Lecture Notes, week 13 and 14
Topology WS 2013/14 (Weiss)

11.1. Homology of the geometric realization

Theorem 11.1. *For every semi-simplicial set* Y *and every* $n \geq 0$*, there is an isomorphism of the* n*-th homology group of the chain complex* $C(Y)$ *with the* n*-th homology group of* $|Y|$*:*

$$H_n(C(Y)) \cong H_n(|Y|)\ .$$

This can be stated in a slightly more precise way: ... *for every* $n \geq 0$ *there is a natural isomorphism*

$$H_n(C(Y)) \xrightarrow{\cong} H_n(|Y|).$$

For this stronger form of the statement, we need to be aware that semi-simplicial sets form a category and that the rule $Y \mapsto C(Y)$ is a functor. How do semi-simplicial sets form a category? One of several definitions of *semi-simplicial set* said that such a thing is a contravariant functor from a certain category $\mathcal{C}$ (with objects $[k] = \{0, 1, \ldots, k\}$ for $k \geq 0$ and with injective order-preserving maps as morphisms) to the category of sets. Therefore it is very reasonable to say that a morphism from one simplicial set X to another, Y, is a natural transformation $X \Rightarrow Y$ between such contravariant functors. In detail, this means that a morphism $\alpha \colon X \to Y$ is given by a sequence $(\alpha_k)_{k\geq 0}$ where each $\alpha_k \colon X_k \to Y_k$ is a map (of sets). The maps α_k are together subject to a strong condition: for each order-preserving injective map $g \colon [k] \longrightarrow [\ell]$ the diagram

$$\begin{array}{ccc} X_\ell & \xrightarrow{\alpha_\ell} & Y_\ell \\ \downarrow g^* & & \downarrow g^* \\ X_k & \xrightarrow{\alpha_k} & Y_k \end{array}$$

is commutative, $g^*\alpha_\ell = \alpha_k g^*$. (It is enough to verify this when $k = \ell - 1$, in which case the custom is to write d_i for g^*, where i is the element of $[\ell]$ not in the image of g. Then the condition becomes $d_i\alpha_\ell = \alpha_{\ell-1}d_i \colon X_\ell \to Y_{\ell-1}$.)

How is the rule $Y \mapsto C(Y)$ a functor from the category of semi-simplicial sets to the category of chain complexes? This is now clear: a morphism $\alpha \colon X \to Y$ of simplicial sets defines a chain map $C(\alpha) \colon C(X) \to C(Y)$ given by homomorphisms

$$C(\alpha_k) \colon C(X)_k \longrightarrow C(Y)_k$$

defined by $\langle x\rangle \mapsto \langle \alpha_k(x)\rangle$, where $\langle x\rangle$ is the generator corresponding to an element $x \in X_k$. The condition for a chain map, $\partial_k \circ C(\alpha_k) = C(\alpha_{k-1}) \circ \partial_k$, is satisfied because ∂_k is defined in terms of the d_i , which α respects.

The naturality part of theorem 11.1 is important because it will help us to prove the theorem. The proof occupies all of this section. We are going to proceed by induction on *skeletons*.

Definition 11.2. The n-*skeleton* of a semi-simplicial set Y is the semi-simplicial set $Y^{\leq n}$ defined by

$$Y_k^{\leq n} = \begin{cases} Y_k & \text{if } k \leq n \\ \emptyset & \text{if } k > n \end{cases}$$

and face operators $Y_\ell^{\leq n} \to Y_k^{\leq n}$ defined like the corresponding ones in Y if $k, \ell \leq n$. (Thus $Y^{\leq n}$ is a semi-simplicial *sub*set of Y. For induction purposes it is useful to allow $n = -1$; we define $Y_k^{\leq -1} = \emptyset$ for all k.)

Lemma 11.3. Let X be a semi-simplicial subset of Y, so that $X_n \subset Y_n$ for all $n \geq 0$ and the face operators $f^*\colon Y_\ell \to Y_k$ take X_ℓ to X_k. Then the map $|X| \to |Y|$ induced by the inclusion $X \to Y$ has closed image and is a homeomorphism onto its image (so that it can be viewed as the inclusion of a closed subspace). In particular, $|Y^{\leq n}|$ can be identified with a closed subspace of $|Y|$.

