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Approximate subgroups

APPROXIMATE SUBGROUPS

by   Jean-Cyrille Massicot & Frank O. Wagner

Abstract. — Given a definably amenable approximate subgroup A of a (local) group in some first-order structure, there is a type-definable subgroup H normalized by A and contained in A^4 such that every definable superset of H has positive measure.

Résumé (Sous-groupes approximatifs). — Étant donné un sous-groupe approximatif A définissablement moyennable d'un groupe (local) dans une structure du premier ordre, il y a un sous-groupe H type-définissable normalisé par A et contenu dans A^4 tel que tout ensemble définissable contenant H est de mesure positive.

Contents

Introduction

Let G be a group and $K > 0$ an integer, a subset $A \subseteq G$ closed under inverse is a *K-approximate subgroup* if there is a finite subset $E \subseteq G$ with $|E| \leqslant K$ such that $A^2 = \{ab : a, b \in A\} \subseteq EA$. Then $A^n \subseteq E^{n-1}A$.

Following work of Hrushovski [6] and many others, Breuillard, Green and Tao [1] have classified finite approximate subgroups of local groups (see [2] for an excellent survey). In particular, they show that there is an approximate subgroup $A^* \subseteq A^4$ and an actual A^*-invariant subgroup $H^* \subseteq A^*$ such that

– finitely many left translates of A^* cover A, and
– $\langle A^* \rangle / H^*$ is nilpotent.

The result and its proof are inspired not only by Gleason's and Yamabe's solution of Hilbert's 5th problem [4, 10] and its extension to the local context by Goldbring [5], but also by Gromov's Theorem on groups with polynomial growth, and is indeed a way to generalize this theorem. The three articles [6, 1, 2] provide some applications to geometric group theory.

The proof proceeds by considering a non-principal ultraproduct of a sequence of finite counterexamples $(A_n : n < \omega)$ with $|A_n| \to \infty$, giving rise to a pseudofinite counterexample A. Then an A-invariant subgroup $H \subseteq A^4$ is constructed such that $\langle A \rangle / H$ is locally compact. From Yamabe's theorem on the approximation of locally compact groups by Lie groups, it follows that there are suitable A^* and H^* such that $\langle A^* \rangle / H^*$ is a real Lie group; using pseudofiniteness, the final result is obtained.

The construction of the locally compact quotient $\langle A \rangle / H$ and the Lie model $\langle A^* \rangle / H^*$ was first shown by Hrushovski [6] by model-theoretic means inspired by and reminiscent of stability theory. Breuillard, Green and Tao use instead a (subsequent) theorem of Sanders [9] from finite combinatorics, constructing successively the traces of the definable supersets of H on the various finite approximate groups A_n. Using the ultraproduct construction, the pseudofinite counting measure and the translation-invariant ideal of measure zero sets, Hrushovski's theorem allows to recover Sanders' result at least qualitatively.

Definability. — The topology of $\langle A \rangle / H$ was constructed analytically in [1], but has a natural model-theoretic interpretation already given in [6]. Recall that a subset of the ultraproduct is *definable* if it is the set of realizations of some first-order formula (usually involving quantifiers); it is *type-definable* if it is given as the intersection of a countable (say) family of definable sets. For instance, the centralizer of a group element g is defined by the formula $xg = gx$, and if G is a group defined by a formula $\varphi(x)$, its centre $Z(G)$ is defined by the formula $\varphi(x) \wedge \forall y\, (\varphi(y) \to xy = yx)$. On the other hand, the group generated by an element g, or the centre of a type-definable group, are in general not even type-definable.

If H is a type-definable normal subgroup of $\langle A \rangle$, it has *bounded index* if any definable superset of H covers any definable subset of $\langle A \rangle$ in finitely many translates. We can then endow the quotient $\langle A \rangle / H$ with the *logic topology* whose proper closed subsets are precisely those subsets whose preimage in $\langle A \rangle$ is type-definable; this will turn it into a locally compact topological group.

Of course, (type-)definability strongly depends on the language: if we *expand* the structure, for instance by adding predicates for certain subsets, there will be more definable sets. While the group H constructed by Hrushovski is naturally type-definable in the structure given, it only becomes so in Sanders' Theorem (either in the ultra-product or in a suitable version using a bi-invariant measure instead of cardinality) after such an expansion of language. Hrushosvki, on the other hand, assumes the existence of a bi-invariant S1 ideal (which should be thought of as the ideal of sets of measure zero) which in addition is automorphism invariant; in order for an ideal to become automorphism invariant, one would generally also have to expand the language.

