10.01.2014

# Lecture Notes, week 11
# Topology WS 2013/14 (Weiss)

### 9.1. Vertex schemes and simplicial complexes

**Definition 9.1.** A *vertex scheme* consists of a set $V$ and a subset $\mathcal{S}$ of the power set $\mathcal{P}(V)$, subject to the following conditions: every $T \in \mathcal{S}$ is finite and nonempty, every subset of $V$ which has exactly one element belongs to $\mathcal{S}$, and if $T'$ is a nonempty subset of some $T \in \mathcal{S}$, then $T' \in \mathcal{S}$.
The elements of $V$ are called *vertices* (singular: *vertex*) of the vertex scheme. The elements of $\mathcal{S}$ are called *distinguished subsets* of $V$.

**Example 9.2.** The following are examples of vertex schemes:

(i) Let $V = \{1, 2, 3, \dots, 10\}$. Define $\mathcal{S} \subset \mathcal{P}(V)$ so that the elements of $\mathcal{S}$ are the following subsets of $V$: all the singletons, that is to say $\{1\}, \{2\}, \dots, \{10\}$, and $\{1, 2\}$, $\{2, 3\}$, ..., $\{9, 10\}$ as well as $\{10, 1\}$.

(ii) Let $V = \{1, 2, 3, 4\}$ and define $\mathcal{S} \subset \mathcal{P}(V)$ so that the elements of $\mathcal{S}$ are exactly the subsets of $V$ which are nonempty and not equal to $V$.

(iii) Let $V$ be any set and define $\mathcal{S}$ so that the elements of $\mathcal{S}$ are exactly the nonempty finite subsets of $V$.

(iv) Take a regular icosahedron. Let $V$ be the set of its vertices (which has 12 elements). Define $\mathcal{S} \subset \mathcal{P}(V)$ in such a way that the elements of $\mathcal{S}$ are all singletons, all doubletons which are connected by an edge, and all tripletons which make up a triangular face of the icosahedron. (There are twenty such tripletons, which is supposed to explain the name *icosahedron*.)

The *simplicial complex* determined by a vertex scheme $(V, \mathcal{S})$ is a topological space $X = |V|_{\mathcal{S}}$. We describe it first as a set. An element of $X$ is a function $f \colon V \to [0, 1]$ such that

$$\sum_{v \in V} f(v) = 1$$

and the set $\{v \in V \mid f(v) > 0\}$ is an element of $\mathcal{S}$.
It should be clear that $X$ is the union of certain subsets $\Delta(T)$, where $T \in \mathcal{S}$. Namely, $\Delta(T)$ consists of all the functions $f \colon V \to [0, 1]$ for which $\sum_{v \in V} f(v) = 1$ and $f(v) = 0$ if $v \notin T$. The subsets $\Delta(T)$ of $X$ are not always disjoint. Instead we have $\Delta(T) \cap \Delta(T') = \Delta(T \cap T')$ if $T \cap T'$ is nonempty; also, if $T \subset T'$ then $\Delta(T) \subset \Delta(T')$.
The subsets $\Delta(T)$ of $X$, for $T \in \mathcal{S}$, come equipped with a preferred topology. Namely, $\Delta(T)$ is (identified with) a subset of a finite dimensional real

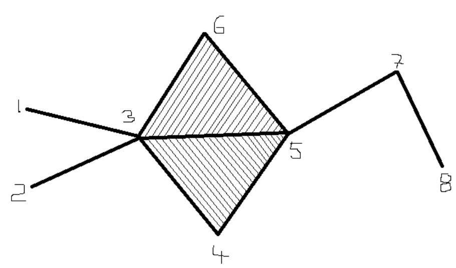
vector space, the vector space of all functions from $T$ to $\mathbb{R}$, and as such gets a subspace topology. (For example, $\Delta(T)$ is a single point if $T$ has one element; it is homeomorphic to an edge or closed interval if $T$ has two elements; it looks like a compact triangle if $T$ has three elements; etc. We say that $\Delta(T)$ is a *simplex* of dimension $m$ if $T$ has cardinality $m+1$.) These topologies are compatible in the following sense: if $T \subset T'$, then the inclusion $\Delta(T) \to \Delta(T')$ makes a homeomorphism of $\Delta(T)$ with a subspace of $\Delta(T')$. We decree that a subset $W$ of $X$ shall be *open* if and only if $W \cap \Delta(T)$ is open in $\Delta(T)$, for every $T$ in $\mathcal{S}$. Equivalently, and perhaps more usefully: a map $g$ from $X$ to another topological space $Y$ is continuous if and only if the restriction of $g$ to $\Delta(T)$ is a continuous from $\Delta(T)$ to $Y$, for every $T \in \mathcal{S}$.