Proof. It follows from lemma 9.2 for example that $|X| \to |Y|$ is injective. The image is a closed subspace of $|Y|$ by the definition of the topology in $|Y|$. (This is easier to see if we reformulate that definition as follows: *a subset* A *of* $|Y|$ *is closed if its preimage under each of the characteristic maps* $c_y\colon \Delta^k \to |Y|$ *is closed in* Δ^k, *where* $y \in Y_k$. When $A = |X|$, the preimage of A under c_y is the union of some faces of Δ^k. This is obviously a closed subset of Δ^k.) By the same reasoning, applied to $|X|$ and to $|Y|$, we see that a subset of $|X|$ is closed if and only if its image in $|Y|$ is closed. □

Lemma 11.4. *Let* X *and* Y *be spaces,* X *compact Hausdorff. For any mapping cycle* α *from* X *to* Y, *there exists a compact subspace* $K \subset Y$ *such that* α *factors through* K.

Proof. Choose a finite open cover $(U_i)_{i=1,2,\ldots,k}$ of X such that α restricted to any U_i can be written as a formal linear combination, with integer coefficients, of (finitely many) continuous maps: $\sum_j a_{ij} f_{ij}$ where $a_{ij} \in \mathbb{Z}$ and the $f_{ij}\colon U_i \to Y$ are continuous maps. Choose another finite open cover $(V_i)_{i=1,2,\ldots,k}$ of X such that the closure $\bar{V}_i$ of V_i in X is contained in U_i. (This is possible because X is compact Hausdorff.) Let $K \subset Y$ be the union of the finitely many compact sets $f_{ij}(\bar{V}_i)$. □

Lemma 11.5. *Let* Y *be a semi-simplicial set. For any compact subset* K *of* $|Y|$*, there exists a finite semi-simplicial subset* X *of* Y *such that* $K \subset |X| \subset |Y|$*.*

Proof. Suppose that K is not contained in any subspace of the form $|X|$, where X is a finite semi-simplicial subset of Y. Then it must be possible to choose elements $z_j \in K$ for $j = 1, 2, 3, 4, \ldots$ (infinitely many) such that z_j has reduced presentation of the form $c_{y_j}(u)$, where the $y_j \in Y_{n_j}$ are all *distinct* (and $c_{y_j} \colon \Delta^{n_j} \to |Y|$ is the characteristic map associated with y_j, and we are assuming that $u \in \Delta^{n_j}$ does not belong to the boundary). Let

$$W_i = |Y| \smallsetminus \{z_i, z_{i+1}, \ldots\} \,.$$

By construction and by the definition of the topology in $|Y|$, the sets W_i are open in $|Y|$. Clearly $W_i \subset W_{i+1}$ and $\bigcup_i W_i = |Y|$. Therefore the union of all the sets $W_i \cap K$ is all of K. We have found an open covering of K which does not have a finite subcover; contradiction. □

Corollary 11.6. *For every element* u *of* $H_k(|Y|)$*, there exists a finite semi-simplicial subset* X *of* Y *such that* u *is in the image of the inclusion-induced homomorphism* $H_k(|X|) \to H_k(|Y|)$*. Moreover, if* X *and* X' *are finite semi-simplicial subsets of* Y *and* $v \in H_k(|X|)$*,* $w \in H_k(|X'|)$ *have the same image in* $H_k(|Y|)$*, then there exists another finite semi-simplicial subset* X'' *of* Y *such that* $X \subset X''$*,* $X' \subset X''$ *and* v, w *have the same image in* $H_k(|X''|)$*.*

