Patterns and Approx. Subgps.

§1. Beyond the Lascar Group.

Recall: Def. A Quasi-homomorphism $\phi: G \to H:K$ with G, H gps, $K \subseteq H$ is a map s.t. (1) $\phi(1)=1$

(2) $\phi(xy^{-1}) \in \phi(x)\phi(y^{-1})K$.

Rmk: $1 = \phi(y \cdot y^{-1}) \in \phi(y)\phi(y^{-1})K$ $\quad \phi(y)^{-1} = \phi(y^{-1})k$ for some $k \in K$.

$$\phi(y^{-1}) = \phi(y)^{-1} \cdot k^{-1} \subseteq \phi(y)^{-1} \cdot K^{-1}$$

$$(2) \Rightarrow \phi(xy^{-1}) \in \phi(x)\phi(y)^{-1}K^{-1}K.$$

Prop: (Prop 5.12 Lascar). Let H be a locally compact top. gp. K a compact subset of H, assume $f: G \to H:K$ is a quasi-homomorphism with $f(xy^{-1}) \in f(x)f(y)^{-1}K$.

Then all sets $X = f^{-1}(UK)$ with U compact nbd of id in H are commesurable with each other (can cover each other with finitely right translates).

And any symmetric set sandwiched between two such sets ~~are~~ an approx. subgp. In particular $X \cdot X^{-1}$ is one.

Pf: Claim 1: For any $V \subset H$ and $a \in H$, $f^{-1}(Va) \subset f^{-1}(VV^{-1}K)c$ for some c.

WMA $f^{-1}(Va) \neq \emptyset$, take c s.t. $f(c) \in Va$. If $x \in f^{-1}(Va)$, then $f(x) = Va$ and $f(xc^{-1}) \in (Va)(Va)^{-1} \cdot K = V \cdot V^{-1}K$. so. $xc^{-1} \in f^{-1}(VV^{-1}K)$ and $x \in f^{-1}(VV^{-1}K)c$.

Claim 2: Let $Z \subseteq H$ be compact and U be a nbd. of 1 in H. Then finitely-many right cosets of $f^{-1}(UK)$ cover $f^{-1}(Z)$.

Take V nbd of 1 s.t. $VV^{-1} \subset U$. $Z \subseteq \bigcup_{i \leq N} Va_i$: It suffices to show $f^{-1}(Va_i)$ is covered by a right coset of $f^{-1}(VV^{-1}K) \subseteq f^{-1}(UK)$.

Which follows from Claim 1.

Let U be compact nbd of 1 and $X_U = f^{-1}(UK)$.

Claim 2 $\Rightarrow$ any two X_U's are commesurable. ~~with each~~

If $XU_1 \subseteq Y \subseteq XU_2$, ~~the~~ and Y symmetric, then

$$Y \cdot Y^{-1} \subseteq XU_2 (XU_2)^{-1} = f^{-1}(U_2K)(f^{-1}(U_2K))^{-1} \subseteq f^{-1}(U_2K(U_2K)^{-1}K)$$

by $f(f^{-1}(U_2K)\cdot(f^{-1}(U_2K))^{-1}) \subseteq f(U_2K)\cdot(ff^{-1}(U_2K))^{-1}\cdot K$.

Let $Z = UK(UK)^{-1}K$ the compact set in claim 2.

We get $Y \cdot Y^{-1} \subseteq XU_1 M \subseteq Y \cdot M$ for finite set M.

(~~Same~~ $XU_2(XU_2)^{-1} \subseteq XU_2 \cdot M$ by the same reason).

<u>Thm</u> (Thm 4.2 Lascar).

Let G be a gp, generated by an approx. subgp Λ. ($G = \bigcup_{n\in\mathbb{N}} \Lambda^n$).
Then there exists a second ctble locally compact top. gp H, a compact normal subset $\Delta \subseteq H$ and a quasi-homomorphism

$$f: G \to H : \Delta$$

s.t.

(1) For $C \subseteq H$ compact, $f^{-1}(C)$ is contained in some Λ^i.

(2) For each i, $\exists$ a compact $C \subseteq H$ with $\Lambda^i \subseteq f^{-1}(C)$.

(3) Specialy, $f^{-1}(\Delta) \subseteq \Lambda^{12}$.

(4) Let X, X' be compact subsets of H with $\Delta^2 X \cap \Delta^2 X' = \emptyset$, then $\exists$ disjoint definable subsets D, D' of Λ^k for some k, s.t. $f^{-1}(X) \subseteq D$ and $f^{-1}(X') \subseteq D$.