**Example 9.3.** The simplicial complex associated to the vertex scheme (i) in example 9.2 is homeomorphic to $S^1$. In (ii) and (iv) of example 9.2, the associated simplicial complex is homeomorphic to $S^2$.

**Example 9.4.** The simplicial complex associated to the vertex scheme $(V, \mathcal{S})$ where $V = \{1,2,3,4,5,6,7,8\}$ and

$$\mathcal{S} = \left\{ \begin{array}{l} \{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\},\{1,3\},\{2,3\},\{3,4\}, \\ \{3,5\},\{3,6\},\{4,5\},\{5,6\},\{5,7\},\{7,8\},\{3,4,5\},\{3,5,6\} \end{array} \right\}$$

looks like this:




**Lemma 9.5.** *The simplicial complex* $X = |V|_{\mathcal{S}}$ *associated with a vertex scheme* $(V, \mathcal{S})$ *is a Hausdorff space.*

*Proof.* Let $f$ and $g$ be distinct elements of $X$. Keep in mind that $f$ and $g$ are functions from $V$ to $[0,1]$ subject to certain conditions. Choose $v_0 \in V$ such that $f(v_0) \neq g(v_0)$. Let $\varepsilon = |f(v_0) - g(v_0)|$. Let $U_f$ be the set of all $h \in X$ such that $|h(v_0) - f(v_0)| < \varepsilon/2$. Let $U_g$ be the set of all $h \in X$ such that $|h(v_0) - g(v_0)| < \varepsilon/2$. From the definition of the topology on $X$, the sets $U_f$ and $U_g$ are open. They are also disjoint, for if $h \in U_f \cap U_g$ then $|f(v_0) - g(v_0)| \leq |f(v_0) - h(v_0)| + |h(v_0) - g(v_0)| < \varepsilon$, contradiction. Therefore $f$ and $g$ have disjoint neighborhoods in $X$. $\square$

**Lemma 9.6.** *Let* $(V, \mathcal{S})$ *be a vertex scheme and* $(W, \mathcal{T})$ *a vertex sub-scheme, that is,* $W \subset V$ *and* $\mathcal{T} \subset \mathcal{S} \cap \mathcal{P}(W)$*. Then the evident map* $\iota\colon |W|_{\mathcal{T}} \to |V|_{\mathcal{S}}$ *is a closed, continuous and injective map and therefore a homeomorphism onto its image.*

*Proof.* The map $\iota$ is obtained by viewing functions from $W$ to $[0,1]$ as functions from $V$ to $[0,1]$ by defining the values on elements of $V \smallsetminus W$ to be $0$. A subset $A$ of $|V|_{\mathcal{S}}$ is closed if and only if $A \cap \Delta(T)$ is closed for the standard topology on $\Delta(T)$, for every $T \in \mathcal{S}$. Therefore, if $A$ is a closed subset of $|V|_{\mathcal{S}}$, then $\iota^{-1}(A)$ is a closed subset of $|W|_{\mathcal{T}}$; and if $C$ is a closed subset of $|W|_{\mathcal{S}}$, then $\iota(C)$ is closed in $|V|_{\mathcal{S}}$. $\square$