Proof. Given $u \in H_k(|Y|)$, represent it by a mapping cycle from S^k to $|Y|$. By lemmas 11.4 and 11.5, that mapping cycle factors through $|X|$ for some finite semi-simplicial subset X of Y. This proves the first part. Now suppose that $v \in H_k(|X|)$ and $w \in H_k(|X'|)$ have the same image in $H_k(|Y|)$, for finite semi-simplicial subsets X and X' of Y. Recall that $H_k(|X|)$ can be defined as the cokernel of $[[\star, |X|]] \to [[S^k, |X|]]$ or alternatively as the kernel of $[[S^k, |X|]] \to [[\star, |X|]]$, where the first homomorphism is induced by the projection $S^k \to \star$ and the other is induced by the inclusion of the base point in the sphere, $\star \to S^k$. Here the second definition is more useful, so we represent v by a mapping cycle $\alpha \colon S^k \to |X|$ such that the composition of α with $\star \to S^k$ is homotopic to 0 as a mapping cycle. In the same way, we represent w by a mapping cycle $\beta \colon S^k \to |X'|$ such that the composition of β with $\star \to S^k$ is homotopic to 0 as a mapping cycle. Since v and w have the same image in $H_k(|Y|)$, there exists a suitable homotopy, i.e., a mapping cycle $\gamma \colon S^k \times [0, 1] \to |Y|$ which restricts to α on $S^k \times \{0\} \cong S^k$ and to β on $S^k \times \{1\} \cong S^k$. By lemma 11.4 and 11.5, that mapping cycle γ factors through $|X''|$ for some finite semi-simplicial subset X'' of Y, and we can enlarge X if necessary to ensure $X'' \supset X$ and $X'' \supset X'$. Then clearly v and w have the same image in $H_k(|X''|)$. □

Lemma 11.7. *For every element* u *of* $H_k(|Y|)$*, there exists* $n \geq 0$ *such that* u *is in the image of the inclusion-induced homomorphism from* $H_k(|Y^{\leq n}|)$ *to* $H_k(|Y|)$*. Moreover, if* $v \in H_k(|Y^{\leq m}|)$*,* $w \in H_k(|Y^{\leq n}|)$ *have the same image in* $H_k(|Y|)$*, then there exists* $\ell \geq m, n$ *such that* v *and* w *have the same image already in* $H_k(|Y^{\leq \ell}|)$*.*

This can be deduced from corollary 11.6, or proved in the same way. — Let's now ask how $H_*(|Y^{\leq n-1}|)$ is related to $H_*(|Y^{\leq n}|)$.

Lemma 11.8. *The homomorphism*

$$H_k(|Y^{\leq n-1}|) \to H_k(|Y^{\leq n}|)$$

induced by the inclusion of $|Y^{\leq n-1}|$ *in* $|Y^{\leq n}|$ *is an isomorphism for* $k < n-1$*, whereas* $H_k(|Y^{\leq n-1}|) = 0$ *for* $k > n-1$*. For* $k = n-1$ *this homomorphism is part of an exact sequence*

$$0 \longrightarrow H_n(|Y^{\leq n}|) \xrightarrow{\gamma_n} \bigoplus_{x \in Y_n} \mathbb{Z} \xrightarrow{\beta_n} H_{n-1}(|Y^{\leq n-1}|) \longrightarrow H_{n-1}(|Y^{\leq n}|) \longrightarrow 0\ .$$

Proof. We can assume $n \geq 1$. Let $W = |Y^{\leq n}| \smallsetminus |Y^{\leq n-1}|$. In other words, W consists of all points which can be written in the form $c_y(u)$ for some $y \in Y_n$ (with characteristic map $c_y \colon \Delta^n \to |Y^{\leq n}|$) and some $u \in \Delta^n$ whose barycentric coordinates are all nonzero: $u_0, \dots, u_n > 0$. Let V be the subset of $|Y^{\leq n}|$ obtained by taking out all points of the form $c_y(b)$ where $y \in Y_n$ and $b \in \Delta^n$ is the barycenter, that is, $b_0 = b_1 = \cdots = b_n = 1/(n+1)$. Then V and W are open subsets and $V \cup W = |Y^{\leq n}|$.
Clearly W is homeomorphic to a disjoint union of copies of $\mathbb{R}^n$, one copy for each $y \in Y_n$. Therefore clearly $V \cap W$ (viewed as a subspace of W, if you wish) is homeomorphic to a disjoint union of copies of $\mathbb{R}^n \smallsetminus \{0\}$, and consequently homotopy equivalent to a disjoint union of copies of S^{n_1}, one copy for each $y \in Y_n$. This means that we know the homology groups of W and of $V \cap W$.
Regarding V, we show that the inclusion $\iota \colon |Y^{\leq n-1}| \to V$ is a homotopy equivalence. A map in the opposite direction, $r \colon V \to |Y^{\leq n-1}|$, is given by $r(z) = z$ if $r \notin V \cap W$ and