Such an expansion does not matter much if the structure to start with is arbitrary, but should be avoided if the initial structure has particular model-theoretic properties one wants to preserve.

For example, Eleftheriou and Peterzil [3] construct H type-definably without expanding the language in the case when A is definable in an o-minimal expansion of an ordered group (such as the field of real numbers with exponentiation), provided that $\langle A \rangle$ is abelian. Pillay [8], generalizing an argument in [7], generalizes this result if A is definable in a theory without the independence property, and is definably amenable (see below). In fact, in this setting there is a unique minimal choice for H, namely the unique minimal type-definable subgroup of bounded index, $\langle A \rangle^{00}$.

DEFINABLE AMENABILITY. — We shall call A *definably amenable* if $\langle A \rangle$ carries a finitely additive left-invariant measure μ on its definable subsets such that $\mu(A) = 1$. We can now state the main theorem of our paper.

THEOREM 1. — *In any group G, a definably amenable approximate subgroup A gives rise to a type-definable subgroup $H \subseteq A^4$, such that finitely many left translates of any definable superset of H cover A. Hence a definably amenable approximate subgroup allows a real Lie model without expanding the language.*

Our proof follows the ideas of Sanders, except that we use the measure not to define the subgroup we obtain, but only to show that the formulas we construct in the original language have the necessary properties. We conjecture that even without the definable amenability assumption a suitable Lie model exists.

The classification of approximate subgroups of real Lie groups is still an open problem. Since in a real Lie group any compact neighbourhood of the identity is an approximate subgroup, in particular no nilpotency (or even solubility) result can hold in general. We hope that under additional model-theoretic assumptions on the original structure, a partial classification might be easier to achieve.

We will end this introduction with two useful remarks. The first one concerns essentially the only (but crucial) use of model theory in this paper. The second one is an easy generalization which played a key role in the conclusion of [1], and thus seems worth noticing.

We shall assume that all structures under consideration are ω^+-*saturated*, which means that any countable intersection of definable sets is non-empty as soon as all finite subintersections are. All non-principal ultraproducts are ω^+-saturated; the compactness theorem of model theory implies that we can replace any structure M by a superstructure M^* satisfying the same first-order sentences with parameters in M (an elementary extension) which in addition is ω^+-saturated.

As in [1], the results in this paper remain true if G is only a *local group*, i.e. a set closed under inverse and endowed with a multiplication such that the product of up to 100 elements is well-defined and fully associative. For this, one can check throughout the proofs that one never needs to multiply more than 100 elements of A.

1. A type-definable version of Sanders' Theorem

Definition 2. — A subset A of a (local) group G is *symmetric* if $1 \in A$, and $a^{-1} \in A$ for all $a \in A$.

If $K < \omega$, a symmetric subset A of G is a K-*approximate subgroup* if

$$A^2 = \{aa' : a, a' \in A\}$$

is contained in K left cosets of A. An *approximate subgroup* is a symmetric subset which is a K-approximate subgroup for some $K < \omega$.

From a model-theoretic point of view, a definable approximate subgroup A is just a symmetric *generic set* in $\langle A \rangle$, i.e. a definable symmetric subset of $\langle A \rangle$ such that every definable subset of $\langle A \rangle$ is covered by finitely many left translates of A.

Definition 3. — A definable approximate subgroup A is *definably amenable* if there is a left translation-invariant finitely additive measure μ on the definable subsets of $\langle A \rangle$ with $\mu(A) = 1$.

Note that by ω^+-saturation, for any definable subset X of $\langle A \rangle$ there is $n < \omega$ with $X \subseteq A^n$. So if A is a definably amenable approximate subgroup, then $\mu(X) \leqslant \mu(A^n) < \infty$.

Remark 4. — If $\lim_{n \to \infty} \mu(A^n) < \infty$, then there is $n < \omega$ with $A^n = \langle A \rangle$.