**Remark 9.7.** The notion of a simplicial complex is old. Related vocabulary comes in many dialects. I have taken the expression *vertex scheme* from Dold's book *Lectures on algebraic topology* with only a small change (for me, $\emptyset \notin \mathcal{S}$). It is in my opinion a good choice of words, but the traditional expression for that appears to be *abstract simplicial complex.* Most authors agree that a *simplicial complex* (non-abstract) is a topological space with additional data. For me, a simplicial complex is a space of the form $|V|_{\mathcal{S}}$ for some vertex scheme $(V, \mathcal{S})$; other authors prefer to write, in so many formulations, that a simplicial complex is a topological space $X$ together with a homeomorphism $|V|_{\mathcal{S}} \to X$, for some vertex scheme $(V, \mathcal{S})$.

## 9.2. Semi-simplicial sets and their geometric realizations

Semi-simplicial sets are closely related to vertex schemes. A semi-simplicial set has a *geometric realization*, which is a topological space; this is similar to the way in which a vertex scheme determines a simplicial complex.

**Definition 9.8.** A semi-simplicial set $Y$ consists of a sequence of sets

$$(Y_0, Y_1, Y_2, Y_3, \ldots)$$

(each $Y_k$ is a set) and, for each injective order-preserving map

$$f\colon \{0,1,2,\ldots,k\} \longrightarrow \{0,1,2,\ldots,\ell\}$$

where $k, \ell \geq 0$, a map $f^*\colon Y_\ell \to Y_k$. The maps $f^*$ are called *face operators* and they are subject to conditions:

- if $f$ is the identity map from $\{0,1,2,\ldots,k\}$ to $\{0,1,2,\ldots,k\}$ then $f^*$ is the identity map from $Y_k$ to $Y_k$.
- $(g \circ f)^* = f^* \circ g^*$ when $g \circ f$ is defined (so $f\colon \{0,1,\ldots,k\} \to \{0,1,\ldots,\ell\}$ and $g\colon \{0,1,\ldots,\ell\} \to \{0,1,\ldots,m\}$).

Elements of $Y_k$ are often called $k$-*simplices* of $Y$. If $x \in Y_k$ has the form $f^*(y)$ for some $y \in Y_\ell$, then we may say that $x$ is a *face* of $y$ corresponding to face operator $f^*$.

**Remark 9.9.** The definition of a semi-simplicial set can be reformulated in category language as follows. There is a category $\mathcal{C}$ whose objects are the sets $[n] = \{0, 1, \dots, n\}$, where $n$ can be any non-negative integer. A morphism in $\mathcal{C}$ from $[m]$ to $[n]$ is an order-preserving injective map from the set $[m]$ to the set $[n]$. Composition of morphisms is, by definition, composition of such order-preserving injective maps.
A semi-simplicial set is a contravariant functor $Y$ from $\mathcal{C}$ to the category of sets. We like to write $Y_n$ instead of $Y([n])$. We like to write $f^*\colon Y_n \to Y_m$ instead of $Y(f)\colon Y([n]) \to Y([m])$, for a morphism $f\colon [m] \to [n]$ in $\mathcal{C}$.
Nota bene: if you wish to define (invent) a semi-simplicial set $Y$, you need to invent sets $Y_0, Y_1, Y_2, \dots$ (one set $Y_n$ for each integer $n \geq 0$) *and* you need to invent maps $f^*\colon Y_n \to Y_m$, one for each order-preserving injective map $f\colon [m] \to [n]$. *Then* you need to convince yourself that $(g \circ f)^* = f^* \circ g^*$ whenever $f\colon [k] \to [\ell]$ and $g\colon [\ell] \to [m]$ are order-preserving injective maps.

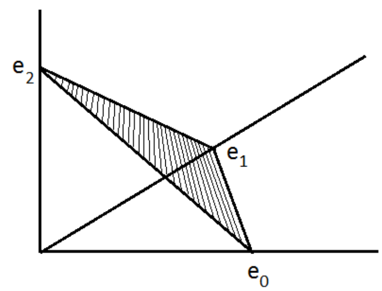
**Example 9.10.** Let $(V, \mathcal{S})$ be a vertex scheme as in the preceding (sub)section. Choose a total ordering of $V$. From these data we can make a semi-simplicial set $Y$ as follows.