$$r(c_y(u)) \mapsto c_y(\rho(u))$$

where we assume $y \in Y_n$ and $u \in \Delta^n$ not equal to the barycenter b , and $\rho(u)$ is the point in the boundary of Δ^n where the straight line through b and u meets the boundary. Then the composition $r \circ \iota$ is the identity on $|Y^{\leq n-1}|$, and $\iota \circ r$ is homotopic to the identity on V.
Now we are in a good position to understand the Mayer-Vietoris sequence:

$$\cdots \to H_k(V \cap W) \to H_k(V) \oplus H_k(W) \to H_k(V \cup W) \to H_{k-1}(V \cap W) \to \cdots$$

Since $V \cap W$ is homotopy equivalent to a disjoint union of copies of S^{n-1}, the homology groups $H_k(V \cap W)$ are nonzero only for $k = n-1$ and $k = 0$. The homology groups $H_k(W)$ are nonzero only for $k = 0$. It is routine to make the following deduction from the exactness of the Mayer-Vietoris sequence:

> The inclusion $V \to V \cup W$ induces an isomorphism
> $$H_k(V) \to H_k(V \cup W)$$
> when $k < n-1$. (This is trivially true when $n = 1$. In the cases $n > 1$, the inclusion $V \cap W \to W$ induces an isomorphism from $H_0(V \cap W)$ to $H_0(W)$, and this should be used, too.)

Exactness of the Mayer-Vietoris sequence also permits us to show by induction on n that

> $H_k(|Y^{\leq n}|) = H_k(V \cup W)$ is zero for $k > n$.

Therefore, if $n > 1$, the interesting part of the Mayer-Vietoris sequence (where k is close to n) is an exact sequence

$$0 \to H_n(V \cup W) \to \bigoplus_{x \in Y_n} \mathbb{Z} \to H_{n-1}(V) \to H_{n-1}(V \cup W) \to 0$$

where the 0 on the right is justified as the kernel of the (injective) homomorphism $H_{n-2}(V \cap W) \to H_{n-2}(V) \oplus H_{n-2}(W)$. If $n = 1$ we have instead an exact sequence

$$0 \to H_1(V \cup W) \to \bigoplus_{x \in Y_n} (\mathbb{Z} \oplus \mathbb{Z}) \to H_0(V) \oplus \bigoplus_{x \in Y_n} \mathbb{Z} \to H_0(V \cup W) \to 0 \,.$$

Here the composite homomorphism

$$\bigoplus_x (\mathbb{Z} \oplus \mathbb{Z}) \longrightarrow H_0(V) \oplus \bigoplus_x \mathbb{Z} \xrightarrow{\text{proj.}} \bigoplus_x \mathbb{Z}$$

is onto by inspection and its kernel is the antidiagonal $\bigoplus_x \mathbb{Z}_a$, where $\mathbb{Z}_a \subset \mathbb{Z} \oplus \mathbb{Z}$ consists of all pairs of integers of the form $(r, -r)$. Therefore we can remove some terms and obtain an exact sequence

$$0 \to H_1(V \cup W) \to \bigoplus_{x \in Y_n} \mathbb{Z} \to H_0(V) \to H_0(V \cup W) \to 0$$

in the case $n = 1$, too. Finally, using that $V \cup W = |Y^{\leq n}|$ and $V \simeq |Y^{\leq n-1}|$, we have the exact sequence that we wanted. □

Corollary 11.9. *The inclusion* $|Y^{\leq n}| \to |Y|$ *induces an isomorphism in* H_k *for* $k < n$*, and a surjection for* $k = n$*.*

Proof. We have a sequence of inclusion-induced homomorphisms

$$H_k(|Y^{\leq n}|) \to H_k(|Y^{\leq n+1}|) \to H_k(|Y^{\leq n+2}|) \to \cdots$$

If $k < n$, all homomorphisms in the sequence are isomorphisms by lemma 11.8. Combine this with lemma 11.7 to deduce that $H_k(|Y^{\leq n}|) \to H_k(|Y|)$ is an isomorphism. If $k = n$, the first homomorphism in the sequence is surjective, the others are isomorphisms. Again combine his with lemma 11.7 to deduce that $H_k(|Y^{\leq n}|) \to H_k(|Y|)$ is surjective.

