Principles for Deforming Nonnegative Curvature

joint work with Peter Petersen

June 4, 2009

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- Goal–to isolate some abstract principles that are used to prove this theorem, and to understand how they might be put together to make a proof.

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- Scaling the fibers to integrally positive curvature (and leaving nonnegative curvature)
- Exploiting rigidity of totally geodesic flat tori, to redistribute curvatures,
- Exploiting rigidity of totally geodesic, to make a "partial conformal change" that behaves (much) like a conformal change

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- The quotient is diffeomorphic to M. Give G a biinvariant metric and $G \times M$ the product metric. Then the quotient inherits a new metric g_{Cheeg} of nonnegative curvature.

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- $Hat(v)$

$$Hat(v) \equiv \hat{v}$$

is the unique vector in $T(G \times M)$ that is horizontal for $d\pi_{\text{Cheeg}}$ has the form

$$\hat{v} = (???, v)$$

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- So if $G = S^3$, any plane whose projection to the orbits of G is nondegenerate becomes positively curved.

Applications

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- Notably in [Wilh] and [Esch-Ker] to get positive curvature almost everywhere on the Gromoll-Meyer sphere.

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- By scaling the bi-invariant metric on the G–factor in $G \times M$ we obtain a one parameter familly of metrics on M.
- If "l" is the scaling factor, then as $l \longrightarrow \infty$, the new metrics converge to the old one.

Long Term Cheeger Principle

- If v is fixed, and if the ??? in

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is k_v, when $l = 1$, then for general l,

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- So $\mathrm{curv}_M d\pi_{\mathrm{Cheeg}} (\hat{P}) > 0$ if l is sufficiently small.
- Aside–if either k_v or k_w is nonzero, then the sectional curvature of $\hat{P}$ is "almost nonnegative" as $l \to 0$.

Fiber Scaling

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- However, over any of the totally geodesic flat tori of g_{old}, the integral of the curvature of g_{new} is positive.

Structure of the zero planes of the GM-Sphere

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- On S^4, ζ can be viewed as the gradient of the distance from two points in S^1 (its mulitvalued in places and not in a zero plane in others).
- W is typically neither horizontal nor vertical for p_{GM}. Its horizontal part $H = W^{horiz}$ is a killing field for the isometric $SO(3)$–action along any fixed geodesic tangent to a particular ζ.

Curvature after fiber scaling

- If g_{new} is obtained from g_{old} by scaling the lengths of the fibers by $\sqrt{1-s^2}$, then

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- and
$$\int_c \mathrm{curv}_{g_s}(\zeta, W) = \int_c s^4 \left(D_\zeta \, |H_w| \right)^2 > 0.$$

Point-wise Curvature

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  - negative near the singular circle
  - and positive away from this circle.
- Its much larger than the term $s^4 \left(D_\zeta |H_w|\right)^2$ whose integral is positive.

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- Since the ratio $\frac{w_h}{|W|}$ varies from torus to torus, we actually do a modification of a conformal change called a "partial confomral change".

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- Some long term Cheeger Deformations allow us to simplify this problem, however,
- It seems impossible to get positive curvature with the deformations we have discussed so far.
- We therefore need a further deformation which we called the "redistribution".

Rigidty of Totally Geodesic Flat Tori

- **Exercise**(5.4 in [Pet]) Let γ be a geodesic in (M, g). Let $\tilde{g}$ be another metric on M which satisfies

$$g(\dot{\gamma}, \cdot) = \tilde{g}(\dot{\gamma}, \cdot) : TM \longrightarrow \mathbb{R}.$$

Then γ is also a geodesic with respect to $\tilde{g}$.

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- **Proposition** Let $\mathcal{S}$ be a family of totally geodesic submanifolds of (M, g). Let $\tilde{g}$ be another metric on M which satisfies

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for all vectors tangent to a totally geodesic submanifold in $\mathcal{S}$, then $\mathcal{S}$ is also a family of totally geodesic submanifolds of $(M, \tilde{g})$.

