is easy to check that M is a generalized n-gon. Clearly, M is a bipartite graph, which does not contain ordinary k-gons for $k < n$. Next, we have to show that any two elements $a, b \leqslant M$ are connected by a path of length at most n (and are of the same colour if n is even, and of different colours in cases where n is odd). Say $a, b \in A \leqslant M$ (such a set A always exists since $y(B) > n-2$ for all finite subgraphs B of M). If a, b are connected in A by a path of length at most n, we are done. Otherwise, by Lemma 3.14, $B = A \cup \{b_1, \ldots, b_{n-1}\}$ with incidence $(a, b_1), \ldots, (b_{n-1}, b)$ is a strong extension of A and also in $\mathscr{K}$, so there is an embedding $f: B \longrightarrow M$ with $f(B) \leqslant M$. By condition (H3) the restriction of f to A extends to an automorphism of M. However, since $f(a)$ and $f(b)$ are connected by a path of length at most n, the same must be true of a and b.

To see that M is thick, it suffices to observe that the graph $\{(a, b), (a, c), (a, d)\}$ with $P(a)$ is in $\mathscr{K}$. Since all single elements are strong in M, all $x \in M$ with $P(x)$ are conjugate under an automorphism of M. It follows that all elements in P are incident with at least three elements, and similarly for all elements not in P. It is left to show that all ordered $(n+1)$-gons are conjugate under an automorphism of M. However, this follows easily from the fact that by condition (K2) any ordinary $(n+1)$-gon contained in M is a strong subgraph of M. Since any two ordered ordinary $(n+1)$-gons are isomorphic, this isomorphism lifts to an automorphism of M.

Now Theorem 3.1 follows from Theorem 3.15 and the preceding paragraphs. □

REMARKS 3.18. (i) The n-gons with this property are far from being unique: in Section 4 we will show that there are at least countably many non-isomorphic countable n-gons that are $(n+1)$-gon transitive.

(ii) Clearly, if the automorphism group acts transitively on the set of ordered ordinary $(n+1)$-gons, it also acts transitively on the flags, and on the set of ordered ordinary n-gons. Thus, as pointed out in the introduction, this class of n-gons gives rise to a new class of BN-pairs.

(iii) The action of the automorphism group is not regular on the ordered ordinary $(n+1)$-gons; in fact, the stabilizer of any $(n+1)$-gon is at least countable. This is easily seen through the following observation. Consider an ordinary $(n+1)$-gon $\Gamma = (x_0, \dots, x_{2n-1})$ and k new elements $b_1, \dots, b_k$ incident with x_0. This graph can be identified with some $A \leqslant M$, which easily implies that Γ extended by each of the b_i is strongly embedded into M. Clearly, $\Gamma b_i \cong \Gamma b_j$ for all i, j and thus there is an automorphism of M fixing Γ and taking b_i to b_j.

(iv) It is a surprising result due to Feit and Higman that finite generalized n-gons exist only for $n = 3, 4, 6$ and 8. It is curious to notice that the n-gons constructed in this section do not contain any finite generalized n-gons. This can easily be seen by computing the y-value of a finite generalized n-gon using the coordinatization as described, for example, in [**11**] or [**12**, 1.6] to compute the cardinalities of the sets of points, lines and *flags*, that is, ordered pairs (a, ℓ) with a and ℓ incident. However, it might be possible to embed the smallest generalized quadrangle (with three points per line and three lines through a point) into the projective plane just constructed, and the smallest finite generalized hexagon into the generalized k-gons with $k < 6$. For $n = 8$, the smallest octagon (with three points per line and five lines through a point) cannot be embedded in any of these k-gons for $k > 2$. This observation is due to Norbert Knarr and Hendrik Van Maldeghem.

4. *Finite Morley rank*

In this section we modify the construction by putting an extra condition on the class $\mathcal{K}$ to ensure that the n-gons have finite Morley rank. We need some control over how algebraic closure behaves, and for this reason we are introducing the following multiplicity function and condition (K3) from [**1**, **3**].

DEFINITION 4.1. Fix a function μ from pairs (A, B) of finite graphs with B minimally 0-simple over A into the natural numbers, which satisfies the following properties.