Corollary 11.10. $H_k(Y) \cong H_k(D(Y))$ *where* $D(Y)$ *is the chain complex defined as follows:*

$$D(Y)_n = C(Y)_n = \bigoplus_{x \in Y_n} \mathbb{Z}$$

for $n \geq 0$ *and the differential* $D(Y)_n \to D(Y)_{n-1}$ *is* $\gamma_{n-1} \circ \beta_n$ *with notation as in lemma 11.8.*

(We take the view that γ_0 is defined and is an isomorphism from $H_0(|Y^{\leq 0}|)$ to $\bigoplus_{y \in Y_0} \mathbb{Z}$. This is in good agreement with lemma 11.8. Note also that

$$(\gamma_{n-1} \circ \beta_n) \circ (\gamma_n \circ \beta_{n+1}) = 0$$

because already $\beta_n \circ \gamma_n$ is zero, as can be seen in lemma 11.8. Therefore $D(Y)$ is indeed a chain complex.)

Proof. From lemma 11.8 we obtain

$$H_n(|Y|) \cong H_n(|Y^{\leq n+1}|) \cong \frac{H_n(|Y^{\leq n}|)}{\mathrm{im}(\beta_{n+1})} \cong^a \frac{\ker(\beta_n)}{\mathrm{im}(\gamma_n \circ \beta_{n+1})} =^b \frac{\ker(\gamma_{n-1} \circ \beta_n)}{\mathrm{im}(\gamma_n \circ \beta_{n+1})}$$

if $n > 0$. The isomorphism with superscript a is obtained by using the injective homomorphism γ_n to identify $H_n(|Y^{\leq n}|)$ with the subgroup $\ker(\beta_n)$ of $\bigoplus_{y \in Y_n} \mathbb{Z}$. The equality with superscript b is based on the observation that γ_{n-1} is injective. For $n = 0$ we get

$$H_0(|Y|) \cong H_0(|Y^{\leq 1}|) \cong \frac{H_0(|Y^{\leq 0}|)}{\mathrm{im}(\beta_1)} \cong^a \frac{\bigoplus_{y \in Y_0} \mathbb{Z}}{\mathrm{im}(\gamma_0 \circ \beta_1)}$$

which is again exactly what we want. ∎

To finish the proof of theorem 11.1 we need to show that the chain complexes $D(Y)$ and $C(Y)$ are the same, or at least isomorphic. We already have $C(Y)_n = D(Y)_n$ by construction. It would be wonderful to know that the boundary homomorphism $\gamma_{n-1} \circ \beta_n \colon D(Y)_n \to D(Y)_{n-1}$ agrees with the boundary homomorphism $C(Y)_n \to C(Y)_{n-1}$ from the definition of the chain complex $C(Y)$.

Lemma 11.11. *These two boundary homomorphisms agree up to a sign; in other words*

$$(\gamma_{n-1} \circ \beta_n)\langle x \rangle = \pm \sum_{i=0}^{n} (-1)^i \langle d_i x \rangle$$

for $x \in Y_n$ *with corresponding basis element* $\langle x \rangle \in D(Y)_n$. *The sign* $\pm$ *depends on* n, *but not on* Y *or* $x \in Y_n$.