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  for all vectors tangent to a totally geodesic submanifold in $\mathcal{S}$, then $\mathcal{S}$ is also a family of totally geodesic submanifolds of $(M, \tilde{g})$.
- **Proof:** If γ is any geodesic in $S \in \mathcal{S}$ with respect to g, then by the preceding exercise, γ is a geodesic of $(M, \tilde{g})$.

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- **Corollary** If the totally geodesic family $\mathcal{S}$ of the preceding proposition consists of totally geodesic flat submanifolds for (M, g), then it also consists of totally geodesic flat submanifolds for $(M, \tilde{g})$.

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- **Corollary** If the totally geodesic family $\mathcal{S}$ of the preceding proposition consists of totally geodesic flat submanifolds for (M, g), then it also consists of totally geodesic flat submanifolds for $(M, \tilde{g})$.
- **Proof:** The intrinsic metric on members of $\mathcal{S}$ does not change. In particular, totally geodesic flats are preserved.

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- **Theroem:** Suppose that (M, g) is compact and nonnegatively curved and all of its zero planes are contained in a family $\mathcal{S}$ of totally geodesic flat submanifolds. Let g_{new} be obtained from g as in the preceding proposition.

Rigidty of Totally Geodesic Flat Tori

- **Theroem:** Suppose that (M, g) is compact and nonnegatively curved and all of its zero planes are contained in a family $\mathcal{S}$ of totally geodesic flat submanifolds. Let g_{new} be obtained from g as in the preceding proposition.

- Then (M, g_{new}) is nonnegatively curved along the union of the family $\mathcal{S}$ with precisely the same 0 curvature planes as g, provided g_{new} is sufficiently close to g in the C^2–topology.

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- **Idea of Proof:** If ζ and W are tangent to one of the submanifolds $S \in \mathcal{S}$, then

$$R^{g_{\mathrm{new}}}(\zeta, W)W = R^{g_{\mathrm{new}}}(W, \zeta)\zeta = 0$$

and all other components of $R^{g_{\mathrm{new}}}$ are close to the corresponding components of R^g.

Shortcomings

- The above result only tells about the new curvature at points where the old curvature is 0.

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- I will address the second issue here, in outline.

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  into a small neighborhood, T_0, of the singular circle of $S^4 = S^1 * S^2$.
- This has several advantages.

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- In addition, the smallness of T_0 is crucial for keeping the deformation C^1–small globally.
- Since the whole project of [PetWilh] is about evening out

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it is extremely important that we can redistribute some preexisting positive curvature into the region that counts, while simultaneously having only a negligible effect on curvatures elsewhere.

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  for some other smooth functions ψ^i and λ^i.
- If the functions ϕ^i are close to 1 in the C^1–topology, then the only components of $R^{g_{\mathrm{new}}}\left(\tilde{E}_i, \tilde{E}_j, \tilde{E}_k, \tilde{E}_l\right)$ that are not close to R^g are the terms that up to symmetries of the curvature tensor can be reduced to $R^{g_{\mathrm{new}}}\left(\tilde{E}_1, \tilde{E}_i, \tilde{E}_i, \tilde{E}_1\right)$.

Limited C-2 Effect

- **Proposition (continued)**

$$R^{g_{\mathrm{new}}}\left(\tilde{E}_1, \tilde{E}_i, \tilde{E}_i, \tilde{E}_1\right) = R^{g_{\mathrm{old}}}\left(E_1, E_i, E_i, E_1\right) - \left(\left(\phi^i\right)''\right) + O\left(C^1\right)$$

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- When applied to the Gromoll-Meyer sphere it is precisely the curvatures,

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- The functions ϕ^i are either 1, or all the same function, φ, that is concave down (with big second derivatire) on the small set T_0, and concave up with small second derivative on the complment of T_0.

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  - Rigidity of tori allows us to preserve nonnegative curvature
  - The smallness of T_0 allows us to make certain curvatures much larger on T_0,while not changing curvatures much off of T_0.

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