(M1) If the isomorphism types of (A, B) and (A', B') are the same, then $\mu(A', B') = \mu(A, B)$.

(M2) If $A = \{a_0, a_{n-1}\}$ and $B = \{a_1, \dots, a_{n-2}\}$ with AB having edges

$$(a_0, a_1), \dots, (a_{n-2}, a_{n-1}),$$

then $\mu(A, B) = 1$; otherwise $\mu(A, B) \geqslant \max\{y(A), n\}$.

For any finite graph N and a pair (A, B) where $A \subseteq N$ and B is minimally 0-simple over A, we define $\chi_N(A, B)$ to be the number of pairwise disjoint graphs $B' \subseteq N$ such that $B' \cong_A B$.

Definition 4.2. Now let $\mathcal{K}^{\mu}$ be the class of finite graphs bipartite with respect to P, and satisfying conditions (K1) and (K2) and the following extra condition.

(K3) For every pair of subgraphs (A, B) of N with B minimally 0-simple over A, we have $\chi_N(A, B) \leqslant \mu(A, B)$.

Clearly, $\mathcal{K}^{\mu}$ is closed under substructures and, as before, the empty set and singletons are strong in any graph. We have to check that condition (K3) survives the amalgamation and the extension by an arc of length $n-1$ or $n-2$ as in Lemma 3.14.

Lemma 4.3. *Let $A \in \mathcal{K}^{\mu}$ and let B be an extension of A by an arc of length $n-1$ or $n-2$ as in Lemma* 3.14. *Then $A \leqslant B$ and $B \in \mathcal{K}^{\mu}$.*

Proof. Of course, conditions (K1) and (K2) still hold for B and $A \leqslant B$ as before. For condition (K3) we have to check that for any pair $(F, C) \subseteq B$ with C minimally 0-simple over F we have $\chi_B(F, C) \leqslant \mu(F, C)$. This is the content of [**3**, Lemma 9]. □

Theorem 4.4. *The class $(\mathcal{K}^{\mu}, \leqslant)$ has the amalgamation property.*

Proof. Following the proof of Theorem 3.15 we have to check that Lemma 3.17 and Lemma 3.16 still work for $\mathcal{K}^{\mu}$.

Thus suppose that $B_0 \leqslant B_1, B_2$ and $\hat{B}_1 = B_1 \backslash B_0$ is a 1-simple extension of B_0, say $\hat{B}_1 = \{b\}$. Conditions (K1) and (K2) have already been checked in Lemma 3.16. For condition (K3) it suffices to note that no 0-simple extension inside $D = B_1 \otimes_{B_0} B_2$ can contain b, and if $C \subseteq D$ is minimally 0-simple over $F \subseteq D$ with $b \in F$, then $1 = \chi_D(C, D) \leqslant \mu(C, D)$ since b is incident with a unique element of B_0.

If $\hat{B}_1 = B_1 \backslash B_0$ is 0-simple over B_0, then it was shown in Lemma 3.16 that $D = B_1 \otimes_{B_0} B_2$ is in $\mathcal{K}$ unless $\hat{B}_1$ is an arc of length n_2 of which there is an isomorphic copy over B_0 inside B_2. In this case B_2 is the required amalgam, and we do not have to worry about condition (K3). If $D \in \mathcal{K}$, it was shown in [**3**, Lemma 13] that either D also satisfies condition (K3), or there is an isomorphic copy of $\hat{B}_1$ over B_0 inside B_2. □

Obviously, as in Section 3, the $(\mathcal{K}^{\mu}, \leqslant)$-homogeneous universal model M whose existence follows from Theorem 4.4 is a generalized n-gon.

It is a well-known fact that any generalized n-gon is in the definable closure of a *point row* and a *line pencil* plus a finite number of further elements (see, for example, [**12**, 1.6]). This is due to coordinates that can be introduced for n-gons in much the same way as for projective planes. The set $D(a) = \{b \in M; bIa\}$ is called a *line pencil* if a is a point, and is known as a *point row* if a is a line. Note that there exist definable bijections between any two line pencils (or, respectively, between any two point rows). For odd n, the situation is even better: there are definable bijections between any point row and any line pencil (given by so-called '*projectivities*'). (See, for example, [**11**, **12**] for more background and model-theoretic properties.) Therefore, for odd n, any n-gon is in the definable closure of a single point row and two additional points.

