07.11.2013

Lecture Notes, week 3 and 4
Topology WS 2013/14 (Weiss)

3.1. Pullbacks of fiber bundles

Let $p\colon E \to B$ be a fiber bundle. Let $g\colon X \to B$ be any continuous map of topological spaces.

Definition 3.1. The pullback of $p\colon E \to B$ along g is the space

$$g^*E := \{\,(x,y) \in X \times E \mid g(x) = p(y)\}.$$

It is regarded as a subspace of $X \times E$ with the subspace topology.

Lemma 3.2. *The projection* $g^*E \to X$ *given by* $(x,y) \mapsto x$ *is a fiber bundle.*

Proof. First of all it is helpful to write down the obvious maps that we have in a commutative diagram:

$$\begin{tikzcd} g^*E \arrow[r, "r"] \arrow[d, "q"] & E \arrow[d, "p"] \\ X \arrow[r, "g"] & B \end{tikzcd}$$

Here q and r are the projections given by $(x,y) \mapsto x$ and $(x,y) \mapsto y$. *Commutative* means that the two compositions taking us from g^*E to B agree. Suppose that we have an open set $V \subset B$ and a bundle chart

$$h\colon p^{-1}(V) \xrightarrow{\cong} V \times F.$$

Now $U := g^{-1}(V)$ is open in X. Also $q^{-1}(U)$ is an open subset of g^*E and we describe elements of that as pairs (x,y) where $x \in U$ and $y \in E$, with $g(x) = p(y)$. We make a homeomorphism

$$q^{-1}(U) \to U \times F$$

by the formula $(x,y) \mapsto (x, h_{(2)}(y))$, where $h_{(2)}(y)$ is the F-coordinate of $h(y) \in V \times F$. It is a homeomorphism because the inverse is given by

$$(x,z) \mapsto (x, h^{-1}(g(x),z))$$

for $x \in U$ and $z \in F$, so that $(g(x),z) \in V \times F$. Its is also clearly a bundle chart. In this way, every bundle chart

$$h\colon p^{-1}(V) \xrightarrow{\cong} V \times F$$

for $p\colon E \to B$ determines a bundle chart

$$q^{-1}(U) \xrightarrow{\cong} U \times F$$

with the same F, where U is the preimage of V under g. Since $p\colon E \to B$ is a fiber bundle, we have many such bundle charts $p^{-1}(V_j) \to V_j \times F_j$ such that the union of the V_j is all of B. Then the union of the corresponding U_j is all of X, and we have bundle charts $q^{-1}(U_j) \to U_j \times F_j$. This proves that q is a fiber bundle. □

This proof was too long and above all too formal. Reasoning in a less formal way, one should start by noticing that the fiber of q over $z \in X$ is essentially the same (and certainly homeomorphic) to the fiber of p over $g(z) \in B$. Namely,

$$q^{-1}(z) = \{(x,y) \in X \times E \mid g(x) = p(y),\ x = z\} \ = \ \{z\} \times p^{-1}(\{g(z)\}) \ .$$

Now recall once again that a bundle chart $h\colon p^{-1}(U) \to U \times F$ for p is just a way to specify, simultaneously and continuously, homeomorphisms h_x from the fibers of p over elements $x \in U$ to F. If we have such a bundle chart for p, then for any $z \in g^{-1}(U)$ we get a homeomorphism from the fiber of q over z, which "is" the fiber of p over $g(z)$, to F. And so, by letting z run through $g^{-1}(U)$, we get a bundle chart for q.
(The notation has become mildly unsystematic because I allowed both h_x and $h_{(2)}$. Sorry. In fact h_x is the restriction of $h_{(2)}$ to $p^{-1}(\{x\})$.)

