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Lecture Notes, week 1
Topology WS 2013/14 (Weiss)

1.1. The homotopy relation

Let X and Y be topological spaces. (If you are not sufficiently familiar with topological spaces, you should assume that X and Y are metric spaces.) Let f and g be continuous maps from X to Y. Let $[0,1]$ be the unit interval with the standard topology, a subspace of $\mathbb{R}$.

Definition 1.1. A *homotopy* from f to g is a continuous map

$$h : X \times [0,1] \to Y$$

such that $h(x,0) = f(x)$ and $h(x,1) = g(x)$ for all $x \in X$. If such a homotopy exists, we say that f and g are *homotopic*, and write $f \simeq g$. We also sometimes write $h : f \simeq g$ to indicate that h is a homotopy from the map f to the map g.

Remark 1.2. If you made the assumption that X and Y are metric spaces, then you should use the product metric on $X \times [0,1]$ and $Y \times [0,1]$, so that for example

$$d((x_1,t_1),(x_2,t_2)) := \max\{d(x_1,x_2), |t_1 - t_2|\ \}$$

for $x_1, x_2 \in X$ and $t_1, t_2 \in [0,1]$. If you were happy with the assumption that X and Y are "just" topological spaces, then you need to know the definition of *product of two topological spaces* in order to make sense of $X \times [0,1]$ and $Y \times [0,1]$.

Remark 1.3. A homotopy $h : X \times [0,1] \to Y$ from $f : X \to Y$ to $g : X \to Y$ can be seen as a "family" of continuous maps

$$h_t : X \to Y\ ;\ h_t(x) = h(x,t)$$

such that $h_0 = f$ and $h_1 = g$. The important thing is that h_t depends continuously on $t \in [0,1]$.

Example 1.4. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the identity map. Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be the map such that $g(x) = 0 \in \mathbb{R}^n$ for all $x \in \mathbb{R}^n$. Then f and g are homotopic. The map $h : \mathbb{R}^n \times [0,1]$ defined by $h(x,t) = tx$ is a homotopy from f to g.

Example 1.5. Let $f : S^1 \to S^1$ be the identity map, so that $f(z) = z$. Let $g : S^1 \to S^1$ be the antipodal map, $g(z) = -z$. Then f and g are homotopic. Using complex number notation, we can define a homotopy by $h(z,t) = e^{\pi i t} z$.

Example 1.6. Let $f : S^2 \to S^2$ be the identity map, so that $f(z) = z$. Let $g : S^2 \to S^2$ be the antipodal map, $g(z) = -z$. Then f and g are *not* homotopic. We will prove this later in the course.

Example 1.7. Let $f : S^1 \to S^1$ be the identity map, so that $f(z) = z$. Let $g : S^1 \to S^1$ be the constant map with value 1. Then f and g are *not* homotopic. We will prove this quite soon.

Proposition 1.8. *"Homotopic" is an equivalence relation on the set of continuous maps from X to Y.*

Proof. Reflexive: For every continuous map $f : X \to Y$ define the *constant homotopy* $h : X \times [0,1] \to Y$ by $h(x,t) = f(x)$.
Symmetric: Given a homotopy $h : X \times [0,1] \to Y$ from a map $f : X \to Y$ to a map $g : X \to Y$, define the *reverse homotopy* $\bar{h} : X \times [0,1] \to Y$ by $\bar{h}(x,t) = h(x, 1-t)$. Then $\bar{h}$ is a homotopy from g to f.
Transitive: Given continuous maps $e, f, g : X \to Y$, a homotopy h from e to f and a homotopy k from f to g, define the *concatenation homotopy* $k * h$ as follows:

$$(x,t) \mapsto \begin{cases} h(x,2t) & \text{if } 0 \leqslant t \leqslant 1/2 \\ k(x,2t-1) & \text{if } 1/2 \leqslant t \leqslant 1 \,. \end{cases}$$

Then $k * h$ is a homotopy from e to g. $\square$

Definition 1.9. The equivalence classes of the above relation "homotopic" are called *homotopy classes*. The homotopy class of a map $f : X \to Y$ is often denoted by $[f]$. The set of homotopy classes of maps from X to Y is often denoted by $[X, Y]$.