Rmk: Thm 4.2 says all approx subgps are roughly (comm. with) pullback of compact nbd of id. containing Δ along some quasi homomorphism with error Δ.

(1) + (2) $\Rightarrow$ Let U be compact nbd. of id, then $\exists\, i_0$,

$$\phi^{-1}(U\cdot\Delta) \subseteq \Lambda^{i_0} \subseteq \phi^{-1}(C) \subseteq \phi^{-1}(C\Delta) \text{ for some compact } C.$$

§2. Patterns.

Let T be a complete first-order theory. $M \models T$. We will define ~~an~~ a rel. structure L_p on the type space $S_x(M)$ for x a tuple (can be infinite) and morphisms between L_p-structures. The <u>core</u> of T, $\mathcal{J}(T)$ will be a universal minimal object under these morphisms.

(We will follow Pierre Simon's notes). T $\mathcal{L}$-theory, $M \models T$. suff. saturated.

- Def 1: Let $\phi_1(x_1,t), \cdots, \phi_n(x_n,t)$, $\theta(t)$ be $\mathcal{L}$-formulas without parameters. Write $\bar{\phi} := (\phi_1, \cdots, \phi_n)$ and ~~also~~ define $R(\bar{\phi},\theta) \subseteq S_{x_1}(M) \times \cdots \times S_{x_n}(M)$ be ~~tuples~~ $(p_1, \cdots, p_n) \in R(\bar{\phi},\theta)$ iff.

  there exists no $b \in M^t$, s.t. (1) $M \models \theta(b)$

  (2) $p_i \models \phi_i(x_i, b)$ for all $1 \le i \le n$.

Examples: ① T stable, p a definable type over ~~ϕ~~ $\emptyset$. in $S_x(M)$.

For $\phi(x,y)$ Let $d^p_\phi(y)$ be the defining formular of p.

Then $p \in R(\phi, \neg d^p_\phi(y))$ for all ϕ.

② $\phi(x)$ formula over $\emptyset$. Then $p \in R(\neg\phi(x), t=t)$ iff

$\neg\exists b$ $M \models b=b$ and $p \models \neg\phi(x)$ iff $p \not\models \neg\phi(x)$ iff $\phi(x) \in p$.

Rmk: 1. Union of two sets $R(\bar{\phi},\theta)$, $R(\bar{\phi}',\theta')$ is still of the form:

$R(\bar{\phi}'',\theta'')$ where $\phi''_i(x_i; t_1, t_2) := \phi_i(x_i; t_1) \wedge \phi'_i(x_i, t_2)$

and $\theta''(t_1,t_2) := \theta(t_1) \wedge \theta(t_2)$.

2. $R(\bar{\phi},\theta)$ is a closed subset of $S_{x_1}(M) \times \cdots \times S_{x_n}(M)$ (with the product topology), and $\mathrm{aut}(M)$-invariant.

- Def 2: A subset of $S_{x_1}(M) \times \cdots \times S_{x_n}(M)$ ~~is~~ of the form $R(\bar{\phi},\theta)$ is called an <u>atomic p-closed set</u>. A <u>p-closed set</u> is an intersection of atomic p-closed set.

Rmk: 1. p-closed sets form top. on $S_{x_1}(M) \times \cdots \times S_{x_n}(M)$. (Rmk 1).

2. p-closed sets are closed in $S_{x_1}(M) \times \cdots \times S_{x_n}$ in product. top.

3. Equality is a p closed set in $S_x(M) \times S_x(M)$ $\left(\bigwedge_{\phi \in L} R(\phi(x,t), \neg\phi(x,t), t=t) \right)$

Prop ~~1~~ 2: The projection of a p closed set is p-closed.

Def 3: Fix a variable x, define a new relational language $L_p(x)$.

For each tuple $\bar{\phi} = (\phi_1(x,t), \dots, \phi_n(x,t))$ and all $\theta(t)$ formula (over $\emptyset$)

Let $R(\bar{\phi},\theta)(\gamma_1,\dots,\gamma_n)$ be a rel. symbol. ~~$L_p(x) := \{R(\bar{\phi},\theta) : \text{all } \phi_i, \theta \in L\}$~~

$L_p(x) := \{R(\bar{\phi},\theta) : \phi = (\phi_1,\dots,\phi_n),\ n \in \mathbb{N},\ t,\ \phi_i \in L,\ \theta(t) \in L\}$

Def 4: If A, B are L_p-structures (in particular subsets of $S_x(M)$)

<u>a morphism</u> from A to B is a map $f: A \to B$ which is an L_p-morph.

i.e. if $R_\cdot(\bar{a})$ holds for $\bar{a} \in A$ then $R_\cdot(f(\bar{a}))$ holds in B.