Example 3.3. Restriction of fiber bundles is a special case of pullback, up to isomorphism of fiber bundles. More precisely, suppose that $p\colon E \to B$ is a fiber bundle and let $A \subset B$ be a subspace, with inclusion $g\colon A \to B$. Then there is an isomorphism of fiber bundles from $p_A\colon E_{|A} \to A$ to the pullback $g^*E \to A$. This takes $y \in E_{|A}$ to the pair $(p(y), y) \in g^*E \subset A \times E$.

3.2. Homotopy invariance of pullbacks of fiber bundles

Theorem 3.4. *Let $p\colon E \to B$ be a fiber bundle. Let $f, g\colon X \to B$ be continuous maps, where X is a compact Hausdorff space. If f is homotopic to g, then the fiber bundles $f^*E \to X$ and $g^*E \to X$ are isomorphic.*

Remark 3.5. The compactness assumption on X is unnecessarily strong; *paracompact* is enough. But paracompactness is also a more difficult concept than compactness. Therefore we shall prove the theorem as stated, and leave a discussion of improvements for later.

Remark 3.6. Let X be a compact Hausdorff space and let $U_0, U_1, \dots, U_n$ be open subsets of X such that the union of the U_i is all of X. *Then there exist continuous functions*

$$\varphi_0, \varphi_1, \dots, \varphi_n\colon X \to [0,1]$$

such that $\sum_{j=0}^{n} \varphi_j \equiv 1$ *and such that* $\mathrm{supp}(\varphi_j)$, *the support of* φ_j , *is contained in* U_j *for* $j = 0, 1, \dots, n$. Here $\mathrm{supp}(\varphi_j)$ is the closure in X of the open set

$$\{x \in X \mid \varphi_j(x) > 0\}.$$

A collection of functions $\varphi_0, \varphi_1, \dots, \varphi_n$ with the stated properties is called a *partition of unity subordinate to the open cover of* X *given by* $U_0, \dots, U_n$. For readers who are not aware of this existence statement, here is a reduction (by induction) to something which they might be aware of.

First of all, if X is a compact Hausdorff space, then it is a *normal* space. This means, in addition to the Hausdorff property, that any two disjoint closed subsets of X admit disjoint open neighborhoods. Next, for any normal space X we have the *Tietze-Urysohn extension lemma*. This says that if A_0 and A_1 are disjoint closed subsets of X, then there is a continuous function $\psi\colon X \to [0,1]$ such that $\psi(x) = 1$ for all $x \in A_1$ and $\psi(x) = 0$ for all $x \in A_0$.

Now suppose that a normal space X is the union of two open subsets U_0 and U_1. Because X is normal, we can find an open subset $V_0 \subset U_0$ such that the closure of V_0 in X is contained in U_0 and the union of V_0 and U_1 is still X. Repeating this, we can also find an open subset $V_1 \subset U_1$ such that the closure of V_1 in X is contained in U_1 and the union of V_1 and V_0 is still X. Let $A_0 = X \smallsetminus V_0$ and $A_1 = X \smallsetminus V_1$. Then A_0 and A_1 are disjoint closed subsets of X, and so by Tietze-Urysohn there is a continuous function $\psi\colon X \to [0,1]$ such that $\psi(x) = 1$ for all $x \in A_1$ and $\psi(x) = 0$ for all $x \in A_0$. This means that supp(ψ) is contained in the closure of $X \smallsetminus A_0 = V_0$, which is contained in U_0. We take $\varphi_1 = \psi$ and $\varphi_0 = 1 - \psi$. Since $1 - \psi$ is zero on A_1, its support is contained in the closure of V_1, which is contained in U_1. This establishes the induction beginning (case $n = 1$).