Proposition 1.10. *Let X, Y and Z be topological spaces. Let $f : X \to Y$ and $g : X \to Y$ and $u : Y \to Z$ and $v : Y \to Z$ be continuous maps. If f is homotopic to g and u is homotopic to v, then $u \circ f : X \to Z$ is homotopic to $v \circ g : X \to Z$.*

Proof. Let $h : X \times [0,1] \to Y$ be a homotopy from f to g and let $w : Y \times [0,1] \to Z$ be a homotopy from u to v. Then $u \circ h$ is a homotopy from $u \circ f$ to $u \circ g$ and the map $X \times [0,1] \to Z$ given by $(x,t) \mapsto w(g(x),t)$ is a homotopy from $u \circ g$ to $v \circ g$. Because the homotopy relation is transitive, it follows that $u \circ f \simeq v \circ g$. $\square$

Definition 1.11. Let X and Y be topological spaces. A (continuous) map $f : X \to Y$ is a *homotopy equivalence* if there exists a map $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.
We say that X is *homotopy equivalent* to Y if there exists a map $f : X \to Y$ which is a homotopy equivalence.

Definition 1.12. If a topological space X is homotopy equivalent to a point, then we say that X is *contractible*. This amounts to saying that the identity map $X \to X$ is homotopic to a constant map from X to X.

Example 1.13. $\mathbb{R}^m$ is contractible, for any $m \geq 0$.

Example 1.14. $\mathbb{R}^m \smallsetminus \{0\}$ is homotopy equivalent to S^{m-1}.

Example 1.15. The general linear group of $\mathbb{R}^m$ is homotopy equivalent to the orthogonal group $\mathrm{O}(m)$. The Gram-Schmidt orthonormalisation process leads to an easy proof of that.

1.2. Homotopy classes of maps from the circle to itself

Let $p\colon \mathbb{R} \to S^1$ be the (continuous) map given in complex notation by $p(t) = \exp(2\pi i t)$ and in real notation by $p(t) = (\cos(2\pi t), \sin(2\pi t))$. In the first formula we think of S^1 as a subset of $\mathbb{C}$ and in the second formula we think of S^1 as a subset of $\mathbb{R}^2$.
Note that p is surjective and $p(t+1) = p(t)$ for all $t \in \mathbb{R}$. We are going to use p to understand the homotopy classification of continuous maps from S^1 to S^1. The main lemma is as follows.

Lemma 1.16. *Let $\gamma\colon [0,1] \to S^1$ be a continuous map and let $a \in \mathbb{R}$ be such that $p(a) = \gamma(0)$. Then there exists a unique continuous map $\tilde{\gamma}\colon [0,1] \to \mathbb{R}$ such that $\gamma = p \circ \tilde{\gamma}$ and $\tilde{\gamma}(0) = a$.*

Proof. The map γ is *uniformly continuous* since $[0,1]$ is compact. It follows that there exists a positive integer n such that $d(\gamma(x), \gamma(y)) < 1/100$ whenever $|x - y| \leq 1/n$. Here d denotes the standard (euclidean) metric on S^1 as a subset of $\mathbb{R}^2$. We choose such an n and write

$$[0,1] = \bigcup_{k=1}^{n} [t_{k-1}, t_k]$$

where $t_k = k/n$. We try to define $\tilde{\gamma}$ on $[0, t_k]$ by induction on k. For the induction beginning we need to define $\tilde{\gamma}$ on $[0, t_1]$ where $t_1 = 1/n$. Let $U \subset S^1$ be the open ball of radius $1/100$ with center $\gamma(0)$. (Note that *open ball* is a metric space concept.) Then $\gamma([0,t_1]) \subset U$. Therefore $\tilde{\gamma}([0,t_1])$ must be contained in $p^{-1}(U)$. Now $p^{-1}(U) \subset \mathbb{R}$ is a disjoint union of open intervals which are mapped homeomorphically to U under p. One of these, call it V_a , contains a, since $p(a) = \gamma(0) \in U$. The others are translates of the form $\ell + V_a$ where $\ell \in \mathbb{Z}$. Since $[0, t_1]$ is connected, its image under $\tilde{\gamma}$ will also be connected, whatever $\tilde{\gamma}$ is, and so it must be contained entirely in exactly one of the intervals $\ell + V_a$. Since we want $\tilde{\gamma}(0) = a$, we must have $\ell = 0$, that is, image of $\tilde{\gamma}$ contained in V_a. Since the map p restricts to a homeomorphism from V_a to U, we must have $\tilde{\gamma} = q\gamma$ where q is the

