26.11.2013

Lecture Notes, week 7
Topology WS 2013/14 (Weiss)

6.1. Homotopies in $\mathcal{A}\mathcal{T}op$

Definition 6.1. Let X and Y be topological spaces. We call two mapping cycles f and g from X to Y *homotopic* if there exists a mapping cycle h from $X\times[0,1]$ to Y such that $f=h\circ\iota_0$ and $g=h\circ\iota_0$. Here $\iota_0,\iota_1\colon X\to X\times[0,1]$ are defined by $\iota_0(x)=(x,0)$ and $\iota_1(x)=(x,1)$ as usual. Such a mapping cycle h is a *homotopy* from f to g.

Remark. In that definition, $X\times[0,1]$ still means the product of X and $[0,1]$ in $\mathcal{T}op$. We saw some evidence suggesting that in $\mathcal{A}\mathcal{T}op$ this does not have the properties that we might expect from a product (in a category sense).

Lemma 6.2. *"Homotopic" is an equivalence relation on the set of mapping cycles from* X *to* Y. The set of equivalence classes will be denoted by $[[X,Y]]$ and the equivalence class of a mapping cycle f will be denoted by $[[f]]$.

Proof. Reflexivity and symmetry are fairly obvious. Transitivity is more interesting. Let h be a homotopy from e to f and k a homotopy from f to g, where e,f and g are mapping cycles from X to Y. Without loss of generality, h and k are *stationary* at times 0 and 1, in the following precise sense: for some small positive ε, the mapping cycle h agrees on the open subset $X\times[0,\varepsilon[$ of $X\times[0,1]$ with $e\circ p_1$ where $p_1(x,t)=x$, and agrees on the open subset $X\times]1-\varepsilon,1]$ with $f\circ p_1$; and similarly for k. (If not, choose a continuous map $\lambda\colon[0,1]\to[0,1]$ such that $\lambda(t)=0$ and $\lambda(1-t)=1$ for $t<\varepsilon$. Re-define h and k by pre-composing with the continuous map $X\times[0,1]\to X\times[0,1]$ given by $(x,t)\mapsto(x,\lambda(t))$.) Then by the sheaf property for mapping cycles, there is a mapping cycle

$$X\times[0,2]\longrightarrow Y$$

which on the open set $U_1=X\times[0,1+\varepsilon[$ agrees with the composition $h\circ q_1$ where $q_1(x,t)=(x,\min\{1,t\})$, and which on the open set $U_2=X\times[1-\varepsilon,2[$ agrees with the composition $k\circ q_2$ where $q_2(x,t)=(x,\max\{0,t-1\})$. (These mapping cycles agree on $U_1\cap U_2$ by our assumptions on h and k.) Pre-compose this mapping cycle from $X\times[0,2]$ to Y with the homeomorphism $X\times[0,1]\to X\times[0,2]$ given by stretching, $(x,t)\mapsto(x,2t)$. ☐

Proposition 6.3. *The set* $[[X,Y]]$ *is an abelian group.*

Proof. This amounts to observing that the homotopy relation is compatible with addition of mapping cycles. In other words, if f is homotopic to g and u is homotopic to v, where f, g, u, v are mapping cycles from X to Y, then $f+u$ is homotopic to $g+v$. Indeed, if h is a homotopy from f to g and k is a homotopy from u to v, then $h+k$ is a homotopy from $f+u$ to $g+v$. $\square$

Lemma 6.4. *A composition map* $[[Y, Z]] \times [[X, Y]] \to [[X, Z]]$ *can be defined by* $([[f]], [[g]]) \mapsto [[f \circ g]]$. *Composition is bilinear, i.e., for fixed* $[[g]]$ *the map* $[[f]] \mapsto [[f \circ g]]$ *is a homomorphism of abelian groups and for fixed* $[[f]]$ *the map* $[[g]] \mapsto [[f \circ g]]$ *is a homomorphism of abelian groups.* $\square$

As a result there is a homotopy category $\mathcal{H}o\mathcal{AT}op$ whose objects are (still) the topological spaces, while the set of morphisms from X to Y is $[[X, Y]]$.