For the induction step, suppose that we have an open cover of X given by $U_0, \dots, U_n$ where $n \geq 2$. By inductive assumption we can find a partition of unity subordinate to the cover $U_0 \cup U_1, U_2, \dots, U_n$ and by the induction beginning, another partition of unity subordinate to $U_0, U_1 \cup U_2 \cup \cdots U_n$. Call the functions in the first partition of unity $\varphi_{01}, \varphi_2, \dots, \varphi_n$ and those in the second ψ_0, ψ_1, we see that the functions $\psi_0\varphi_{01}, \psi_1\varphi_{01}, \varphi_2, \dots, \varphi_n$ form a partition of unity subordinate to the cover by $U_0, \dots, U_n$. □

Proof of theorem 3.4. Let $h\colon X \times [0,1] \to B$ be a homotopy from f to g, so that $h_0 = f$ and $h_1 = g$. Then $h^*E \to X \times [0,1]$ is a fiber bundle. We give this a new name, say $q\colon L \to X \times [0,1]$. Let ι_0 and ι_1 be the maps from X to $X \times [0,1]$ given by $\iota_0(x) = (x, 0)$ and $\iota_1(x) = (x, 1)$. It is not hard to verify that the fiber bundle $f^*E \to X$ is isomorphic to $\iota_0^*L \to X$ and $g^*E \to X$ is isomorphic to $\iota_1^*L \to X$. Therefore all we need to prove is the following.

Let $q\colon L \to X \times [0,1]$ *be a fiber bundle, where* X *is compact Hausdorff. Then the fiber bundles* $\iota_0^*L \to X$ *and* $\iota_1^*L \to X$ *obtained from* q *by pullback along* ι_0 *and* ι_1 *are isomorphic.* To make this even more explicit: given the fiber bundle $q\colon L \to X \times [0,1]$, we need to produce a homeomorphism from $L_{|X \times \{0\}}$

to $L_{|X\times\{1\}}$ which fits into a commutative diagram

$$\begin{tikzcd} L_{|X\times\{0\}} \arrow[r, "\text{our homeom.}"] \arrow[d, "\text{res. of } q"'] & L_{|X\times\{1\}} \arrow[d, "\text{res. of } q"] \\ X\times\{0\} \arrow[r, "{(x,0)\mapsto(x,1)}"] & X\times\{1\} \end{tikzcd}$$

Here $L_{|K}$ means $q^{-1}(K)$, for any $K\subset X\times[0,1]$.
By a lemma proved last week (lecture notes week 2), we can find a covering of X by open subsets U_i such that that $q_{U_i\times[0,1]}\colon L_{|U_i\times[0,1]}\to U_i\times[0,1]$ is a trivial bundle, for each i. Since X is compact, finitely many of these U_i suffice, and we can assume that their names are $U_1,\dots,U_n$. Let $\varphi_1,\dots,\varphi_n$ be continuous functions from X to $[0,1]$ making up a partition of unity subordinate to the open covering of X by $U_1,\dots,U_n$. For $j=0,1,2,\dots,n$ let $\nu_j=\sum_{k=1}^{j}\varphi_k$ and let $\Gamma_j\subset X\times[0,1]$ be the graph of ν_j. Note that Γ_0 is $X\times\{0\}$ and Γ_n is $X\times\{1\}$. It suffices therefore to produce a homeomorphism $e_j\colon L_{|\Gamma_{j-1}}\to L_{|\Gamma_j}$ which fits into a commutative diagram

$$\begin{tikzcd} L_{|\Gamma_{j-1}} \arrow[r, "e_j"] \arrow[d, "\text{res. of } q"'] & L_{|\Gamma_j} \arrow[d, "\text{res. of } q"] \\ \Gamma_{j-1} \arrow[r, "{(x,\nu_{j-1}(x))\mapsto(x,\nu_j(x))}"] & \Gamma_j \end{tikzcd}$$