inverse of the homeomorphism from V_a to U. This formula determines the map $\tilde{\gamma}$ on $[0, t_1]$.
The induction steps are like the induction beginning. In the next step we define $\tilde{\gamma}$ on $[t_1, t_2]$, using a "new" a which is $\tilde{\gamma}(t_1)$ and a "new" U which is the open ball of radius $1/100$ with center $\gamma(t_1)$. □

Now let $g \colon S^1 \to S^1$ be any continuous map. We want to associate with it an integer, the degree of g. Choose $a \in \mathbb{R}$ such that $p(a) = g(1)$. Let $\gamma = g \circ p$ on $[0, 1]$; this is a map from $[0, 1]$ to S^1. Construct $\tilde{\gamma}$ as in the lemma. We have $p\tilde{\gamma}(1) = \gamma(1) = \gamma(0) = p\tilde{\gamma}(0)$, which implies $\tilde{\gamma}(1) = \tilde{\gamma}(0) + \ell$ for some $\ell \in \mathbb{Z}$.

Definition 1.17. This ℓ is the degree of g, denoted $\deg(g)$.

It looks as if this might depend on our choice of a with $p(a) = g(1)$. But if we make another choice then we only replace a by $m + a$ for some $m \in \mathbb{Z}$, and we only replace $\tilde{\gamma}$ by $m + \tilde{\gamma}$. Therefore our calculation of $\deg(g)$ leads to the same result.

Remark. Suppose that $g \colon S^1 \to S^1$ is a continuous map which is close to the constant map $z \mapsto 1 \in S^1$ (complex notation). To be more precise, assume $d(g(z), 1) < 1/1000$ for all $z \in S^1$. Then $\deg(g) = 0$.
The verification is mechanical. Define $\gamma \colon [0, 1] \to S^1$ by $\gamma(t) = g(p(t))$. Let $V \subset \mathbb{R}$ be the open interval from $-1/100$ to $1/100$. The map p restricts to a homeomorphism from V to $p(V) \subset S^1$, with inverse $q \colon p(V) \to V$. Put $\tilde{\gamma} = q \circ \gamma$, which makes sense because the image of γ is contained in $p(V)$ by our assumption. Then $p \circ \tilde{\gamma} = \gamma$ as required. Now the image of $\tilde{\gamma}$ is contained in V and therefore

$$|\deg(g)| = |\tilde{\gamma}(1) - \tilde{\gamma}(0)| \leq 2/100$$

and so $\deg(g) = 0$.

Remark. Suppose that $f, g \colon S^1 \to S^1$ are continuous maps. Let $w \colon S^1 \to S^1$ be defined by $w(z) = f(z) \cdot g(z)$ (using the multiplication in $S^1 \subset \mathbb{C}$). Then $\deg(w) = \deg(f) + \deg(g)$.
The verification is also mechanical. Define $\varphi, \gamma, \omega \colon [0, 1] \to S^1$ by $\varphi(t) = f(p(t))$, $\gamma(t) = g(p(t))$ and $\omega(t) = w(p(t))$. Construct $\tilde{\varphi} \colon [0, 1] \to \mathbb{R}$ and $\tilde{\gamma} \colon [0, 1] \to \mathbb{R}$ as in lemma 1.16. Put $\tilde{\omega} := \tilde{\varphi} + \tilde{\gamma}$. Then $p \circ \tilde{\omega} = \omega$, so

$$\deg(w) = \tilde{\omega}(1) - \tilde{\omega}(0) = \cdots = \deg(f) + \deg(g).$$

