19.11.2013

Lecture Notes, week 6
Topology WS 2013/14 (Weiss)

5.1. The stalks of a presheaf

Let $\mathcal{F}$ a presheaf on a topological space X. Fix $z \in X$. There are situations where we want to understand the behavior of $\mathcal{F}$ near z, that is to say, in small neighborhoods of z. Then it is a good idea to work with pairs (U, s) where U is an open neighborhood of z and s is an element of $\mathcal{F}(U)$. Two such pairs (U, s) and (V, t) are considered to be *germ-equivalent* if there exists an open neighborhood W of z such that $W \subset U \cap V$ and $s_{|W} = t_{|W}$ in $\mathcal{F}(W)$. It is easy to show that germ equivalence is indeed an equivalence relation.

Definition 5.1. The set of equivalence classes is called the *stalk* of $\mathcal{F}$ at z and denoted by $\mathcal{F}_z$. The elements of $\mathcal{F}_z$ are often called *germs* (at z, of something ... depending on the meaning of $\mathcal{F}$).

Example 5.2. Let $\mathcal{F}$ be the sheaf on X where $\mathcal{F}(U)$ is the set of continuous maps from U to Y, for a fixed Y. An element of $\mathcal{F}_z$ is called a *germ of continuous maps from* (X, z) *to* Y.

Example 5.3. Fix a continuous map $p: Y \to X$. Let $\mathcal{F}$ be the sheaf on X where $\mathcal{F}(U)$ is the set of continuous maps $s: U \to Y$ such that $p \circ s$ is the inclusion $U \to X$. An element of $\mathcal{F}(U)$ can be called a *continuous section of* p *over* U. For $z \in X$, an element of $\mathcal{F}_z$ can be called a *germ at* z *of continuous sections of* $p: X \to Y$.

Example 5.4. Let X be the union of the two coordinate axes in $\mathbb{R}^2$. For open U in X, let $\mathcal{G}(U)$ be the set of connected components of $X \smallsetminus U$. For open subsets U, V of X such that $U \subset V$, define

$$\mathrm{res}_{V,U}: \mathcal{G}(V) \to \mathcal{G}(U)$$

by saying that $\mathrm{res}_{V,U}(C)$ is the unique connected component of $X \smallsetminus U$ which contains C (where C can be any connected component of $X \smallsetminus V$). These definitions make $\mathcal{G}$ into a presheaf on X. For $z \in X$, what can we say about the stalk $\mathcal{G}_z$? If z is the origin, $z = (0, 0)$, then $\mathcal{G}_z$ has four elements. In all other cases $\mathcal{G}_z$ has two elements. (Despite that, for any $z \in X$ and any open neighborhood V of z in X, there exists an open neighborhood W of z in X such that $W \subset V$ and $\mathcal{G}(W)$ has more than 1000 elements.)

Now let $\alpha: \mathcal{F} \to \mathcal{G}$ be a map (morphism) of sheaves on X. Again fix $z \in X$. Then every pair (U, s), where U is an open neighborhood of z and $s \in \mathcal{F}(s)$, determines another pair $(U, \alpha(s))$ where U is still an open neighborhood of z and $\alpha(s) \in \mathcal{G}(U)$. The assignment $(U, s) \mapsto (U, \alpha(s))$ is compatible with germ equivalence. That is, if V is another open neighborhood of z in X, and $t \in \mathcal{F}(V)$, and (U, s) is germ equivalent to (V, t), then $(U, \alpha(s))$ is germ equivalent to $(V, \alpha(t))$. In short, α determines a map of sets from $\mathcal{F}_z$ to $\mathcal{G}_z$ which takes the equivalence class (the germ) of (U, s) to the equivalence class (the germ) of $(U, \alpha(s))$. In category jargon: the assignment

$$\mathcal{F} \mapsto \mathcal{F}_z$$

is a *functor* from $\mathcal{P}\mathrm{re}\mathcal{S}\mathrm{h}(X)$, the category of presheaves on X, to $\mathcal{S}\mathrm{ets}$.