(for $j=1,2,\dots,n$). Since $q_{U_j\times[0,1]}\colon L_{|U_j\times[0,1]}\to U_j\times[0,1]$ is a trivial fiber bundle, we have a single bundle chart for it, a homeomorphism

$$g\colon L_{|U_j\times[0,1]}\longrightarrow(U_i\times[0,1])\times F$$

with the additional good property that we require of bundle charts. Fix j now and write $L=L'\cup L''$ where L' consists of the $y\in L$ for which $q(y)=(x,t)$ with $x\notin\operatorname{supp}(\varphi_j)$, and L'' consists of the $y\in L$ for which $q(y)=(x,t)$ with $x\in U_j$. Both L' and L'' are open subsets of L. Now we make our homeomorphism $e=e_j$ as follows. By inspection, $L_{|\Gamma_{j-1}}\cap L'=L_{|\Gamma_j}\cap L'$, and we take e to be the identity on $L_{|\Gamma_{j-1}}\cap L'$. By restricting the bundle chart g, we have a homeomorphism $L_{|\Gamma_{j-1}}\cap L''\to U_j\times F$; more precisely, a homeomorphism from $L_{|\Gamma_{j-1}}\cap L''$ to $(\Gamma_{j-1}\cap U_j\times[0,1])\times F$. By the same reasoning, we have a homeomorphism $L_{|\Gamma_{j-1}}\cap L''\to U_j\times F$; more precisely, a homeomorphism from $L_{|\Gamma_{j-1}}\cap L''$ to $(\Gamma_j\cap U_j\times[0,1])\times F$. Therefore we have a preferred homeomorphism from $L_{|\Gamma_{j-1}}\cap L''$ to $L_{|\Gamma_j}\cap L''$, and we use that as the definition of e on $L_{|\Gamma_{j-1}}\cap L''$. By inspection, the two definitions of e which we have on the overlap $L_{|\Gamma_{j-1}}\cap L'\cap L''$ agree, so e is well defined. $\square$

Corollary 3.7. *Let $p\colon E\to B$ be a fiber bundle where B is compact Hausdorff and contractible. Then p is a trivial fiber bundle.*

Proof. By the contractibility assumption, the identity map $f\colon B \to B$ is homotopic to a constant map $g\colon B \to B$. By the theorem, the fiber bundles $f^*E \to B$ and $g^*E \to B$ are isomorphic. But clearly $f^*E \to B$ is isomorphic to the original fiber bundle $p\colon E \to B$. And clearly $g^*E \to B$ is a trivial fiber bundle. □

Corollary 3.8. *Let $q\colon E \to B \times [0,1]$ be a fiber bundle, where B is compact Hausdorff. Suppose that the restricted bundle*

$$q_{B\times\{0\}}\colon E_{|B\times\{0\}} \to B \times \{0\}$$

admits a section, i.e., there exists a continuous map $s\colon B\times\{0\} \to E_{|B\times\{0\}}$ such that $q \circ s$ is the identity on $B \times \{0\}$. Then $q\colon E \to B \times [0,1]$ admits a section $\bar{s}\colon B \times [0,1] \to E$ which agrees with s on $B \times \{0\}$.

Proof. Let $f, g\colon B \times [0,1] \to B \times [0,1]$ be defined by $f(x,t) = (x,t)$ and $g(x,t) = (x,0)$. These maps are clearly homotopic. Therefore the fiber bundles $f^*E \to B \times [0,1]$ and $g^*E \to B \times [0,1]$ are isomorphic fiber bundles. Now $f^*E \to B \times [0,1]$ is clearly isomorphic to the original fiber bundle

$$q\colon E \to B \times \{0,1\}$$

and $g^*E \to B \times [0,1]$ is clearly isomorphic to the fiber bundle

$$E_{|B\times\{0\}} \times [0,1] \to B \times [0,1]$$

given by $(y,t) \mapsto (q(y),t)$ for $y \in E_{|B\times\{0\}}$, that is, $y \in E$ with $q(y) = (x,0)$ for some $x \in B$. Therefore we may say that there is a homeomorphism $h\colon E_{|B\times\{0\}} \times [0,1] \to E$ which is over $B \times [0,1]$, in other words, which satisfies

$$(q \circ h)(y,t) = (q(y),t)$$

for all $y \in E_{|B\times\{0\}}$ and $t \in [0,1]$. Without loss of generality, h satisfies the additional condition $h(y,0) = y$ for all $y \in E_{|B\times\{0\}}$. (In any case we have a homeomorphism $u\colon E_{|B\times\{0\}} \to E_{|B\times\{0\}}$ defined by $u(y) = h(y,0)$. If it is not the identity, use the homeomorphism $(y,t) \mapsto h(u^{-1}(y),t)$ instead of $(y,t) \mapsto h(y,t)$.) Now define $\bar{s}$ by $\bar{s}(x,t) = h(s(x),t)$ for $x \in B$ and $t \in [0,1]$. □