Proof. — Suppose not. Then for every $n < \omega$ there is $a_n \in A^{n+1} \smallsetminus A^n$. But then $(a_{3k}A : k \leqslant n)$ is a sequence of disjoint left translates of A inside A^{3n+2}, whence $\mu(A^{3n+2}) \geqslant (n+1)\,\mu(A) = n+1$, a contradiction. □

For the remainder of the paper we fix a K-approximate subgroup A of a (local) group G, and consider the structure whose domain is G, with a predicate for A, and with group multiplication (which is a partial map in case G is only local). We assume that $A^n \subseteq G$ for all $n < \omega$ (in fact $n \leqslant 100$ would be enough). Definability and type-definability will be with respect to this structure.

We assume that A is definably amenable with $\lim_{n \to \infty} \mu(A^n) = \infty$. We also fix a set E of size K with $A^2 \subseteq EA$.

Fact 5 (Ruzsa's covering lemma). — *Let $X, Y \subseteq G$ be definable such that $\mu(XY) \leqslant K\mu(Y)$. Then $X \subseteq ZYY^{-1}$ for some finite $Z \subseteq X$ with $|Z| \leqslant K$.*

Proof. — If $X = \varnothing$ there is nothing to show. Otherwise, consider a finite subset $Z \subseteq X$ such that $zY \cap z'Y = \varnothing$ for all $z \neq z'$ in Z. By left invariance,

$$|Z|\,\mu(Y) = \mu(ZY) \leqslant \mu(XY) \leqslant K\mu(Y),$$

so $|Z| \leqslant K$, and there is a maximal such Z. But then for any $x \in X$ there is $z \in Z$ with $zY \cap xY \neq \varnothing$ by maximality, whence $x \in zYY^{-1}$. □

DEFINITION 6. — A definable subset $B \subset \langle A \rangle$ is *wide in A* if A is covered by finitely many translates of B.

Two approximate subgroups are said to be *equivalent* if each one is contained in finitely many translates of the other.

We will sometimes make explicit the finite constants and say that B is L-wide in A, or that A and A^* are L-equivalent.

LEMMA 7. — *Let $B \subset \langle A \rangle$ be definable.*

(1) *If $\mu(B) > 0$, then BB^{-1} is wide in A and symmetric.*

(2) *If B is wide in A and symmetric, then B is also an approximate subgroup equivalent to A.*

Proof

(1) Clearly, BB^{-1} is symmetric. Since AB is a definable subset of $\langle A \rangle$, we have $\mu(AB) < \infty$ and there is $L < \omega$ with $\mu(AB) \leqslant L\mu(B)$. By Fact 5, at most L translates of BB^{-1} are needed to cover A.

(2) There is $n < \omega$ such that $B^2 \subseteq A^{2n} \subseteq E^{2n-1}A$. Suppose Y is finite with $A \subseteq YB$. Then
$$B^2 \subseteq E^{2n-1}A \subseteq E^{2n-1}YB.$$
Thus B is an approximate subgroup; being wide in A, it must be equivalent to A. $\quad\square$

DEFINITION 8. — A type-definable subgroup H of a (local) group G has *bounded* index if there is some cardinal κ such that in any elementary extension the index $|G : H|$ is bounded by κ.

REMARK 9. — By ω^+-saturation H has bounded index in G if and only if for every definable subset X of G and every definable superset Y of H, finitely many left translates of Y cover X.

LEMMA 10. — *If A and A^* are equivalent, there exists an approximate subgroup in which both are wide, and another one which is wide in both. In particular, $\langle A \rangle \cap \langle A^* \rangle$ will have bounded index in both $\langle A \rangle$ and $\langle A^* \rangle$.*

Proof. — Suppose $A^{*2} \subseteq E^*A^*$, and put $B = AA^*A$, a symmetric set containing A and A^*. If $A \subseteq XA^*$ and $A^* \subseteq X^*A$, then
$$B = AA^*A \subseteq XA^*A^*A \subseteq XE^*A^*A \subseteq XE^*X^*AA \subseteq XE^*X^*EA \subseteq XE^*X^*EXA^*,$$
so A and A^* are wide in B. Moreover
$$B^2 \subseteq XE^*X^*EAB = XE^*X^*EAAA^*A \subseteq XE^*X^*E^2AA^*A = XE^*X^*E^2B,$$
so B is also an approximate subgroup. As $\langle A \rangle$ and $\langle A^* \rangle$ have bounded index in $\langle B \rangle$, the intersection $\langle A \rangle \cap \langle A^* \rangle$ has bounded index in $\langle B \rangle$, and thus in $\langle A \rangle$ and in $\langle A \rangle^*$.