Proof. We begin with the important observation that the construction of $\gamma_{n-1} \circ \beta_n$ was *natural.* To be more explicit, a morphism $f\colon X \to Y$ of semi-simplicial sets determines (for any fixed $n \geq 0$) a homomorphism from $D(X)_n$ to $D(Y)_n$ given by $\langle x \rangle \mapsto \langle f(x) \rangle$ for $x \in X_n$, and the resulting diagram

(11.12)
$$\begin{array}{ccc} D(X)_n & \xrightarrow{\gamma_{n-1}\circ\beta_n} & D(X)_{n-1} \\ \downarrow & & \downarrow \\ D(Y)_n & \xrightarrow{\gamma_{n-1}\circ\beta_n} & D(Y)_{n-1} \end{array}$$

is *commutative.* (So $Y \mapsto D(Y)_n$ and $Y \mapsto D(Y)_{n-1}$ are functors, rather obviously, and now $\gamma_{n-1} \circ \beta_n$ is claimed to be a natural transformation from one to the other.) The reason for this is that

- $f\colon X \to Y$ determines a map $|f^{\leq k}|\colon |X^{\leq k}| \to |Y^{\leq k}|$ for every k;
- the preimages under the map $|f^{\leq k}|$ of certain open subsets $V = V_{Y,k}$ and $W = W_{Y,k}$ of $|Y^{\leq k}|$ which we used to set up a Mayer-Vietoris sequence and so to construct β_k and γ_k are precisely $V_{X,k}$ and $W_{X,k}$, open subsets of $|X^{\leq k}|$.

Therefore naturality of the Mayer-Vietoris sequence applies, and we get the commutativity of (11.12). — It follows almost immediately that it suffices to prove our formula

$$(\gamma_{n-1} \circ \beta_n)\langle x \rangle = \pm \sum_{i=0}^{n} (-1)^i \langle d_i x \rangle$$

in the very special case where $Y = \underline{\Delta}^n$ and x is the unique element in Y_n. (Recall that $Y = \underline{\Delta}^n$ is defined in such a way that Y_k for $k \geq 0$ is the set of monotone injective maps from $\{0, 1, \ldots, k\}$ to $\{0, 1, \ldots, n\}$ and boundary operators are given by composition. We have $|\underline{\Delta}^n| \cong \Delta^n$.) Indeed, if Z is any other semi-simplicial set and $x' \in Z_n$ and we wish to know what $\gamma_{n-1} \circ \beta_n$ does to $\langle x' \rangle$, then we observe that there is precisely one morphism

$$Y = \underline{\Delta}^n \longrightarrow Z$$

which takes the unique $x \in Y_n$ to $x' \in Z_n$. So we know what $\gamma_{n-1} \circ \beta_n$ does to $\langle x' \rangle$ if we know what it does to $\langle x \rangle$, by the commutativity of (11.12).

Having made the observation, we proceed by induction on n. The case $n = 1$ is the induction start. It is not completely trivial. There are exactly two distinct morphisms $\underline{\Delta}^0 \to \underline{\Delta}^1$ of semi-simplicial sets. We know that they induce the same homomorphism $H_0(|\underline{\Delta}^0|) \to H_0(|\underline{\Delta}^1|)$. It follows with lemma 11.8 that they induce the same homomorphism

$$H_0(D(\underline{\Delta}^0)) \longrightarrow H_0(D(\underline{\Delta}^1)).$$

Therefore $\langle d_0x\rangle - \langle d_1x\rangle$ represents zero in $H_0(D(\underline{\Delta}^1))$, and so it must be in the image of

$$\gamma_0 \circ \beta_1 \colon\ D(\underline{\Delta}^1)_1 \cong \mathbb{Z} \ \longrightarrow\ D(\underline{\Delta}^1))_0 = \mathbb{Z} \oplus \mathbb{Z}\,.$$

That can only happen if the generator $\langle x\rangle$ of $D(\underline{\Delta}^1)_1 = \mathbb{Z}$ is taken to

$$\pm(\ \langle d_0x\rangle - \langle d_1x\rangle\ )$$

by $\gamma_0 \circ \beta_1$. This takes care of the case $n = 1$.
For the induction step we assume $n > 1$. The inductive assumption tells us what $\gamma_{n-2} \circ \beta_{n-1} \colon D(\underline{\Delta}^n)_{n-1} \to D(\underline{\Delta}^n)_{n-2}$ is, up to sign. It follows by direct computation that the element

$$\sum_{i=0}^{n} (-1)^i \langle d_ix\rangle \ \in D(\underline{\Delta}^n)_{n-1}$$

is in the kernel of $\gamma_{n-2} \circ \beta_{n-1}$. Therefore it is in the image of $\gamma_{n-1} \circ \beta_n$, since $H_{n-1}(D(\underline{\Delta}^n)) \cong H_{n-1}(|\underline{\Delta}^n|) = 0$ by lemma 11.8. But this can only happen if the generator $\langle x\rangle$ of $D(\underline{\Delta}^n)_n = \mathbb{Z}$ is taken to $\pm(\sum_{i=0}^{n}(-1)^i\langle d_ix\rangle)$ by $\gamma_{n-1} \circ \beta_n$. □