3.3. The homotopy lifting property

Definition 3.9. A continuous map $p\colon E \to B$ between topological spaces is said to have the *homotopy lifting property* (HLP) if the following holds. Given any space X and continuous maps $f\colon X \to E$ and $h\colon X \times [0,1] \to B$ such that $h(x,0) = p(f(x))$ for all $x \in X$, there exists a continuous map $H\colon X \times [0,1] \to E$ such that $p \circ H = h$ and $H(x,0) = f(x)$ for all $x \in X$.
A map with the HLP can be called a *fibration* (sometimes *Hurewicz fibration*).

It is customary to summarize the HLP in a commutative diagram with a dotted arrow:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "x\mapsto(x,0)"'] & E \arrow[d, "p"] \\ X\times[0,1] \arrow[r, "h"] \arrow[ur, dotted, "H"] & B \end{tikzcd}$$

Indeed, the HLP for the map p means that once we have the data in the outer commutative square, then the dotted arrow labeled H can be found, making both triangles commutative.
More associated customs: we think of h as a homotopy between maps h_0 and h_1 from X to B, and we think of $f\colon X\to E$ as a *lift* of the map h_0 , which is just a way of saying that $p\circ f=h_0$.

More generally, or less generally depending on point of view, we say that $p\colon E\to B$ satisfies the HLP for a class of spaces $\mathcal{Q}$ if the dotted arrow in the above diagram can always be supplied when the space X belongs to that class $\mathcal{Q}$.

Corollary 3.10. *Let $p\colon E\to B$ be a fiber bundle. Then p has the HLP for compact Hausdorff spaces.*

Proof. Suppose that we have the data X, f and h as in the above diagram, but we are still trying to construct or find the diagonal arrow H. We are assuming that X is compact Hausdorff. The pullback of p along h is a fiber bundle $h^*E\to X\times[0,1]$. The restricted fiber bundle

$$(h^*E)_{|X\times\{0\}}\to X\times\{0\}$$

has a continuous section s given essentially by f, and if we say it very carefully, by the formula

$$(x,0)\mapsto((x,0),f(x))\in h^*E\subset(X\times[0,1]\,)\times E\,.$$

The section s extends to a continuous section $\bar{s}$ of $h^*E\to X\times[0,1]$ by corollary 3.8. Now we can define $H:=r\circ\bar{s}$, where r is the standard projection from h^*E to E. $\square$

Example 3.11. Let $p\colon S^3\to S^2$ be the Hopf fiber bundle. Assume if possible that p is nullhomotopic; we shall try to deduce something absurd from that. So let $h\colon S^3\times[0,1]\to S^2$ be a nullhomotopy for p. Then $h_0=p$ and h_1 is a constant map. Applying the HLP in the situation

$$\begin{tikzcd} S^3 \arrow[r, "\mathrm{id}"] \arrow[d, "x \mapsto (x,0)"'] & S^3 \arrow[d, "p"] \\ S^3 \times [0,1] \arrow[r, "h"] \arrow[ur, dotted, "H"] & S^2 \end{tikzcd}$$

we deduce the existence of $H\colon S^3 \times [0,1] \to S^3$, a homotopy from the identity map $H_0 = \mathrm{id}\colon S^3 \to S^3$ to a map $H_1\colon S^3 \to S^3$ with the property that $p \circ H_1$ is constant. Since p itself is certainly not constant, this means that H_1 is not surjective. If H_1 is not surjective, it is nullhomotopic. (A non-surjective map to a sphere is nullhomotopic; little exercise.) Consequently $\mathrm{id}\colon S^3 \to S^3$ is also nullhomotopic, being homotopic to H_1. This means that S^3 is contractible.
Is that absurd enough? We shall prove later in the course that S^3 is not contractible. Until then, what we have just shown can safely be stated like this: *if S^3 is not contractible, then the Hopf map $p\colon S^3 \to S^2$ is not nullhomotopic.* (I found this argument in Dugundji's book on topology. Hopf used rather different ideas to show that p is not nullhomotopic.)