Now note that $\langle A \rangle \cap \langle A^* \rangle = \bigcup_{n < \omega}(A^n \cap A^{*n})$. By ω^+-saturation there is $n < \omega$ such that finitely many translates of $A^n \cap A^{*n}$ cover B. So $A^n \cap A^{*n}$ is wide in B, whence in A and in A^*. $\quad\square$

We now turn to the main result. We shall need the following Lemma due to Sanders.

Lemma 11. — *Let $f :]0,1] \to [1,K]$ and $\varepsilon > 0$. Then there exists $n < \omega$ depending only on K, ε and $t > 1/(2K)^{2^n-1}$ such that*

$$f(t^2/2K) \geqslant (1-\varepsilon)f(t).$$

Proof. — Define a sequence (t_n) by $t_0 = 1$ and $t_{k+1} = t_k^2/2K$, so $t_n = 1/(2K)^{2^n-1}$. For $n < \omega$ suppose that for all $i < n$ we have $f(t_{i+1}) < (1-\varepsilon)f(t_i)$. Then

$$f(t_n) < (1-\varepsilon)^n f(t_0) \leqslant (1-\varepsilon)^n K.$$

But $f(t_n) \geqslant 1$, so if $n < \omega$ is such that $(1-\varepsilon)^n K < 1$, there must be some $i < n$ with $f(t_{i+1}) \geqslant (1-\varepsilon)f(t_i)$.   □

Theorem 12. — *Let A be a K-approximate subgroup. For any $m < \omega$ there is a definable L-wide approximate subgroup S with $S^m \subseteq A^4$, where L depends only on K and m.*

Proof. — Let us show first that if $B \subseteq A$ is definable with $\mu(B) \geqslant t\mu(A)$ for some $0 < t \leqslant 1$ and $s = t/2K$, then A is covered by $N = \lfloor 1/s \rfloor$ translates of

$$X = \{g \in A^2 : \mu(gB \cap B) \geqslant st\mu(A)\}$$

by elements of A. So suppose not. Then inductively we find a sequence $(g_i : i \leqslant N)$ of elements of A such that $\mu(g_iB \cap g_jB) < st\mu(A)$ for all $i < j \leqslant N$, since for $j \leqslant N$ the set $\bigcup_{i<j} g_iX$ cannot cover A. But then

$$
\begin{aligned}
K\mu(A) \geqslant \mu(A^2) &\geqslant \mu\Big(\bigcup_{i \leqslant N} g_iB \Big) \\
&\geqslant (N+1)\mu(B) - \sum_{i < j \leqslant N} \mu(g_iB \cap g_jB) \\
&> (N+1)t\mu(A) - \frac{N(N+1)}{2}st\mu(A) = (1 - N\frac{s}{2})(N+1)t\mu(A) \\
&\geqslant (1 - \frac{1}{s}\frac{s}{2})\frac{1}{s}t\mu(A) = \frac{1}{2}\frac{2K}{t}t\mu(A) = K\mu(A),
\end{aligned}
$$

a contradiction.

However, as μ is not supposed to be definable, X need not be definable either. We shall hence look for definable sets with similar properties. To this end, consider the following conditions $P_n^t(X)$ on definable subsets of A, for $n < \omega$ and $0 < t \leqslant 1$:

  – $P_0^t(B)$ if $B \neq \varnothing$.
  – $P_{n+1}^t(B)$ if $P_n^t(B)$, and A is covered by $\lfloor 2K/t \rfloor$ translates of

$$X_{n+1}^t(B) = \{g \in A^2 : P_n^{t^2/2K}(gB \cap B) \text{ and } P_n^{t^2/2K}(g^{-1}B \cap B)\}.$$

Clearly, if $(B_x)_x$ is a family of uniformly definable subsets of A, then $P_n^t(B_x)$ is definable by a formula $\theta_n^t(x)$ for all $n < \omega$ and $0 < t \leqslant 1$. As $X_{n+1}^t(B) \subseteq A^2$, the translating elements for the covering of A must come from A^3, so the P_n^t are definable even in a local group (where we can only quantify over finite powers of A).