- $Y_n$ is the set of all order-preserving injective maps $\beta$ from $\{0, 1, \dots, n\}$ such that $\mathrm{im}(\beta) \in \mathcal{S}$. Note that for each $T \in \mathcal{S}$ of cardinality $n+1$, there is exactly one such $\beta$.
- For an order-preserving injective $f\colon \{0, 1, \dots, m\} \to \{0, 1, \dots, n\}$ and $\beta \in Y_n$ , define $f^*(\beta) = \beta \circ f \in Y_m$.

In order to warm up for geometric realization, we introduce a (covariant) functor from the category $\mathcal{C}$ in remark 9.9 to the category of topological spaces. On objects, the functor is given by

$$\{0, 1, 2, \dots, m\} \mapsto \Delta^m$$

where $\Delta^m$ is the space of functions $u$ from $\{0, 1, \dots, m\}$ to $\mathbb{R}$ which satisfy the condition $\sum_{j=0}^m u(j) = 1$. (As usual we view this as a subspace of the finite-dimensional real vector space of all functions from $\{0, 1, \dots, n\}$ to $\mathbb{R}$. It is often convenient to think of $u \in \Delta^n$ as a vector, $(u_0, u_1, \dots, u_m)$, where all coordinates are $\geq 0$ and their sum is $1$.) Here is a picture of $\Delta^2$ as a subspace of $\mathbb{R}^3$ (with basis vectors $e_0, e_1, e_2$):




For a morphism $f$, meaning an order-preserving injective map

$$f\colon \{0,1,2,\dots,m\} \longrightarrow \{0,1,2,\dots,n\},$$

we want to see an induced map

$$f_*\colon \Delta^m \to \Delta^n.$$

This is easy: for $u = (u_0, u_1, \dots, u_m) \in \Delta^m$ we define

$$f_*(u) = v = (v_0, v_1, \dots, v_n) \in \Delta^n$$

where $v_j = u_i$ if $j = f(i)$ and $v_j = 0$ if $j \notin \mathrm{im}(f)$.
(Keep the following informal conventions in mind. For a covariant functor $G$ from a category $\mathcal{A}$ to a category $\mathcal{B}$, and a morphism $f\colon x \to y$ in $\mathcal{A}$, we often write $f_*\colon G(x) \to G(y)$ instead of $G(f)\colon G(x) \to G(y)$. For a contravariant functor $G$ from a category $\mathcal{A}$ to a category $\mathcal{B}$, and a morphism $f\colon x \to y$ in $\mathcal{A}$, we often write $f^*\colon G(y) \to G(x)$ instead of $G(f)\colon G(y) \to G(x)$.)

The geometric realization $|Y|$ of a semi-simplicial set $Y$ is a topological space defined as follows. Our goal is to have, for each $n \geq 0$ and $y \in Y_n$, a preferred continuous map

$$c_y\colon \Delta^n \to |Y|$$

(the *characteristic map* associated with the simplex $y \in Y_n$). These maps should match in the sense that whenever we have an injective order-preserving

$$f\colon \{0,1,\dots,m\} \to \{0,1,\dots,n\}$$

and $y \in Y_n$ , so that $f^*y \in Y_m$ , then the diagram

$$\begin{tikzcd} \Delta^n \arrow[r, "c_y"] & {|Y|} \\ \Delta^m \arrow[u, "f_*"] \arrow[r, "c_{f^*y}"] & {|Y|} \arrow[u, "="'] \end{tikzcd}$$

is commutative. There is a "most efficient" way to achieve this. As a set, let $|Y|$ be the set of all symbols $\bar{c}_y(u)$ where $y \in Y_n$ for some $n \geq 0$ and $u \in \Delta^n$, modulo the relations[1]

$$\bar{c}_y(f_*(u)) \sim \bar{c}_{f^*y}(u)$$

(notation and assumptions as in that diagram). This ensures that we have maps $c_y \colon \Delta^n \to |Y|$ given in the best tautological manner by

$$c_y(u) := \text{ equivalence class of } \bar{c}_y(u) \text{ .}$$

Also, those little squares which we wanted to be commutative are now commutative because we enforced it. Finally, we say that a subset $U$ of $|Y|$ shall be *open* (definition coming) if and only if $c_y^{-1}(U)$ is open in $\Delta^n$ for each characteristic map $c_y \colon \Delta^n \to |Y|$.