In this example, the HLP was used for something resembling a computation with homotopy classes of maps. Let us try to formalize this in hopes to get hold of some algebra in homotopy theory. So let $p\colon E \to B$ be a continuous map which has the HLP for a class of topological spaces $\mathcal{Q}$. Let $\star \subset B$ be a selected element (which I will also regard as a one-point subspace). Let $F = p^{-1}(*)$ be the fiber of p over $*$ and let $j\colon F \to E$ be the inclusion. Take any space X in the class $\mathcal{Q}$. There is a diagram of sets and maps

$$[X, F] \xrightarrow{[j]\circ} [X, E] \xrightarrow{[p]\circ} [X, B]$$

where the arrows are given by composition with the homotopy classes of $[j]$ and $[p]$, respectively. Also there is a distinguished zero element in $[X, B]$, corresponding to the homotopy class of the map $X \to B$ which maps all of X to $*$. Therefore it makes sense to speak of the *kernel* of the map $[p]\circ\colon [X, E] \to [X, B]$. It is a subset of $[X, E]$, the preimage of the zero element in $[X, B]$.

Proposition 3.12. *The above diagram of sets and maps is "half exact" in the sense that the kernel of $[X, E] \to [X, B]$ coincides with the image of $[X, F]$.*

Proof. Since $p \circ j$ is the constant map with value $*$ from F to B, the composition of $[p]\circ$ and $[j]\circ$ is the map from $[X, F]$ to $[X, B]$ taking all elements of $[X, F]$ to zero. This is equivalent to saying that the kernel of $[p] \circ$ contains the image of $[j] \circ$. For the other inclusion, imagine a continuous $f\colon X \to E$ such that $[p] \circ [f]$ is zero in $[X, B]$. This means that $p \circ f\colon X \to B$ is nullhomotopic, and more precisely, homotopic to the constant map with value $*$ from X to B. Let $h\colon X \times [0,1] \to B$ be a homotopy which has $h_0 = f$ and h_1 equal to the constant map with value $*$. Apply the HLP to find a homotopy $H\colon X \times [0,1] \to E$ such that $p \circ H = h$ and $H_0 = f$. Then $H_1\colon X \to E$ is a map whose image is contained in $F \subset E$. So $[f] = [H_0] = [H_1]$ is in the image of the map $[j] \circ$. □

Looking back, we can say that example 3.11 is an application of proposition 3.12 with $p\colon E \to B$ equal to the Hopf fibration (and $\mathcal{Q}$ equal to the class of compact Hausdorff spaces, say). There are some special features in this example which we used. Firstly, the spaces F, E, B are all path-connected, so the sets $[X, F]$, $[X, E]$ and $[X, B]$ all have a distinguished zero element. Secondly, we saw that $j\colon F \to E$ is nullhomotopic, being a non-surjective map to a sphere. Hence $[j] \circ\colon [X, F] \to [X, E]$ is the zero map.
We made some unusual choices: $X = E$ and $[f] = [\mathrm{id}] \in [X, E]$. Then $[p] \circ [f] = [p] \in [X, B]$, so the assumption that $[p] \in [X, B]$ is zero is equivalent to saying that $[f]$ is in the kernel of $[p] \circ\colon [X, E] \to [X, B]$, which by the proposition is also the image of $[j] \circ$. But the image of $[j]$ only has the zero element in it, so ... if $[p]$ is zero in $[X, B]$ then $[f]$ is zero in $[X, E]$.

