Def A sequence (f_n) of real valued functions on a set X is <u>independent</u> if there are reals $a < b$ such that

$$\bigcap_{n \in P} f_n^{-1}(-\infty, a) \cap \bigcap_{n \in M} f_n^{-1}(b, \infty) \neq \emptyset$$

for all finite disjoint subsets $P, M \subseteq \mathbb{N}$.

Def A bounded family F of real valued functions on a set X is <u>tame</u> if F does not contain an independent sequence.

Examples (of independent sequences, i.e. of nontame families)

1) Projections $\{\pi_n : \{0,1\}^{\mathbb{N}} \to \{0,1\}\}_{n \in \mathbb{N}}$
2) Rademacher functions
   $\{r_n : [0,1] \to \mathbb{R}\}_{n \in \mathbb{N}}$
   $r_n(x) = \operatorname{sgn}(\sin(2^n \pi x))$

Def (Rosenthal)
Let $\{f_n : X \to \mathbb{R}\}$ be a uniformly bounded sequence. It is an <u>l_1-sequence</u> on X if there exist $A > 0$ such that for all $n \in \mathbb{N}$ and $c_1, c_2, \dots, c_n \in \mathbb{R}$ we have

$$A \sum_{i=1}^{n} |c_i| \leq \| \sum_{i=1}^{n} c_i f_i \|_\infty .$$

That means that the closed linear span of (f_n) is isomorphic to l_1

<u>Theorem</u> (Rosenthal, 1974)

Let (f_n) be a bounded sequence in a real Banach space B. Then either

(i) there is a subsequence (f_n') of (f_n), which is weak-Cauchy

(i.e. $\lim_{n\to\infty} b^*(f_n)$ exists for all $b^* \in B^*$, the dual of B)

or

(ii) there is a subsequence (f_n') of (f_n), which is equivalent to the usual l_1-basis.

<u>Def</u> Let (X, τ) be a topol. space.
Y - metric space

a) A function $f : X \to \mathbb{R}$ is <u>fragmented</u> if for every $A \neq \emptyset$, $A \subseteq X$ and every $\varepsilon > 0$ there is open $O \subseteq X$ s.t. $O \cap A \neq \emptyset$ and $\operatorname{diam} f(O \cap A) < \varepsilon$

b) $f : X \to Y$ is <u>barely continuous</u> if for every $\neq \emptyset$ closed $A \subseteq X$ $f \restriction A$ has at least one point of continuity

c) $f: X \to Y$ is of <u>Baire class 1</u> if preimages of open sets are F_σ

<u>Note</u>: The pointwise limit of continuous funtions are of Baire class 1. In case X is seperable metrizable and $Y = \mathbb{R}$, the converse is true (see Kechris (24.10)).

Corollary 2.6 in GM Trans. Glasner-Megrelishvili and (24.15) Kechris

<u>Theorem</u> X - Polish space, $Y = \mathbb{R}$

Then $f: X \to \mathbb{R}$ is fragmented iff it is barely continuous iff it is of Baire class 1

GM "More on tame dyn. systems" Thm. 2.4

<u>Theorem</u> Let X be a compact space and $F \subseteq C(X) = C(X, \mathbb{R})$ a bounded subset. TFAE

(1) F does not contain an l_1-sequence

(2) F is a tame family

(3) Each sequence in F has a pointwise convergent subsequence in $\mathbb{R}^X$
(i.e. F is sequentially precompact)

(4) The pointwise closure of F in $\mathbb{R}^X$ consists of fragmented maps.

we show (1) => (2) (Prop. 4 in Rosenthal)

Every bounded independent sequence is an l_1-seqence:

$r, \delta \in \mathbb{R}$, $\delta > 0$ $\quad a = r$, $\quad b = r + \delta$

Take $(c_i)_{i \in \mathbb{N}}$ with $\sum_i |c_i| = 1$ and only finitely many $\neq 0$

It suffices to show that there is $s \in \mathbb{N}$

$$|\sum c_i f_i(s)| \geqslant \frac{\delta}{2} \qquad (\Delta)$$

(Then take $A = \frac{\delta}{2}$ and note that

$$A \sum_i |c_i| = \frac{\delta}{2} \leqslant |\sum c_i f_i(s)| = \|\sum c_i f_i\|_\infty)$$

Let $G = \{i : c_i > 0\}$ and $B = \{i : c_i < 0\}$.

Let also $A_n = \{x : f_n(x) > \delta + r\}$

and $B_n = \{x : f_n(x) < r\}$

By the independence of (f_i) there are

$x \in \bigcap_{i \in G} A_i \cap \bigcap_{i \in B} B_i$ and

$y \in \bigcap_{i \in B} A_i \cap \bigcap_{i \in G} B_i$.

Suppose first that $r \geqslant 0$ and set $B' = \{i \in B : f_i(x) > 0\}$.