Rmk: morphisms can be non-injective and maps $\neg R_\cdot(\bar{a})$ to $R_\cdot(f(\bar{a}))$

(more relations $R_\cdot$ are realised along morphisms including equality).

Examples & Rmk's: ①. p a def. type over ~~$\emptyset$~~ $\emptyset$.

Then $R(\phi(x,t), \neg d^p_\phi(t))(p)$ holds for all $\phi(x,t)$ over $\emptyset$.

Hence $R(\phi(x,t), \neg d^p_\phi(t))(f(p))$ holds - - - - . . . ,

i.e. $p = f(p)$ in $S_x(M)$.

②. p and $f(p)$ restricts to the same type over $\emptyset$.

Def 5: A subset $A \subseteq S_x(M)$ is <u>p-minimal</u> if any morphism $f: A \to S_x(M)$

is an L_p-isomorphism on its image.

Thm 1: There exists a p-minimal $J \subseteq S_x(M)$ and a morphism $f: S_x(M) \to J$

with $f|_J = \mathrm{id}$ called a <u>retraction</u>. Further more, J is unique up to L_p-isomorphism and

its L_p-isomorphism type does not depend on the choice of model M.

We call the L_p-isomorphism type of the set J the <u>core</u> of T,

denoted core(T). We write J = core(T).

Rmk: $|J| \leq 2^{|T|} \vee \aleph_0$. (~~or~~ J is the existentially closed model ~~$\emptyset$~~ in positive logic in $L_p(x)$).

Examples:

1. T stable, the a core $J \subseteq S_x(M)$ consists of all $acl^{eq}(\phi)$-def. types (equiv. types that do not fork over ϕ).
   (We have seen if p $\emptyset$-def. then $f: S_x(M) \to J$ must have $f(p)=p$. hence $p \in J$.)
   On the other hand, the retraction map: $f: S_x(M) \to J$ is given by sending $p \in S_x(M)$ to the unique non-forking extension of its strong type over $\emptyset$.
2. $T = DLO$ and $|x|=1$, then J consists two 0-def. types $= \{\pm\infty\}$'s.

Now we will define a topology on $\mathfrak{J}$ and its p-automorphism gp G.
(p-topoloy is not good enough to make $\mathfrak{J}$ T_1)

Def 6: The pp-topology.

A subset $C(u) \subseteq \mathfrak{J}$ is <u>pp-closed</u> if it is an intersection of the sets of the form $R(u, q_1, \cdots, q_n)$ with $R \subseteq \mathfrak{J}^{1+n}$ p-closed and $(q_1, \cdots, q_n) \in \mathfrak{J}^n$.

(Namely, a pp-closed set is a fiber of a p-closed set).

Similarly a pp-closed set of $\mathfrak{J}^k$ is an intersection of the sets of the form $R(u_1, \cdots, u_k, q_1, \cdots, q_n)$ with $R \subseteq \mathfrak{J}^{k+n}$ p-closed and $(q_1 \cdots q_n) \in \mathfrak{J}^n$.

Prop: The pp-topology on $\mathfrak{J}^n$ is compact and T_1.

Pf: WMA $n=1$. Let $p \in \mathfrak{J}$, since equality is p-closed in $\mathfrak{J}^2$, the set $x=p$ is pp-closed. Hence $\mathfrak{J}$ is T_1.

Compact: Let $(C_i(u))_{i \in \mathbb{N}}$ be a family of pp-closed subsets of $\mathfrak{J}$ and any finite subfamity has non-empty intersection. WMA: $C_i(u) = R_i(u, \bar{q}_i)$ for some $\bar{q}_i \in \mathfrak{J}$ and p-closed R_i. Fix $\mathfrak{J} \cong J \subseteq S_x(M)$.

Then $R_i(u, \bar{q}_i)$ is closed in $S_x(M)$ in the usual topology.

By compactness, $\exists$ $p \in S_x(M)$ in $\bigcap_i R_i(u, \bar{q}_i)$.