When a presheaf $\mathcal{F}$ on X is a sheaf, the stalks $\mathcal{F}_z$ carry a lot of information about $\mathcal{F}$. The following theorem illustrates that.

Theorem 5.5. *Let $\beta\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Suppose that for every $z \in X$, the map of stalks $\mathcal{F}_z \to \mathcal{G}_z$ determined by β is a bijection. Then β is an isomorphism.*

Proof. The claim that β is an isomorphism means, abstractly speaking, that there exists a morphism $\gamma\colon \mathcal{G} \to \mathcal{F}$ of sheaves such that $\beta \circ \gamma$ is the identity on $\mathcal{G}$ and $\gamma \circ \beta$ is the identity on $\mathcal{F}$. In more down-to-earth language it means simply that $\beta_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is a bijection for every open U in X, so this is what we have to show. To ease notation, we write $\beta\colon \mathcal{F}(U) \to \mathcal{G}(U)$.
We fix U, an open subset of X. First we want to show that $\beta\colon \mathcal{F}(U) \to \mathcal{F}(G)$ is *injective.* For that we set up a commutative square of sets and maps:

$$\begin{array}{ccc} \prod_{z\in U} \mathcal{F}_z & \xrightarrow{\beta} & \prod_{z\in U} \mathcal{G}_z \\ \uparrow & & \uparrow \\ \mathcal{F}(U) & \xrightarrow{\beta} & \mathcal{G}(U) \end{array}$$

The left-hand vertical arrow is obtained by noting that each $s \in \mathcal{F}(U)$ determines a pair (U, s) representing an element of $\mathcal{F}_z$, for each $z \in U$. The right-hand vertical arrow is similar. We show that the left-hand vertical arrow is injective. Suppose that $s, t \in \mathcal{F}(U)$ have the same image in $\prod_{z\in U} \mathcal{F}_z$. It follows that every $z \in U$ admits a neighborhood W_z in U such that $s_{|W_z} = t_{|W_z}$. Selecting such a W_z for every $z \in U$, we have an open cover

$$(W_z)_{z\in U}$$

of U. Since $s_{|W_z} = t_{|W_z}$ for each of the open sets W_z in the cover, the sheaf property for $\mathcal{F}$ implies that $s = t$. Hence the left-hand vertical arrow in our square is injective, and so is the right-hand arrow by the same argument. But the top horizontal arrow is bijective by our assumption. Therefore $\beta\colon \mathcal{F}(U) \to \mathcal{F}(G)$ is injective.

Next we show that $\beta\colon \mathcal{F}(U) \to \mathcal{F}(G)$ is *surjective*. We can use the same commutative square that we used to prove injectivity. An element $s \in \mathcal{G}(U)$ determines an element of $\prod_{z\in U} \mathcal{G}_z$ (right-hand vertical arrow) which comes from an element of $\prod_{z\in U} \mathcal{F}_z$ because the top horizontal arrow is bijective. So for each $z \in U$ we can find an element of $\mathcal{F}_z$ which under β is mapped to the germ of s at z (an element of $\mathcal{G}_z$). In terms of representatives of germs, this means that for each $z \in U$ we can find an open neighborhood V_z of z in U and an element $t_z \in \mathcal{F}(V_z)$ such that $\beta(t_z) = s_{|V_z} \in \mathcal{G}(V_z)$. Selecting such a V_z for every $z \in U$, we have an open cover

$$(V_z)_{z\in U}$$

of U and we have $t_z \in \mathcal{F}(V_z)$. Can we use the sheaf property of $\mathcal{F}$ to produce $t \in \mathcal{F}(U)$ such that $t_{|V_z} = t_z$ for all $z \in U$? We need to verify the matching condition,

$$t_{z|V_z\cap V_y} = t_{y|V_z\cap V_y} \in \mathcal{F}(V_z \cap V_y)$$

whenever $y, z \in U$. By the injectivity of $\beta\colon \mathcal{F}(V_z \cap V_y) \to \mathcal{G}(V_z \cap V_y)$, which we have established, it is enough to show

$$\beta(t_z)_{|V_z\cap V_y} = \beta(t_y)_{|V_z\cap V_y} \in \mathcal{G}(V_z \cap V_y).$$