Proof of theorem 11.1. Write ∂_n for the differentials in $C(Y)$, and $\partial'_n = \gamma_{n-1} \circ \beta_n$ for the differentials in $D(Y)$. By lemma 11.11, we have $\partial'_n = a_n \cdot \partial_n$ where $a_n \in \{-1, +1\}$. This is meaningful because $D(Y)_n = C(Y)_n$ for all n. An isomorphism u of chain complexes from $C(Y)$ to $D(Y)$ can be defined by $u_0 = \mathrm{id} \colon C(Y)_0 \to D(Y)_0$ and $u_n = a_n a_{n-1} \cdots a_2 a_1 \cdot \mathrm{id} \colon C(Y)_n \to D(Y)_n$ for $n > 0$. □

Remark. The undetermined sign in lemma 11.11 is slightly annoying. It can be determined with more work (to me, more annoying). Without a doubt it is also possible to re-set some basic definitions in such a way that the undetermined sign turns out to be always $+$. There are a few places where we had a choice of sign: notably in defining the boundary operator of the Mayer-Vietoris sequence, but also in choosing the order of V and W in the proof of lemma 11.8.

11.2. Semi-simplicial sets and the foundations of homology theory

There is a construction (a functor) which to a topological space X associates a semi-simplicial set SX. This is quite important in the more standard treatments of homology theory, even though in some of those standard treatments it does not appear explicitly.

Definition 11.13. The semi-simplicial set SX has

$$SX_n = \text{ set of continuous maps from } \Delta^n \text{ to } X$$

for $n \geq 0$. Face operators are given by composition. More precisely, if

$$f\colon \{0, 1, \ldots, m\} \to \{0, 1, \ldots, n\}$$

is monotone injective, and $\sigma \in SX_n$, meaning that $\sigma\colon \Delta^n \to X$ is a continuous map, then we let $f^*\sigma := \sigma \circ f_* \in SX_m$, where $f_*\colon \Delta^m \to \Delta^n$ is the "linear" map determined by f.

In most cases SX is huge. For example, if $X = S^1$, then clearly each of the sets SX_n is uncountable. But for every X there is a comparison map

$$\kappa\colon |SX| \longrightarrow X \,.$$

It is defined in such a way that for every $\sigma \in SX_n$, the composition of characteristic map $c_\sigma\colon \Delta_n \to |SX|$ with $\kappa\colon |SX| \to X$ agrees with the map $\sigma\colon \Delta^n \to X$ itself. In many cases the map $\kappa\colon |SX| \to X$ is a homotopy equivalence. (It is not a homotopy equivalence in all cases because there are topological spaces which are not at all homotopy equivalent to the geometric realization of any semi-simplicial set. In fact this is the only source of trouble. If X *is* homotopy equivalent to the geometric realization of some semi-simplicial set, then $\kappa\colon |SX| \to X$ is a homotopy equivalence. But the proof will not be given here.)
Even though $\kappa\colon |SX| \to X$ is not a homotopy equivalence in all cases, it can be shown that the homomorphism $H_n(|SX|) \to H_n(X)$ induced by κ is always an isomorphism, for all $n \geq 0$. (That proof will not be given here either.) Therefore

$$H_n(X) \cong H_n(|SX|) \cong H_n(C(SX))$$

using theorem 11.1. The right-hand expression, $H_n(C(SX))$, is the definition of the n-th homology group of X given in many standard treatments. This means that chain complexes and their homology are prominent from the very beginning in those treatments. The standard simplices and the maps $f_*\colon \Delta^m \to \Delta^n$ are also prominent from the start. Semi-simplicial sets need not make an explicit appearance, because it is easy to describe the chain complex $C(SX)$ is without explaining what a semi-simplicial set is.