3.4. Remarks on paracompactness and fiber bundles

Quoting from many books on point set topology: a topological space $X = (X, \mathcal{O})$ is *paracompact* if it is Hausdorff and every open cover $(U_i)_{i \in \Lambda}$ of X admits a locally finite refinement $(V_j)_{j \in \Psi}$.

There is a fair amount of open cover terminology in that definition. In this formulation, we take the view that an open cover of X is a *family*, i.e., a map from a set to $\mathcal{O}$ (with a special property). This is slightly different from the equally reasonable view that an open cover of X is a subset of $\mathcal{O}$ (with a special property), and it justifies the use of round brackets as in $(U_i)_{i \in \Lambda}$, as opposed to curly brackets. Here the map in question is from Λ to $\mathcal{O}$. There is an understanding that $(V_j)_{j \in \Psi}$ is also an open cover of X, but Ψ need not coincide with Λ. *Refinement* means that for every $j \in \Psi$ there exists $i \in \Lambda$ such that $V_j \subset U_i$. *Locally finite* means that every $x \in X$ admits an open neighborhood W in X such that the set $\{j \in \Psi \mid W \cap V_j \neq \emptyset\}$ is a finite subset of Ψ.

It is wonderfully easy to get confused about the meaning of paracompactness. There is a strong similarity with the concept of compactness, and it is obvious that *compact* (together with Hausdorff) implies paracompact, but it is worth emphasizing the differences. Namely, where compactness has something to do with open covers and *sub*-covers, the definition of paracompactness uses the notion of *refinement* of one open cover by another open cover. We require that every V_j is *contained* in some U_i ; we do not require that every V_j is *equal* to some U_i. And *locally finite* does not just mean that for every $x \in X$ the set $\{j \in \Psi \mid x \in V_j\}$ is a finite subset of Ψ. It means more.

For some people, the Hausdorff condition is not part of *paracompact*, but for me, it is.

An important theorem: every metrizable space is paracompact. This is due to A.H. Stone who, as a Wikipedia page reminds me, is not identical with Marshall Stone of the Stone-Weierstrass theorem and the Stone-Čech compactification. The proof is not very complicated, but you should look it up in a book on point-set topology which is not too ancient, because it was complicated in the A.H.Stone version.

Another theorem which is very important for us: in a paracompact space X, every open cover $(U_i)_{i\in\Lambda}$ admits a subordinate partition of unity. In other words there exist continuous functions $\varphi_i\colon X \to [0,1]$, for $i \in \Lambda$, such that

- every $x \in X$ admits an open neighborhood W such that the set $\{i \in \Lambda \mid W \cap \mathrm{supp}(\varphi_i) \neq \emptyset\}$ is finite;
- $\sum_{i\in\Lambda} \varphi_i \equiv 1$;
- $\mathrm{supp}(\varphi_i) \subset U_i$.

The second condition is meaningful if we assume that the first condition holds. (Then, for every $x \in X$, there are only finitely many nonzero summands in $\sum_{i\in\Lambda} \varphi_i(x)$. The first condition also ensures that for any subset $\Xi \subset \Lambda$, the sum $\sum_{i\in\Xi} \varphi_i$ is a continuous function on X.)
The proof of this theorem (existence of subordinate partition of unity for any open cover of a paracompact space) is again not very difficult, and boils down mostly to showing that paracompact spaces are *normal*. Namely, in a normal space, locally finite open covers admit subordinate partitions of unity, and this is easy.

Many of the results about fiber bundles in this chapter rely on partitions of unity, and to ensure their existence, we typically assumed compactness here and there. But now it emerges that paracompactness is enough.
Specifically, in theorem 3.4 it is enough to assume that X is paracompact. In corollary 3.7 it is enough to assume that B is paracompact (and contractible). In corollary 3.8 it is enough to assume that B is paracompact. In corollary 3.10 we have the stronger conclusion that p has the HLP for paracompact spaces.