This clearly holds as $\beta(t_z) = s_{|V_z}$ by construction, so that both sides of the equation agree with $s_{|V_z\cap V_y}$. So we obtain $t \in \mathcal{F}(U)$ such that $t_{|V_z} = t_z$ for all $z \in U$. Now it is easy to show that $\beta(t) = s$. Indeed we have $\beta(t)_{|V_z} = s_{|V_z}$ by construction, for all open sets V_z in the covering $(V_z)_{z\in U}$ of U, so the sheaf property of $\mathcal{F}$ implies $\beta(t) = s$. Since $s \in \mathcal{G}(U)$ was arbitrary, this means that $\beta\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective. □

5.2. Sheafification

Proposition 5.6. *Let $\mathcal{F}$ be a presheaf on a topological space X. There is a sheaf $\Phi\mathcal{F}$ on X and there is a morphism $\eta\colon \mathcal{F} \to \Phi\mathcal{F}$ of presheaves such that, for every $z \in X$, the map of stalks $\mathcal{F}_z \to (\Phi\mathcal{F})_z$ determined by η is bijective.*

Proof. Let U be an open subset of X. We are going to define $(\Phi\mathcal{F})(U)$ as a subset of the product

$$\prod_{z\in U} \mathcal{F}_z\,.$$

Think of an element of that product as a function s which for every $z \in U$ selects an element $s(z) \in \mathcal{F}_z$. The function s qualifies as an element of $(\Phi\mathcal{F})(U)$ if and only if it satisfies the following *coherence condition.* For every $y \in U$ there is an open neighborhood W of y in U and there is $t \in \mathcal{F}(W)$ such that the pair (W, t) simultaneously represents the germs

$s(z) \in \mathcal{F}_z$ for all $z \in W$.
From the definition, it is clear that there are restriction maps

$$\mathrm{res}_{V,U} \colon (\Phi\mathcal{F})(V) \to (\Phi\mathcal{F})(U)$$

whenever U, V are open in X and $U \subset V$. Namely, a function s which selects an element $s(z) \in \mathcal{F}_z$ for every $z \in V$ determines by restriction a function $s_{|U}$ which selects an element $s(z) \in \mathcal{F}_z$ for every $z \in U$. The coherence condition is satisfied by $s_{|U}$ if it is satisfied by s. With these restriction maps, $\Phi\mathcal{F}$ is a presheaf. Furthermore, it is straightforward to see that $\Phi\mathcal{F}$ satisfies the sheaf condition. Indeed, suppose that $(V_i)_{i\in\Lambda}$ is a collection of open subsets of X, and suppose that elements $s_i \in (\Phi\mathcal{F})(V_i)$ have been selected, one for each $i \in \Lambda$, such that the matching condition

$$s_{i|V_i \cap V_j} = s_{j|V_i \cap V_j}$$

is satisfied for all $i, j \in \Lambda$. Then clearly we get a function s on $V = \bigcup_i V_i$ which for every $z \in V$ selects $s(z) \in \mathcal{F}_z$ by declaring, unambiguously,