6.2. First calculations

Write $\star$ for a singleton, alias one-point space.

Proposition 6.5. *For any space* X *the abelian group* $[[X, \star]]$ *is isomorphic to the set of continuous (=locally constant) functions from* X *to* $\mathbb{Z}$*, where* $\mathbb{Z}$ *has the discrete topology.*

Proof. We saw already in the previous section that the set of mapping cycles from X to $\star$ is identified with the set of continuous functions from X to $\mathbb{Z}$. (It is $(\Phi\mathcal{G})(X)$ where $\Phi\mathcal{G}$ is the sheaf associated to the constant presheaf $\mathcal{G}$ which has $\mathcal{G}(U) = \mathbb{Z}$ for all open $U \subset X$.) Similarly, the set of mapping cycles from $X \times [0, 1]$ to $\star$ is identified with the set of continuous functions from $X \times [0, 1]$ to $\mathbb{Z}$. But a continuous function h from $X \times [0, 1]$ to $\mathbb{Z}$ is constant on $\{x\} \times [0, 1]$ for each $x \in X$, and so will have the form $h(x, t) = g(x)$ for a unique continuous $g \colon X \to \mathbb{Z}$. It follows that the homotopy relation on the set of mapping cycles from X to $\star$ is trivial, i.e., two mapping cycles from X to $*$ are homotopic only if they are equal. $\square$

Example 6.6. Take $X = \mathbb{Q}$, a subspace of $\mathbb{R}$ with the standard topology. The group $[[\mathbb{Q}, \star]]$ is uncountable because the set of continuous maps from $\mathbb{Q}$ to $\mathbb{Z}$ is uncountable.

Lemma 6.7. *For a path-connected (non-empty) space* Y *the abelian group* $[[\star, Y]]$ *is isomorphic to* $\mathbb{Z}$.

Proof. Fix some point $z \in Y$. A mapping cycle from $\star$ to Y is the same thing as a formal linear combination of points in Y, say $\sum_j b_j y_j$ where $b_j \in \mathbb{Z}$ and $y_j \in Y$. In the abelian group $[[\star, Y]]$ we have

$$[[\Sigma_j b_j y_j]] = \Sigma_j b_j [[y_j]] = (\Sigma_j b_j)[[z]].$$

(Here $[[y_j]]$ for example denotes the homotopy class of the mapping cycle determined by the continuous map $\star \to Y$ which has image $\{y_j\}$. As that continuous map is homotopic to the map $\star \to Y$ which has image $\{z\}$, we obtain $[[y_j]] = [[z]]$.) Therefore $[[\star, Y]]$ is cyclic, generated by the element $[[z]]$. To see that it is infinite cyclic we use the homomorphism

$$[[\star, Y]] \to [[\star, \star]]$$

given by composition with the continuous map $Y \to \star$. Now $[[\star, \star]]$ is infinite cyclic by proposition 6.5. It is also clear that the homomorphism just above takes $[[z]]$ to the generator of $[[\star, \star]]$, the class of the identity mapping cycle. Hence it must be an isomorphism and so $[[\star, Y]]$ is infinite cyclic. □

Corollary 6.8. *For any space* Y *the abelian group* $[[\star, Y]]$ *is isomorphic to the free abelian group generated by the set of path components of* Y.