For $0 < t \leqslant 1$ we consider the family $\mathfrak{B}_t$ of definable subsets B of A with $P_n^t(B)$ for all $n < \omega$. The first paragraph implies inductively that for definable $B \subseteq A$, if $\mu(B) \geqslant t\mu(A)$ then $P_n^t(B)$ holds, whence $B \in \mathfrak{B}_t$. In particular, $A \in \mathfrak{B}_t$ so $\mathfrak{B}_t$ is non-empty. Note that P_n^t implies $P_n^{t'}$ for $t \geqslant t'$, so $\mathfrak{B}_t \subseteq \mathfrak{B}_{t'}$.

Define a function $f :]0,1] \to \mathbb{R}$ by

$$f(t) = \inf \left\{ \frac{\mu(BA)}{\mu(A)} : B \in \mathfrak{B}_t \right\}.$$

Fix $\varepsilon > 0$. Since $1 \leqslant f(t) \leqslant K$ for all $0 < t \leqslant 1$, by Lemma 11 there is $t > 0$ depending only on K and ε such that

$$f(t^2/2K) \geqslant (1 - \varepsilon)f(t).$$

Choose $B \in \mathfrak{B}_t$ with $\mu(BA)/\mu(A) \leqslant (1 + \varepsilon)f(t)$. Put

$$X_n = X_n^t(B) = \{ g \in A^2 : P_n^{t^2/2K}(gB \cap B) \text{ and } P_n^{t^2/2K}(g^{-1}B \cap B) \}$$

and $X = \bigcap_{n<\omega} X_n$. Then X_n is symmetric, $X_{n+1} \subseteq X_n$ and $\lfloor 2K/t \rfloor$ translates of X_n cover A, for all $n < \omega$. By ω^+-saturation, $\lfloor 2K/t \rfloor$ translates of X cover A, so X is nonempty. Moreover, for $g \in X$ we have $gB \cap B \in \mathfrak{B}_{t^2/2K}$, whence

$$\mu(gBA \cap BA) \geqslant \mu((gB \cap B)A) \geqslant f(t^2/2K)\mu(A)$$
$$\geqslant (1 - \varepsilon)f(t)\mu(A) \geqslant \frac{1 - \varepsilon}{1 + \varepsilon}\mu(BA).$$

Hence for $g \in X$,

$$\mu(gBA \triangle BA) \leqslant \frac{4\varepsilon}{1 + \varepsilon}\mu(BA) < 4\varepsilon\,\mu(BA).$$

It follows that for $g_1, \ldots, g_m \in X$,

$\mu(g_1 \cdots g_m BA \triangle BA)$
$$\leqslant \mu((BA \triangle g_1 BA) \cup g_1(BA \triangle g_2 BA) \cup \cdots \cup g_1 \cdots g_{m-1}(BA \triangle g_m BA))$$
$$\leqslant \mu(BA \triangle g_1 BA) + \mu(BA \triangle g_2 BA) + \cdots + \mu(BA \triangle g_m BA)$$
$$< 4\,m\,\varepsilon\,\mu(BA).$$

In particular, if $\varepsilon \leqslant 1/4m$, then $g_1 \cdots g_m BA \cap BA \neq \varnothing$, whence $X^m \subseteq A^4$. By ω^+-saturation there is $n < \omega$ such that $X_n^m \subseteq A^4$. Note that $S := X_n$ is $\lfloor 2K/t \rfloor$-wide in A, and thus an approximate subgroup equivalent to A by Lemma 7. $\qquad\square$

COROLLARY 13. — *There is a type-definable subgroup $H \subseteq A^4$ such that every definable superset of H contained in $\langle A \rangle$ is wide in A.*

Proof. — Put $S_0 = A$ and apply inductively Theorem 12 for $m = 8$ with S_i instead of A, in order to obtain a sequence of approximate subgroups $(S_i : i < \omega)$ with S_{i+1} wide in S_i (whence in A) and $S_{i+1}^8 \subseteq S_i^4$. Then $H = \bigcap_{i<\omega} S_i^4$ is a type-definable subgroup of A^4. Any definable superset of H must contain some S_i^4 by ω^+-saturation, and hence be wide in A. $\qquad\square$

2. Normality

Since we want to consider the quotient $\langle A \rangle / H$, we shall look for a stronger version of Theorem 12 where H will be normal.