$$s(z) := s_i(z)$$

for any i such that $z \in V_i$. The coherence condition is satisfied because it is satisfied by each s_i .
The morphism of presheaves $\eta \colon \mathcal{F} \to \Phi\mathcal{F}$ is defined in the following mechanical way. Given $t \in \mathcal{F}(U)$, we need to say what $\eta(t) \in (\Phi\mathcal{F})(U)$ should be. It is the function which to $z \in U$ assigns the element of $\mathcal{F}_z$ represented by the pair (U, t), that is to say, the germ of (U, t) at z.
Last not least, we need to show that for any $z \in X$ the map $\mathcal{F}_z \to (\Phi\mathcal{F})_z$ determined by η is a bijection. We fix z. *Injectivity*: we consider elements a and b of $\mathcal{F}_z$ represented by pairs (U_a, s_a) and (U_b, s_b) respectively, where U_a, U_b are neighborhoods of z and $s_a \in \mathcal{F}(U_a)$, $s_b \in \mathcal{F}(U_b)$. Suppose that a and b are taken to the same element $t \in (\Phi\mathcal{F})_z$ by η. Then in particular $t(z) \in \mathcal{F}_z$ is the germ at z of s_a , and also the germ at z of s_b , so the germs of s_a and s_b (elements of $\mathcal{F}_z$) are equal. *Surjectivity*: let an element of $(\Phi\mathcal{F})_z$ be represented by a pair (U, t) where U is an open neighborhood of z in X and $t \in (\Phi\mathcal{F})(U)$. By the coherence condition, there exists an open neighborhood W of z in U and there exists $s \in \mathcal{F}(W)$ such that $t_{|W}$ is the function which to $y \in W$ assigns the germ at y of (W, s), an element of $\mathcal{F}_y$. But this means that the map of stalks $\mathcal{F}_z \to (\Phi\mathcal{F})_z$ determined by the morphism η takes the element of $\mathcal{F}_z$ represented by (W, s) to the element of $(\Phi\mathcal{F})_z$ represented by (U, t). □

Example 5.7. Let T be any set. Let $\mathcal{F}$ be the constant presheaf on X given by $\mathcal{F}(U) = T$ for all open subsets U of X (and $\mathrm{res}_{V,U} \colon \mathcal{F}(V) \to \mathcal{F}(U)$ is id_T). What does the sheaf $\Phi\mathcal{F}$ look like? This question has quite an interesting

answer. Let's keep a cool head and approach it mechanically. For any $z \in X$, the stalk $\mathcal{F}_z$ can be identified with T. This is easy. Let U be an open subset of X. The elements of $(\Phi\mathcal{F})(U)$ are functions s which for every $z \in U$ select an element $s(z) \in \mathcal{F}_z = T$, subject to a coherence condition. So the elements of $(\Phi\mathcal{F})(U)$ are maps s from U to T subject to a coherence condition. What is the coherence condition? The condition is that s must be locally constant, i.e., every $z \in U$ admits an open neighborhood W in U such that $s_{|W}$ is constant. So the elements of $(\Phi\mathcal{F})(U)$ are the locally constant maps s from U to T. A locally constant map s from U to T is the same thing as a continuous map s from U to T, if we agree that T is equipped with the discrete topology (every subset of T is declared to be open). Summing up, $(\Phi\mathcal{F})(U)$ is the set of continuous functions from U to T. We can say that $\Phi\mathcal{F}$ is the sheaf of continuous functions (from open subsets of X) to T.
To appreciate the beauty of this answer, take a space X which is a little out of the ordinary; for example, $\mathbb{Q}$ with the standard topology inherited from $\mathbb{R}$, or the Cantor set (a subset of $\mathbb{R}$). For T, any set with more than one element is an interesting choice. (What happens if T has exactly one element? What happens if $T = \emptyset$?)

There are a few things of a general nature to be said about proposition 5.6 — not difficult, not surprising, but important. The construction Φ is a functor; we can view it as a functor from the category $\mathcal{P}\mathrm{re}\mathcal{S}\mathrm{h}(X)$ to itself. This means in particular that any morphism of presheaves $\alpha\colon \mathcal{F} \to \mathcal{G}$ on X determines a morphism