Proof. The abelian group of mapping cycles from $\star$ to Y is simply the free abelian group A generated by the underlying set of Y. Write this as a direct sum $\bigoplus_{\lambda \in \Lambda} A_\lambda$ where Λ is an indexing set for the path components Y_λ of Y and A_λ is the free abelian group generated by the underlying set of Y_λ. Now fix some λ. *Claim*: If $f \in A$ is homotopic to $g \in A$, by a mapping cycle $h\colon [0, 1] \to Y$, then the coordinate of f in A_λ is homotopic to the coordinate of g in A_λ , by a mapping cycle $[0, 1] \to Y_\lambda$. To see this, cover the interval $[0, 1]$ by finitely many open subsets U_i such that $h_{|U_i}$ can be represented by a formal linear combination of continuous maps from U_i to Y. This is possible by the coherence condition on h. Choose a subdivision

$$0 = t_0 < t_1 < \cdots t_{N-1} < t_N = 1$$

of $[0, 1]$ such that for each of the the subintervals $[t_r, t_{r+1}]$, where $r = 0, 1, \ldots, N-1$, there exists U_i containing it. Let $h_{t_r} \in A$ be obtained by restricting h to t_r . Then $h_{t_0} = f$ and $h_{t_N} = g$, so it suffices to show that the λ-coordinate of h_{t_r} is homotopic to the λ-coordinate of $h_{t_{r+1}}$, for $r = 0, 1, \ldots, N-1$. But $[t_r, t_{r+1}]$ is contained in some U_i and so there is a formal linear combination

$$\sum_j b_j u_j$$

where $b_j \in \mathbb{Z}$ and the u_j are continuous maps from $[t_r, t_{r+1}]$ to Y, and $\sum_j b_j u_j$ restricts to h_{t_r} on t_r and to $h_{t_{r+1}}$ on t_{r+1}. Each u_j maps to only one path component of Y; in the formal linear combination $\sum_j b_j u_j$, select the terms $b_j u_j$ where u_j is a map to Y_λ and discard the others. Then the selected linear sub-combination is a homotopy from the λ-component of h_{t_r} to the λ-component of $h_{t_{r+1}}$. This proves the claim.
Therefore $[[\star, Y]]$ is the direct sum of the $[[\star, Y_\lambda]]$. By the lemma above, each $[[\star, Y_\lambda]]$ is isomorphic to $\mathbb{Z}$. □

Proposition 6.9. *For topological spaces* X *and* Y *where* X *is a topological disjoint union* $X_1 \amalg X_2$*, there is an isomorphism*

$$[[X, Y]] \longrightarrow [[X_1, Y]] \times [[X_2, Y]] \ ; \ [[f]] \mapsto ([[f_{|X_1}]], [[f_{|X_2}]]) \ .$$

For topological spaces X *and* Y *where* Y *is a topological disjoint union* $Y_1 \amalg Y_2$*, there is an isomorphism*

$$[[X, Y_1]] \oplus [[X, Y_2]] \longrightarrow [[X, Y]] \ ; [[f]] \oplus [[g]] \mapsto [[j_1 \circ f + j_2 \circ g]]$$

where $j_1 \colon Y_1 \to Y$ *and* $j_2 \colon Y_2 \to Y$ *are the inclusions.*

Proof. First statement: the set $\mathrm{mor}_{\mathcal{A}\mathcal{T}\mathrm{op}}(X, Y)$ of mapping cycles breaks up as a product $\mathrm{mor}_{\mathcal{A}\mathcal{T}\mathrm{op}}(X_1, Y) \times \mathrm{mor}_{\mathcal{A}\mathcal{T}\mathrm{op}}(X_2, Y)$ by restriction to X_1 and X_2, and a similar statement holds for the set $\mathrm{mor}_{\mathcal{A}\mathcal{T}\mathrm{op}}(X \times [0, 1], Y)$. Second statement: the set $\mathrm{mor}_{\mathcal{A}\mathcal{T}\mathrm{op}}(X, Y)$ of mapping cycles breaks up as a direct sum $\mathrm{mor}_{\mathcal{A}\mathcal{T}\mathrm{op}}(X, Y_1) \times \mathrm{mor}_{\mathcal{A}\mathcal{T}\mathrm{op}}(X, Y_2)$, and a similar statement holds for $\mathrm{mor}_{\mathcal{A}\mathcal{T}\mathrm{op}}(X \times [0, 1], Y)$. □