As in [**1**, **3**], for M to have finite Morley rank it therefore suffices to show that for any element $a \in M$ the set of elements incident with it is strongly minimal. For n odd, this is already enough to conclude that the n-gon is almost strongly minimal, as pointed out above.

The author has observed that it follows easily from a criterion given in [**6**] that M

is countably saturated; the argument was later included in [**3**]. The same proof as that given in [**1**] now shows that the point rows and line pencils are strongly minimal.

THEOREM 4.5. *Let M be the $(\mathcal{K}^\mu, \leqslant)$-homogeneous-universal model. Let $b \in M$ and $\ell = \{a \in M; aIb\}$. Then ℓ is strongly minimal. Hence there are generalized n-gons of finite Morley rank for all $n < \omega$.*

For even n, the strong minimality of the point rows and line pencils is not enough to imply that the n-gon is almost strongly minimal. However, the almost strong minimality follows from the following, more-involved, geometric argument.

THEOREM 4.6. *Let M be the $(\mathcal{K}^\mu, \leqslant)$-homogeneous universal model. Then M is almost strongly minimal.*

Proof. We already know that the point rows and line pencils are strongly minimal. As pointed out above, for odd n, it follows that the n-gon is almost strongly minimal for trivial reasons.

Suppose now that n is even. We will show that there is a definable surjective finite-to-1 map from a line pencil to a point row, showing that the line pencil is in the algebraic closure of the point row. By the remarks preceding Theorem 4.5 this is enough for it to be concluded that the n-gon is almost strongly minimal.

First, notice that we always have the following special situation of a minimally 0-simple pair (A, B): choose $p_1, p_2, p_3 \in P(M)$ with $d(p_i, p_j) = n$, for $1 \leqslant i \neq j \leqslant 3$. Let A be the graph consisting of these three points (without any incidence holding), so $y(A) = 3(n-1)$. Now extend A by an element $a \in P(M)$ with $d(a, p_1) = 2$ and $d(a, p_2) = d(a, p_3) = n-2$, and let B be the graph containing Aa and the shortest paths between a and the p_i, for $i = 1, 2, 3$. It is easy to calculate $y(B)$, that is, B can be described as a path of length n from p_1 through a to p_2 with a path of length $n-2$ from a to p_3 attached, such that the path from a to p_2 and the path from a to p_3 are disjoint. Thus, using the formula described in the set-up, $y(B) = (2n-1)+(2n-4)-(n-2) = 3n-3$, showing that $\hat{B} = B \backslash A$ is a 0-simple expansion of A. Clearly, $\hat{B}$ is even *minimally* 0-simple over A. Since M is saturated, there are $m = \mu(\hat{B}, A)$ copies of $\hat{B}$ over A in M, that is, M contains exactly m elements b lying in the same configuration with p_1, p_2, p_3 as a. Note also that m does not depend on the choice of p_1, p_2, p_3 and a because μ depends only on the configuration.

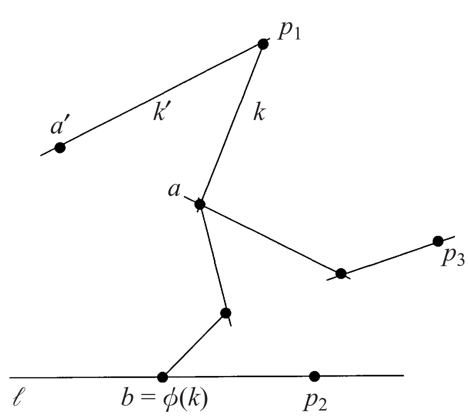

FIGURE 1.

Now let p_1, p_2, p_3 be as above, and fix a line ℓ through p_2 with $d(\ell, p_1) = d(\ell, p_3) = n-1$ (see Figure 1 for $n = 6$, where for clarity we have represented points by points and lines by edges, rather than drawing the incidence graph). Consider the following map ϕ from $D(p_1)$ into $D(\ell)$. For any line $k \in D(p_1)$ we have $d(k, p_3) = n-1$, so there is a unique point a on the line k with $d(a, p_3) = n-2$. Similarly, there is a unique point b on the line ℓ with $d(a, b) = n-2$. Put $\phi(k) = b$.