$$\Phi\alpha\colon \Phi\mathcal{F} \to \Phi\mathcal{G} \,.$$

Namely, for $s \in \Phi\mathcal{F}(V)$ we define $t = (\Phi\alpha)(s) \in \Phi\mathcal{G}(V)$ in such a way that $t(z) \in \mathcal{G}_z$ is the image of $s(z) \in \mathcal{F}_z$ under the map $\mathcal{F}_z \to \mathcal{G}_z$ induced by α. (It is easy to verify that t satisfies the coherence condition.)
Furthermore η is a natural transformation from the identity functor id on $\mathcal{P}\mathrm{re}\mathcal{S}\mathrm{h}(X)$ to the functor $\Phi\colon \mathcal{P}\mathrm{re}\mathcal{S}\mathrm{h}(X) \to \mathcal{P}\mathrm{re}\mathcal{S}\mathrm{h}(X)$. This means that, for a morphism of presheaves $\alpha\colon \mathcal{F} \to \mathcal{G}$ on X as above, the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\alpha"] \arrow[d, "\eta"] & \mathcal{G} \arrow[d, "\eta"] \\ \Phi\mathcal{F} \arrow[r, "\Phi\alpha"] & \Phi\mathcal{G} \end{tikzcd}$$

in $\mathcal{P}\mathrm{re}\mathcal{S}\mathrm{h}(X)$ is commutative. That is also easily verified.
There is one more thing of a general nature which must be mentioned. Let $\mathcal{F}$ be any presheaf on X. What happens if we apply the functor Φ to the morphism $\eta_{\mathcal{F}}\colon \mathcal{F} \to \Phi\mathcal{F}$? The result is obviously a morphism of sheaves

$$\Phi(\eta_{\mathcal{F}})\colon \Phi\mathcal{F} \to \Phi(\Phi\mathcal{F}).$$

It is an *isomorphism* of sheaves. The verification is easy using theorem 5.5.

The sheaf $\Phi\mathcal{F}$ is *the sheafification* (or the *associated sheaf*) of the presheaf $\mathcal{F}$; also Φ may be called the sheafification functor, or the associated sheaf functor.

Corollary 5.8. *Let* $\beta\colon \mathcal{F} \to \mathcal{G}$ *be any morphism of presheaves on* X*. If* $\mathcal{G}$ *is a sheaf, then* β *has a unique factorization* $\beta = \beta_1 \circ \eta_{\mathcal{F}}$ *where* $\eta_{\mathcal{F}}\colon \mathcal{F} \to \Phi\mathcal{F}$ *is the morphism of proposition 5.6:*

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\beta"] \arrow[d, "\eta_{\mathcal{F}}"] & \mathcal{G} \\
\Phi\mathcal{F} \arrow[ur, dotted, "\beta_1"'] &
\end{tikzcd}$$

Proof. Apply Φ and η to $\mathcal{F}$, $\mathcal{G}$ and β to obtain a commutative diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\beta"] \arrow[d, "\eta_{\mathcal{F}}"] & \mathcal{G} \arrow[d, "\eta_{\mathcal{G}}"] \\
\Phi\mathcal{F} \arrow[r, "\Phi\beta"] & \Phi\mathcal{G}
\end{tikzcd}$$

By proposition 5.6, the vertical arrows determine bijections $\mathcal{F}_z \to (\Phi\mathcal{F})_z$ and $\mathcal{G}_z \to (\Phi\mathcal{G})_z$ for every $z \in X$. Both $\mathcal{G}$ and $\Phi\mathcal{G}$ are sheaves, so theorem 5.5 applies and we may deduce that the right-hand vertical arrow is an isomorphism of sheaves on X. Let $\lambda\colon \Phi\mathcal{G} \to \mathcal{G}$ be an inverse for that isomorphism. The factorization problem has a solution, $\beta_1 = \lambda \circ \Phi\beta$.
To see that the solution is unique, apply Φ and η to the commutative diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\beta"] \arrow[d, "\eta_{\mathcal{F}}"] & \mathcal{G} \\
\Phi\mathcal{F} \arrow[ur, dotted] &
\end{tikzcd}$$

in $\mathcal{P}\mathrm{re}\mathcal{S}\mathrm{h}(X)$. The result is a commutative diagram in $\mathcal{P}\mathrm{re}\mathcal{S}\mathrm{h}(X)$ in the shape of a prism:

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\beta"] \arrow[d, "\eta_{\mathcal{F}}"] \arrow[dr] & \mathcal{G} \arrow[dr, "\eta_{\mathcal{G}}"] & \\
\Phi\mathcal{F} \arrow[ur, dotted] \arrow[dr] & \Phi\mathcal{F} \arrow[r, "\Phi\beta"] \arrow[d, "\Phi(\eta_{\mathcal{F}})"] & \Phi\mathcal{G} \\
 & \Phi(\Phi\mathcal{F}) \arrow[ur, dotted] &
\end{tikzcd}$$

Here the arrow labeled $\Phi(\eta_{\mathcal{F}})$ is an isomorphism of sheaves, as noted above under things of a general nature. This makes the lower dotted arrow unique. But the arrow labeled $\eta_{\mathcal{G}}$ is also an isomorphism by theorem 5.5 and the property of $\eta_{\mathcal{G}}$ stated in proposition 5.6. This ensures that the upper dotted arrow is determined by the lower dotted arrow. □

5.3. Mapping cycles

Let X and Y be topological spaces. One of the first examples of a sheaf that we saw was the sheaf $\mathcal{F}$ on X such that

$$\mathcal{F}(U) = \text{set of continuous maps from } U \text{ to } Y$$

etc., for open U in X. From that we constructed a presheaf $\mathcal{G}$ on X such that that

$$\mathcal{G}(U) = \text{free abelian group generated by } \mathcal{F}(U)$$

etc., for open U in X. In other words, $\mathcal{G}(U)$ is the set of formal linear combinations (with coefficients in $\mathbb{Z}$) of continuous functions from X to Y. It turned out that $\mathcal{G}$ is never a sheaf, and for many reasons. The stalk $\mathcal{G}_z$ at $z \in X$ can be described (after some unraveling) as the set of formal linear combinations, with integer coefficients, of germs of continuous maps from (X, z) to Y. (Recall that *germ of continuous maps from* (X, z) *to* Y means an equivalence class of pairs (U, f) where U is an open neighborhood of z in X and $f \colon U \to Y$ is continuous.) Of course, we ask what $\mathcal{G}_z$ is because it feeds into the construction of $\Phi\mathcal{G}$, the sheafification of $\mathcal{G}$. It is permitted and even exciting to evaluate $\Phi\mathcal{G}$ on X, since X is an open subset of X.

Definition 5.9. An element of $(\Phi\mathcal{G})(X)$ will be called a *mapping cycle* from X to Y.

So what is a mapping cycle from X to Y ?

First answer. A mapping cycle from X to Y is a function s which for every $z \in X$ selects $s(z)$, a formal linear combination with integer coefficients of germs[1] of continuous maps from (X, z) to Y. There is a coherence condition to be satisfied: it must be possible to cover X by open sets W_i such that all values $s(z)$, where $z \in W_i$, can be simultaneously represented by one formal linear combination

$$\sum_j b_{ij} f_{ij}$$

where $f_{ij} \colon W_i \to Y$ are continuous maps and the b_{ij} are integers.

Second answer. A mapping cycle from X to Y can be specified (described, constructed) by choosing an open cover $(U_i)_{i\in\Lambda}$ of X and for every $i \in \Lambda$ a formal linear combination s_i with integer coefficients of continuous maps[2] from U_i to Y. There is a matching condition to be satisfied[3]: for any $i, j \in \Lambda$ and any $x \in U_i \cap U_j$, there should exist an open neighborhood W of x in $U_i \cap U_j$ such that $s_{i|W} = s_{j|W}$.

(The second answer is in some ways less satisfactory than the first because it does not say explicitly what a mapping cycle *is*, only how we can construct mapping cycles. But it can indeed be useful when we need to construct mapping cycles.)

Some of the “counter”examples which we saw previously now serve as illustrations of the concept of mapping cycle.

Example 5.10. If S is a set with 6 elements and T is a set with 2 elements, both to be viewed as topological spaces with the discrete topology, then the abelian group of mapping cycles from S to T is isomorphic to $\mathbb{Z}^{12} \cong \prod_{i=1}^{6}(\mathbb{Z} \oplus \mathbb{Z})$. Do not confuse with $\mathbb{Z}/12$.