Let $f: S_x(M) \to J$ be the retraction. Then $R_i(f(p), f(\bar{q}_i))$ namely $R_i(f(p), \bar{q}_i)$ holds for all i. hence $f(p) \in \bigcap_i C_i(u)$ as required.

Def 7: Let $G = \mathrm{Aut}(\mathfrak{J})$ be the gp of Lp-automorphisms, of the core.
/endomorphisms.

Equip ~~with~~ G with the top. for which a basic open set is of the form $\{g \in G : \neg R(ga_1, \dots, ga_k, b_1, \dots, b_n)\}$ for some $a_1, \dots, a_k, b_1, \dots, b_n \in \mathfrak{J}$ and R a p-closed set in $\mathfrak{J}^{k+n}$.

Prop: The gp G is compact T_1. For $g \in G$, left and right translation by g are continuous on G; also inverse is continuous.

Def 8: There is a map $\mathfrak{g}: \mathrm{Aut}(M) \to G$ which is in general not a homomorphism.

$\sigma \in \mathrm{Aut}(M)$, then σ induces $\sigma: S_x(M) \to S_x(M)$ automorphism.

Let $i: \mathfrak{J} \hookrightarrow \cancel{\mathrm{Aut}(M)}\ S_x(M)$ and $f: \cancel{\mathrm{Aut}(M)}\ S_x(M) \to \mathfrak{J}$ retraction.

Then $\sigma \mapsto \mathfrak{d}(\sigma) := f \circ \sigma \circ i : \mathfrak{J} \to \mathfrak{J} \in G$.

~~Def: Let $p, q \in S_x(M)$ ($a, b \in M$). Define $p \triangle q$ if for some elem. extension M' of M and $N \preceq M'$ with $N \cong M$, we have:~~

Let $\mathbb{M} \models T$ be a monster model. For $a, b \in \mathbb{M}^{|x|}$, define $a \triangle b$ if $\exists$ small model $M_0 \preceq \mathbb{M}$, s.t. $tp(a/M_0) = tp(b/M_0)$.

We may push this relation on any $S_x(N)$ with $N \models T$ as:

$L_1(p, q)$ iff $\forall \theta(y)$ and finite Δ, $\exists a \models p$, $b \models q$ in the monster $\mathbb{M}$, and $d \in \theta(\mathbb{M})$ s.t. $tp_\Delta(a/d) = tp(b/d)$.

It induces the same relation on the core $\mathfrak{J}$.

Let $\Delta^{\mathfrak{J}} := \{\sigma \in \mathrm{Aut}(G) = G : \forall p \in \mathfrak{J}, L_1(\sigma(p), p)\}$ a closed normal set in G.

Thm: $\mathfrak{d}: \mathrm{Aut}(M) \to G : \Delta^{\mathfrak{J}}$ is a quasi-homomorphism.

§3. Compact $\Rightarrow$ loc. compact. (Not reliable).

~~Local~~ Let $G = \bigcup_{n \in \mathbb{N}} \Lambda^n$ Λ an approx. subgp. Define a local str. $\mathcal{M}$ as:

$D_1 := (\Lambda, G, P_1(x,y), Q_1(x,y,z,w))$ $P_1(x,y)$ iff $yx^{-1} \in \Lambda$

$\vdots$ and $Q_1(x,y,z,w)$ iff $yx^{-1} = zw^{-1} \in \Lambda$.

$D_n := (\Lambda^n, G, P_n(x,y), Q_n(x,y,z,w))$ $P_n(x,y)$ iff $yx^{-1} \in \Lambda^n$

4.

and $Q_n(x, y, z, w)$ iff $yx^{-1} = zw^{-1} \in \Lambda^n$.

For $g \in G$, the gp action on right $g(d) := d \cdot g^{-1}$ is an automorphism of M.
Hence there exists a homomorphism $\delta : G \to \mathrm{Aut}(M)$.

M is a local structure, $Th(M)$ admits a core $\mathcal{J}$ and $\mathrm{Aut}(\mathcal{J})$ is a loc. compact space. The Hausdorff quotient $\mathcal{G} := \mathrm{Aut}(\mathcal{J}) / g_D$ is a loc. compact top. gp. And there is a compact, normal, symmetric subset $\Delta \subseteq \mathcal{G}$, ~~s.t.~~ and a quasi-morphism $\eta : \mathrm{Aut}(M) \to \mathcal{G} : \Delta$.

Then $\eta \circ \delta : G \to \mathcal{G} : \Delta$ is the quasi-morphism from G to a loc. compact space with compact error as we want.