Lemma 1.18. *If $f, g \colon S^1 \to S^1$ are continuous maps which are homotopic, $f \sim g$, then they have the same degree.*

Proof. Let $h \colon S^1 \times [0, 1] \to S^1$ be a homotopy from f to g. As usual let $h_t \colon S^1 \to S^1$ be the map defined by $h_t(z) = h(z, t)$, for fixed $t \in [0, 1]$. For

fixed $t \in [0,1]$ we can find $\delta > 0$ such that $d(h_t(z), h_s(z)) < 1/1000$ for all $z \in S^1$ and all s which satisfy $|s-t| < \delta$. Therefore $h_s(z) = g_s(z) \cdot h_t(z)$ for such s, where $g_s \colon S^1 \to S^1$ is a map which satisfies $d(g_s(z), 1) < 1/1000$ for all $z \in S^1$. Therefore $\deg(g_s) = 0$ by the remarks above and so $\deg(h_s) = \deg(g_s) + \deg(h_t) = \deg(h_t)$.
We have now shown that the the map $[0,1] \to \mathbb{Z}$ given by $t \mapsto \deg(h_t)$ is locally constant (equivalently, *continuous* as a map of metric spaces) and so it is constant (since $[0,1]$ is connected). In particular $\deg(f) = \deg(h_0) = \deg(h_1) = \deg(g)$. □

Lemma 1.19. *If $f, g \colon S^1 \to S^1$ are continuous maps which have the same degree, then they are homotopic.*

Proof. Certainly f is homotopic to a map which takes 1 to 1 and g is homotopic to a map which takes 1 to 1 (using complex notation, $1 \in S^1 \subset \mathbb{C}$). Therefore we can assume without loss of generality that $f(1) = 1$ and $g(1) = 1$.
Let $\varphi \colon [0,1] \to S^1$ and $\gamma \colon [0,1] \to S^1$ be defined by $\varphi(t) = f(p(t))$ and $\gamma(t) = g(p(t))$. Construct $\tilde{\varphi}$ and $\tilde{\gamma}$ as in the lemma, using $a = 0$ in both cases, so that $\tilde{\varphi}(0) = 0 = \tilde{\gamma}(0)$. Then

$$\tilde{\varphi}(1) = \deg(f) = \deg(g) = \tilde{\gamma}(1).$$

Note that f can be recovered from $\tilde{\varphi}$ as follows. For $z \in S^1$ choose $t \in [0,1]$ such that $p(t) = z$. Then $f(z) = f(p(t)) = \varphi(t) = p\tilde{\varphi}(t)$. If $z = 1 \in S^1$, we can choose $t = 0$ or $t = 1$, but this ambiguity does not matter since $p\tilde{\varphi}(1) = p\tilde{\varphi}(0)$. Similarly, g can be recovered from $\tilde{\gamma}$. Therefore we can show that f is homotopic to g by showing that $\tilde{\varphi}$ is homotopic to $\tilde{\gamma}$ *with endpoints fixed.* In other words we need a continuous

$$H \colon [0,1] \times [0,1] \to \mathbb{R}$$

where $H(s,0) = \tilde{\varphi}(s)$, $H(s,1) = \tilde{\gamma}(s)$ and $H(0,t) = 0$ for all $t \in [0,1]$ and $H(1,t) = \tilde{\varphi}(1) = \tilde{\gamma}(1)$ for all $t \in [0,1]$. This is easy to do: let $H(s,t) = (1-t)\tilde{\varphi}(s) + t\tilde{\gamma}(s)$. □

Summarizing, we have shown that the degree function gives us a well defined map from $[S^1, S^1]$ to $\mathbb{Z}$, and moreover, that this map is injective. It is not hard to show that this map is also surjective! Namely, for arbitrary $\ell \in \mathbb{Z}$ the map $f \colon S^1 \to S^1$ given by $f(z) = z^\ell$ (complex notation) has $\deg(f) = \ell$. (Verify this.)

Corollary 1.20. *The degree function is a bijection from $[S^1, S^1]$ to $\mathbb{Z}$.* □