Example 5.11. Let X and Y be two topological spaces related by a covering map $p \colon Y \to X$ with finite fibers. In other words, p is a fiber bundle whose fibers are finite sets (viewed as topological spaces with the discrete topology). For simplicity, suppose also that X is connected. Choose an open covering $(W_j)_{j\in\Lambda}$ of X such that p admits a bundle chart over W_j for each j:

$$h_j \colon p^{-1}(W_j) \to W_j \times F$$

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[1]Grown-up formulation: selects an element in the free abelian group generated by the set of germs ...

[2]Grown-up formulation: for every $i \in \Lambda$ an element s_i in the free abelian group generated by the set of continuous maps ...

[3]Did you expect to see the condition

$$s_{i|U_i\cap U_j} = s_{j\,U_i\cap U_j} \ ?$$

Sheaf theory dictates a weaker condition!

where F is a finite set (with the discrete topology). For $j \in \Lambda$ and $z \in F$ there is a continuous map $\sigma_{j,z} \colon W_j \to Y$ given by $\sigma_{j,z}(x) = h_j^{-1}(x,z)$ for $x \in W_j$. Define

$$s_j = \sum_{z \in F} \sigma_{j,z} \,.$$

This is a formal linear combination of continuous maps from W_j to Y. Clearly

$$s_{i|W_i \cap W_j} = s_{j|W_i \cap W_j}$$

(yes, this is more than we require). Therefore, by *"second answer"*, we have specified a mapping cycle from X to Y (which agrees with s_j on W_j).

Example 5.12. Let X and Y be topological spaces. Suppose that $X = V_1 \cup V_2$ where V_1 and V_2 are open subsets of X. Let continuous maps $f, g \colon V_1 \to Y$ be given such that

$$f_{|V_1 \cap V_2} = g_{|V_1 \cap V_2} \,.$$

Then it makes (some) sense to view the formal linear combination $f - g = 1 \cdot f + (-1) \cdot g$ as a mapping cycle from X to Y. How? We have the open cover of X consisting of V_1 and V_2 , and we specify $s_1 = f - g$ (a mapping cycle from V_1 to Y), and $s_2 = 0$ (a mapping cycle from V_2 to Y). Then $s_{1|V_1 \cap V_2} = 0 = s_{2|V_1 \cap V_2}$. So the matching condition is satisfied, and so by *"second answer"* we have specified a mapping cycle from X to Y.

Mapping cycles are complicated beasts, but I am hoping that readers having survived the excursion into sheaf theory remain sufficiently intoxicated to find the definition obvious and unavoidable. With that, the excursion into sheaf theory is over (for now, though I do not say *never again*). Now we shall try to develop a comfortable relationship with mapping cycles. Here is a list of some of their good uses and properties.

(1) Every continuous map from X to Y determines a mapping cycle from X to Y.
(2) The mapping cycles from X to Y form an abelian group.
(3) A mapping cycle from X to Y can be composed with a (continuous) map from Y to Z to give a mapping cycle from X to Z. A mapping cycle from Y to Z can be composed with a (continuous) map from X to Y to give a mapping cycle from X to Z. But more remarkably, a mapping cycle from X to Y can be composed with a mapping cycle from Y to Z to give a mapping cycle from X to Z.
(4) Composition of mapping cycles is bilinear.
(5) Mapping cycles satisfy a sheaf property: if $(U_i)_{i \in \Lambda}$ is an open covering of X and $s_i \colon U_i \to Y$ is a mapping cycle, for each $i \in \Lambda$, such that

$$s_{i|U_i \cap U_j} = s_{j|U_i \cap U_j}$$

for all $i, j \in \Lambda$, then there is a unique mapping cycle s from X to Y such that $s_{|U_i} = s_i$ for all $i \in \Lambda$.

(6) There is exactly one mapping cycle from X to $\emptyset$. And there is exactly one mapping cycle from $\emptyset$ to Y, for any space Y.