As we vary k, we vary a and it follows from the previous paragraph that ϕ is an m-to-1 function with $m = \mu(\hat{B}, A)$ as above. By the $(\mathscr{K}^\mu, \leqslant)$-homogeneity of M, the image of ϕ must be all of $D_1(\ell)$, and therefore $D_1(p_1)$ is in the algebraic closure of $D_1(\ell)$. Thus M is almost strongly minimal. □

Given any elements p, q with $d(p, q) = n$, the *trace* of p with respect to q is defined as $p^q = \{x \mid d(x, p) = 2 \wedge d(x, q) = n-2\}$. The following corollary is an immediate consequence of the proof of Theorem 4.6.

Corollary 4.7. *The generalized n-gon has the property that any two traces p^q and $p^{q'}$ meet in exactly $\mu(\hat{B}, A)$ elements* (*where $\hat{B}, A \subseteq M$ are as in the proof of Theorem* 4.6).

Note that we have countably many choices for $\mu(\hat{B}, A) \geqslant 3n-3$ with $\hat{B}$ and A as in the previous paragraph. Since the $(\mathscr{K}^\mu, \leqslant)$-homogeneous universal model is saturated and satisfies $\chi_{M^\mu}(\hat{B}, A) = \mu(\hat{B}, A)$ (see [**1**, 5.1]), for different choices of $\mu(\hat{B}, A)$ the corresponding models M^μ cannot be isomorphic. Therefore we have the following corollary.

Corollary 4.8. *For all n, there exist at least countably many non-isomorphic almost strongly minimal generalized n-gons for which the automorphism group acts transitively on the set of ordered ordinary* $(n+1)$*-gons contained in it.*

Clearly, these n-gons are also flag-transitive (in contrast to [**3**, Proposition 28]) and homogeneous for the set of ordered ordinary n-gons. Thus, this class of n-gons gives rise to a new class of BN-pairs acting on almost strongly minimal generalized polygons. However, it can be shown exactly as in [**1**, 6.4] that none of these n-gons interprets an infinite group. Therefore, the automorphism group and the BN-pair cannot be definable in the polygon. As pointed out in the introduction, if a generalized n-gon has strongly minimal point rows and line pencils and a definable automorphism group acting transitively on ordered ordinary n-gons, then $n = 3$, 4 or 6 and the group is a simple algebraic group over an algebraically closed field. Thus the construction cannot be modified in order to obtain examples of (automorphism) groups of finite Morley rank which are not algebraic.

All the proofs in this section obviously also work for the class of bipartite graphs satisfying conditions (K1) and (K3) and the following weakened form of condition (K2).

(K2′) If $B \subseteq A$ is non-empty, then $y(B) \geqslant n-1$.

This class gives rise to almost strongly minimal generalized n-gons that are no longer flag transitive (by [**3**, Proposition 28]). For this class it can be shown, similarly to [**1**, **3**], that by varying the multiplicity function μ we obtain $2^{\aleph_0}$ many non-isomorphic such n-gons. For odd n, the proof in [**3**] suffices. For even n, the graph in [**3**, p. 506] with $k = n/2$ is minimally 0-simple over the graph consisting of c and a vertex incident with c. Then the same argument shows that for even n we also obtain

$2^{\aleph_0}$ many non-isomorphic n-gons, and that none of these interprets an infinite group. Therefore we now have the following result.

THEOREM 4.9. *There are $2^{\aleph_0}$ many non-isomorphic almost strongly minimal generalized n-gons for all $n \geqslant 3$ not interpreting an infinite group.*

REMARK 4.10. It is worth noticing that all the known examples of generalized polygons of finite Morley rank are almost strongly minimal. If the Morley rank of any point row and line pencil is necessarily 1, this would have strong implications. In particular, any split sharply 2-transitive group of finite Morley rank would then be of the form $K_+ \rtimes K^*$ for some algebraically closed field K (see [**12**]).

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Mathematisches Institut
Universität Würzburg
Am Hubland
97074 Würzburg
Germany

tent@mathematik.uni-wuerzburg.de