Then

$$\sum_{i \in B} c_i f_i(x) \geqslant \sum_{i \in B'} c_i f_i(x) > -r \sum_{i \in B'} |c_i| \geqslant \sum_{i \in B} |c_i| (-r)$$

Similarly $\quad -\sum_{i \in G} c_i f_i(y) \geqslant \sum_{i \in G} |c_i| (-r)$.

Thus by defs of x and y

$$\sum c_i f_i(x) \geqslant \sum_{i \in G} |c_i| (\delta + r) + \sum_{i \in B} |c_i| (-r) \qquad (*)$$

and $$-\sum c_i f_i(y) \geqslant \sum_{i \in B} |c_i| (\delta + r) + \sum_{i \in G} |c_i| (-r) \qquad (**)$$

We check that if $r<0$, then $(*)$ and $(**)$ hold as well.

Sum of RHS of $(*)$ and $(**)$ $= \delta$, so max of LHS is $\geq \frac{\delta}{2}$

So (Δ) holds for $s=x$ or $s=y$. ∎

For $(1) \Rightarrow (2)$, from Rosenthal's paper, if (f_n) has no independent subsequence then it has a pointwise convergent subsequence. From Lebesgue convergence theorem a bdd pointwise conv. sequence is weak Cauchy. Now use Rosenthal's dychotomy.

$(1) \Leftrightarrow (3) \Leftrightarrow (4)$ is in Talagrand 14.1.7, book from 1984

Let G be a top. group acting on compact X by homeomorphisms. For short write $G \curvearrowright X$

<u>Def</u> $f \in C(X)$ is <u>tame</u> iff the family $G \cdot f$ is tame. $\} g \cdot f(x) =: f(g^{-1}x)$

<u>Def</u> (Ellis semigroup) G - top. group, X - compact $G \curvearrowright X$ by homeom.

For every $g \in G$ we have a function

$f_g : X \to X$ given by $x \longmapsto g \cdot x$.

Then the <u>Ellis semigroup</u> is the pointwise closure of $\{f_g\}_g$ in X^X.

Def A compact space K is *Rosenthal* if it is homeomorphic to a subspace of $B_1(Z)$, Baire class one functions of some Polish space Z.

Def A $G \curvearrowright X$ is *tame* if $E(X)$ is Rosenthal.

Proposition Glasner-Megrelishvili-Uspenskij Thm 6.3

X - compact metric

A $G \curvearrowright X$ is tame (i.e $E(X) \underset{\text{embeds}}{\hookrightarrow} B_1(\underset{\text{Polish}}{Z}) = B_1(Z, \mathbb{R})$)

iff

$E(X) \subseteq \underset{\text{Baire class 1 functions}}{B_1(X, X)}$

Proof

$(\Leftarrow)$ $B_1(X,X) \subseteq X^X$

$X \hookrightarrow \mathbb{R}^{\mathbb{N}}$

So $B_1(X,X) \subseteq B_1(X, \mathbb{R}^{\mathbb{N}}) = B_1(X \times \mathbb{N}, \mathbb{R}) = B_1(X \times \mathbb{N})$

So $E(X) \hookrightarrow B_1(X \times \mathbb{N})$.

Take $Z = X \times \mathbb{N}$.

$(\Rightarrow)$ If $E(X)$ is Rosenthal, then by Bourgain-Fremlin-Talagrand $E(X)$ is Fréchet.

(Compact K is *Fréchet* if for every $A \subseteq K$ and every $x \in \bar{A}$ there exists a sequence of elements of A which converges to x.). So every $p \in E(X)$ is the pointwise limit of continuous functions (recall: $G \curvearrowright X$ by homeom.).

Theorem (Glasner - Megrelishvili) A dynamical BFT - dichotomy

Let X be a compact metric space,

$G \curvearrowright X$, $E = E(X)$.

$\uparrow$
top. group, we act by homeomorphisms

Then either

(1) E is a separable Rosenthal compact space

In that case $|E| \leq 2^{\aleph_0}$

or

(2) E contains a homeomorphic copy of $\beta\mathbb{N}$ (i.e. the space of all ultrafilters on $\mathbb{N}$)

In that case $E = 2^{2^{\aleph_0}}$.

Theorem (Bourgain - Fremlin - Talagrand)

X - Polish $(f_n) \subseteq C(X)$ pointwise bdd (i.e. $\forall x \in X$ $(f_n(x))$ is bdd in $\mathbb{R}$). K = pointwise closure of (f_n) in $\mathbb{R}^X$. Then either

(i) $K \subseteq B_1(X)$ or

(ii) K contains a homeomorphic copy of $\beta\mathbb{N}$

Examples

$H(2^{\mathbb{N}}) \curvearrowright 2^{\mathbb{N}}$ is not tame

$H(S^1) \curvearrowright S^1$ is tame