(7) Mapping cycles from a topological disjoint union $X_1 \coprod X_2$ to Y are in bijection with pairs (s_1, s_2) where s_i is a mapping cycle from X_i to Y for $i = 1, 2$. Mapping cycles from X to a topological disjoint union $Y_1 \coprod Y_2$ are in bijection with pairs (s_1, s_2) where s_i is a mapping cycle from X to Y_i for $i = 1, 2$.

Some comments on that.
(1) A continuous map $f \colon X \to Y$ determines a mapping cycle $s = s_f$ where $s(z)$ is the germ of f at z. Interesting observation: the map $f \mapsto s_f$ from the set of continuous maps from X to Y to the set of mapping cycles from X to Y is injective.
(2) Obvious.
(3) Given a mapping cycle s from X to Y and a continuous map $g \colon Y \to Z$ we get a mapping cycle $g \circ s$ from X to Z by $x \mapsto \sum b_j(g \circ f_j)$ when $x \in X$ and $s(x) = \sum b_j f_j$. Given a mapping cycle s from Y to Z and a continuous map $g \colon X \to Y$ we get a mapping cycle $s \circ g$ from X to Z by $x \mapsto \sum b_j(f_j \circ g)$ when $x \in X$ and $s(x) = \sum b_j f_j$. Given a mapping cycle s from X to Y and a mapping cycle t from Y to Z we get a mapping cycle $t \circ s$ from X to Z by the formula

$$x \mapsto \sum (b_j c_{ij})(f_{ij} \circ g_j)$$

when $x \in X$ and $s(x) = \sum_j b_j g_j$ and $t(g_j(x)) = \sum_i c_{ij} f_{ij}$. (The notation is not fantastically precise; in any case b_j, c_{ij} etc. are meant to be integers while f_j , g_j etc. are meant to be germs of continuous functions. Note that f_{ij} in the last formula is a germ at $g_j(x)$, while g_j is a germ at x.)
(4) Should be clear from the last formula in the comment on (3).
(5) By construction.
(6) Mapping cycles from $\emptyset$ to Y: there is exactly one by construction. A mapping cycle s from X to $\emptyset$ is a function which for each $x \in X$ selects a formal linear combination of germs of continuous maps from (X, x) to $\emptyset$, etc.; since there no such germs, the only possible formal linear combination is the zero linear combination. This does satisfy the coherence condition.
(7) By construction and by inspection.

In category language, we can say that there is a category $\mathcal{A}\mathcal{T}\mathrm{op}$ whose objects are the topological spaces and where a morphism from space X to space Y is a mapping cycle from X to Y. There is an "inclusion" functor

$$\mathcal{T}\mathrm{op} \to \mathcal{A}\mathcal{T}\mathrm{op}$$

taking every object X to the same object X, and taking a morphism $\mathsf{f}\colon \mathsf{X} \to \mathsf{Y}$ (continuous map) to the corresponding mapping cycle as explained in (1). For each X and Y, the set $\mathrm{mor}_{\mathcal{A}\mathfrak{T}\mathrm{op}}(\mathsf{X}, \mathsf{Y})$ is equipped with the structure of an abelian group. Composition of morphisms is bilinear. There is a zero object X in $\mathcal{A}\mathfrak{T}\mathrm{op}$, i.e., an object with the property that $\mathrm{mor}_{\mathcal{A}\mathfrak{T}\mathrm{op}}(\mathsf{X}, \mathsf{Y})$ has exactly one element and $\mathrm{mor}_{\mathcal{A}\mathfrak{T}\mathrm{op}}(\mathsf{Y}, \mathsf{X})$ has exactly one element for arbitrary Y. Indeed, $\mathsf{X} = \emptyset$ is a zero object in $\mathcal{A}\mathfrak{T}\mathrm{op}$. The property expressed in (7) can also be formulated in category language, but we must postpone it because the vocabulary for that has not been introduced so far. In all, we can say that $\mathcal{A}\mathfrak{T}\mathrm{op}$ is an *additive category*.