Lemma 14. — *Let $X_1, \ldots, X_n$ be definable subsets of A with $N_i \, \mu(X_i) \geqslant \mu(A)$ for some $N_i < \omega$. Then there is a definable $D \subseteq A$ such that*

$$D^{-1}D \subseteq (X_1 X_1^{-1})^2 \cap \cdots \cap (X_n X_n^{-1})^2 \quad and \quad K^{n-1} N_1 \cdots N_n \, \mu(D) \geqslant \mu(A).$$

Proof. — Since $\mu(AX_2) \leqslant K \, \mu(A) \leqslant K N_2 \, \mu(X_2)$, by Fact 5 there are $g_1, \ldots, g_{KN_2}$ such that

$$A \subseteq \bigcup_{i=1}^{KN_2} g_i X_2 X_2^{-1}.$$

Then there is an i such that

$$K N_1 N_2 \, \mu(X_1 \cap g_i X_2 X_2^{-1}) \geqslant \mu(A).$$

We set $D_0 = X_1 \cap g_i X_2 X_2^{-1}$ and note that $D_0^{-1} D_0 \subseteq X_1^{-1} X_1 \cap (X_2 X_2^{-1})^2$.

Then we can iterate the construction, replacing X_1 by D_0 and X_2 by X_3. Inductively we obtain a suitable D with $K^{n-1} N_1 \cdots N_n \, \mu(D) \geqslant \mu(A)$ such that

$$D^{-1}D \subseteq X_1^{-1} X_1 \cap (X_2 X_2^{-1})^2 \cap \cdots \cap (X_n X_n^{-1})^2.$$

Notice that $D^{-1}D$ is $K^{n-1} N_1 \cdots N_n$-wide in A by Fact 5. □

Theorem 15. — *Let A be a K-approximate subgroup, and R a definable N-wide symmetric subset with $R^4 \subseteq A^4$. Then there exists a definable L-wide symmetric subset S with $(S^8)^A \subseteq R^4$, where L depends only on K and N.*

Proof. — If $A \subseteq XR$, then

$$R^2 \subseteq A^4 \subseteq E^3 A \subseteq E^3 X R,$$

so R is a $K^3 N$-approximate subgroup. Theorem 12 yields the existence of some $T \subseteq R^4$ equivalent to R with $T^{48} \subseteq R^4$. Then T is wide in A and there exists $n < \omega$ depending only on K and N and some elements a_i of A such that

$$A \subseteq \bigcup_{i=1}^{n} a_i T.$$

Consider the measure $\overline{\mu}$ on definable subsets of $\langle A \rangle$ defined by

$$\overline{\mu}(X) := \frac{1}{n} \sum_{i=1}^{n} \mu(X a_i).$$

Clearly $\overline{\mu}$ is left translation invariant, we have

$$\overline{\mu}(A) = \frac{1}{n} \sum_{i=1}^{n} \mu(A a_i) \leqslant \frac{n}{n} \mu(A^2) \leqslant K \mu(A),$$

and

$$\overline{\mu}(a_i T a_i^{-1}) \geqslant \frac{1}{n} \mu(T) \geqslant \frac{1}{n^2} \mu(A) \geqslant \frac{1}{K n^2} \overline{\mu}(A).$$

Since all the $a_i T a_i^{-1}$ are subsets of A^6 and

$$K^6 n^2 \overline{\mu}(a_i T a_i^{-1}) \geqslant K^5 \overline{\mu}(A) \geqslant \overline{\mu}(A^6),$$

Lemma 14 applied to the K^6-approximate subgroup A^6 yields a subset $D \subseteq A^6$ with

$$(K^6)^{n-1}(K^6 n^2)^n \overline{\mu}(D) \geqslant \overline{\mu}(A^6)$$

such that for $i = 1, 2, \ldots, n$ we have

$$S := D^{-1} D \subseteq a_i T^4 a_i^{-1}.$$

Then S is symmetric, wide in A and $S^{a_i} \subseteq T^4$ for $i = 1, \ldots, n$. Since $A \subseteq \bigcup a_i T$, this means that $S^A \subseteq T^6$, so $(S^8)^A \subseteq T^{48} \subseteq R^4$. □

COROLLARY 16. — *There is a type-definable normal subgroup H of $\langle A \rangle$ contained in A^4 such that every definable superset of H contained in $\langle A \rangle$ is wide in A.*

Proof. — As Corollary 13, using Theorem 15 instead of Theorem 12. □

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