Proof of variant of thm. 3.4 with weaker assumption that X is paracompact. By analogy with the case of compact X, we can easily reduce to the following statement. *Let $q\colon L \to X \times [0,1]$ be a fiber bundle, where X is paracompact. Then the fiber bundles $\iota_0^* L \to X$ and $\iota_1^* L \to X$ obtained from q by pullback along ι_0 and ι_1 are isomorphic.* And to make this more explicit: given the fiber bundle $q\colon L \to X \times [0,1]$, we need to produce a homeomorphism h from $L_{|X\times\{0\}}$ to $L_{|X\times\{1\}}$ which fits into a commutative diagram

$$\begin{tikzcd} L_{|X\times\{0\}} \arrow[r, "h"] \arrow[d, "\text{res. of } q"'] & L_{|X\times\{1\}} \arrow[d, "\text{res. of } q"] \\ X \times \{0\} \arrow[r, "(x,0)\mapsto(x,1)"] & X \times \{1\} \end{tikzcd}$$

By a lemma proved in lecture notes week 2, we can find an open cover $(U_i)_{i\in\Lambda}$ of X such that that $q_{U_i\times[0,1]}\colon L_{|U_i\times[0,1]} \to U_i \times [0,1]$ is a trivial bundle, for each $i \in \Lambda$. Let $(\varphi_i)_{i\in\Lambda}$ be a partition of unity subordinate to $(U_i)_{i\in\Lambda}$. So $\varphi_i\colon X \to [0,1]$ is a continuous function with $\mathrm{supp}(\varphi_i) \subset U_i$, and $\sum_i \varphi_i \equiv 1$. Every $x \in X$ admits a neighborhood W in X such that the set

$$\{i \in \Lambda \mid \mathrm{supp}(\varphi_i) \cap W \neq \emptyset\}$$

is finite.

Now choose a total ordering on the set Λ. (A total ordering on Λ is a relation $\leq$ on Λ which is transitive and reflexive, and has the additional property that for any distinct $i, j \in \Lambda$, precisely one of $i \leq j$ or $j \leq i$ holds. We need to assume something here to get such an ordering: for example the Axiom of Choice in set theory is equivalent to the Well-Ordering Principle, which states that every set can be well-ordered. A well-ordering is also a total ordering.) Given $x \in X$, choose an open neighborhood W of x such that the set of $i \in \Lambda$ having $\mathrm{supp}(\varphi_i) \cap W \neq \emptyset$ is finite; say it has n elements. We list these elements in their order (provided by the total ordering on Λ which we selected):

$$i_1 \leq i_2 \leq i_3 \leq \cdots i_n \ .$$

The functions $\varphi_{i_1}, \varphi_{i_2}, \dots, \varphi_{i_n}$ (restricted to W) make up a partition of unity on W which is subordinate to the covering by open subsets $W \cap U_{i_1}, W \cap U_{i_2}, \dots W \cap U_{i_n}$. Now we can proceed exactly as in the proof of theorem 3.4 to produce (in n steps) a homeomorphism h_W which makes the following diagram commute:

$$\begin{tikzcd}
L_{|W\times\{0\}} \arrow[r, "h_W"] \arrow[d, "\text{res. of } q"'] & L_{|W\times\{1\}} \arrow[d, "\text{res. of } q"] \\
W\times\{0\} \arrow[r, "(x,0)\mapsto(x,1)"] & W\times\{1\}
\end{tikzcd}$$

Finally we can regard W or x as variables. If we choose, for every $x \in X$, an open neighborhood W_x with properties like W above, then the W_x for all $x \in X$ constitute an open cover of X. For each W_x we get a homeomorphism h_{W_x} as above. These homeomorphisms agree with each other wherever this is meaningful, and so define together a homeomorphism $h\colon L_{|X\times\{0\}} \to L_{|X\times\{1\}}$ with the property that we require. $\square$