A slightly different way (shorter but possibly less intelligible) to say the same thing is as follows:

$$|Y| := \left( \coprod_{n \geq 0} Y_n \times \Delta^n \right) \Big/ \sim$$

where $\sim$ is a certain equivalence relation on $\coprod_n Y_n \times \Delta^n$. It is the smallest equivalence relation which has $(y, f_*(u))$ equivalent to $(f^*y, u)$ whenever $f \colon \{0, 1, \ldots, m\} \to \{0, 1, \ldots, n\}$ is injective order-preserving and $y \in Y_n$ , $u \in \Delta^m$. Note that, where it says $Y_n \times \Delta^n$ , the set $Y_n$ is regarded as a topological space with the discrete topology, so that $Y_n \times \Delta^n$ has meaning; we could also have written $\coprod_{y \in Y_n} \Delta^n$ instead of $Y_n \times \Delta^n$ .
This new formula for $|Y|$ emphasizes the fact that $|Y|$ is a *quotient space* of a topological disjoint union of many standard simplices $\Delta^n$ (one simplex for every pair $(n, y)$ where $y \in Y_n$). Go ye forth and look up *quotient space* or *identification topology* in your favorite book on point set topology.— To match the second description of $|Y|$ with the first one, let the element of $|Y|$ represented by $(y, u) \in Y_n \times \Delta^n$ in the second description correspond to the element which we called $c_y(u)$ in the first description of $|Y|$.

[1] *Modulo the relations* is short for the following process: form the smallest equivalence relation on the set of all those symbols $\bar{c}_y(u)$ which contains the stated relation. Then pass to the set of equivalence classes for that equivalence relation. That set of equivalence classes is $|Y|$.

**Example 9.11.** Fix an integer $n \geq 0$. We might like to invent a semi-simplicial set $Y$ such that $|Y|$ is homeomorphic to $\Delta^n$. The easiest way to achieve that is as follows. Define $Y_k$ to be the set of all order-preserving injective maps from $\{0, 1, \ldots, k\}$ to $\{0, 1, \ldots, n\}$. So $Y_k$ has $\binom{n+1}{k+1}$ elements (which implies $Y_k = \emptyset$ if $k > n$). For an injective order-preserving map

$$g \colon \{0, 1, \ldots, k\} \to \{0, 1, \ldots, \ell\},$$

define the face operator $g^* \colon Y_\ell \to Y_k$ by $g^*(f) = f \circ g$. This makes sense because $f \in Y_\ell$ is an order-preserving injective map from $\{0, 1, \ldots, \ell\}$ to $\{0, 1, \ldots, n\}$.
There is a unique element $y \in Y_n$, corresponding to the identity map of $\{0, 1, \ldots, n\}$. It is an exercise to verify that the characteristic map $c_y \colon \Delta^n \to |Y|$ is a homeomorphism.

**Example 9.12.** Up to relabeling there is a unique semi-simplicial set $Y$ such that $Y_0$ has exactly one element, $Y_1$ has exactly one element, and $Y_n = \emptyset$ for $n > 1$. Then $|Y|$ is homeomorphic to $S^1$. More precisely, let $z \in Y_1$ be the unique element; then the characteristic map

$$c_z \colon \Delta^1 \longrightarrow |Y|$$

is an identification map. (Translation: it is surjective and a subset of the target is open in the target if and only if its preimage is open in the source.) The only identification taking place is $c_z(a) = c_z(b)$, where $a$ and $b$ are the two boundary points of $\Delta^1$.