Proposition 6.10. *For any topological space* X *we have*

$$[[\emptyset, X]] = 0 = [[X, \emptyset]] \, .$$

Proof. The abelian group of mapping cycles from X to $\emptyset$ is a trivial group and the abelian group of mapping cycles from $\emptyset$ to X is a trivial group. □

6.3. Homology and cohomology: the definitions

Definition 6.11. For $n \geq 0$, the n-th *homology group* of a topological space X is the abelian group

$$H_n(X) := [[S^n, X]]/[[\star, X]] \ .$$

The n-th *cohomology group* of X is the abelian group

$$H^n(X) := [[X, S^n]]/[[X, \star]].$$

Comments. There is an understanding here that $[[\star, X]]$ is a subgroup of $[[S^n, X]]$. How? By pre-composing mapping cycles from $\star$ to X with the unique continuous map $S^n \to \star$, we obtain a (well defined) homomorphism $[[\star, X]] \to [[S^n, X]]$. Conversely, by pre-composing mapping cycles from S^n to X with a selected continuous map $\star \to S^n$, inclusion of the base point, we obtain a homomorphism $[[S^n, X]] \to [[\star, X]]$. The composition $[[\star, X]] \to [[S^n, X]] \to [[\star, X]]$ is the identity on $[[\star, X]]$. So we can say that $[[\star, X]]$ is a direct summand of $[[S^n, X]]$. We remove it, suppress it etc., when we form $H_n(X)$.
Similarly, by post-composing mapping cycles from X to S^n with the unique continuous map $S^n \to \star$, we obtain a homomorphism $[[X, S^n]] \to [[X, \star]]$. Conversely, by post-composing mapping cycles from X to $\star$ with a selected

continuous map $\star \to S^n$, inclusion of the base point, we obtain a homomorphism $[[X,\star]] \to [[X,S^n]]$. The composition $[[X,\star]] \to [[X,S^n]] \to [[X,\star]]$ is the identity on $[[X,\star]]$. Therefore $[[X,\star]]$ is a direct summand of $[[X,S^n]]$. We remove it, suppress it etc., when we form $H^n(X)$.

You will be unsurprised to hear that H_n is a functor from $\mathcal{T}op$ to the category of abelian groups. We can also say that it is a functor from $\mathcal{AT}op$ to abelian groups. Both statements are obvious from the definition. Equally clear from the definition, but important to keep in mind: if $f,g\colon X \to Y$ are homotopic maps, then the induced homomorphisms $f_*\colon H_n(X) \to H_n(Y)$ and $g_*\colon H_n(X) \to H_n(Y)$ are the same. (Therefore we might say that H_n is a functor from $\mathcal{H}o\mathcal{T}op$ to the category of abelian groups. Indeed it is a functor from $\mathcal{H}o\mathcal{AT}op$ to abelian groups ...)
Similarly H^n is a contravariant functor from $\mathcal{T}op$ (or from $\mathcal{AT}op$, or from $\mathcal{H}o\mathcal{T}op$, or from $\mathcal{H}o\mathcal{AT}op$) to the category of abelian groups.

So far we have few tools available for computing $H_n(X)$ and $H^n(X)$ in general. But in the cases $n=0$, arbitrary X, we are ready for it, and in the case where n is arbitrary and $X=\star$ we are also ready for it.

Example 6.12. Take $n=0$ and X arbitrary. Then $H_0(X) = [[S^0,X]]/[[\star,X]]$. For S^0 we can write $\star \amalg \star$ (disjoint union of two copies of $\star$), and using the first part of proposition 6.9, we get $[[S^0,X]] \cong [[\star,X]] \times [[\star,X]]$. Therefore $H_0(X) \cong [[\star,X]]$. Using corollary 6.8, it follows that $H_0(X)$ is identified with the free abelian group generated by the set of path components of X. For example, if X is path connected, then $H_0(X)$ is isomorphic to $\mathbb{Z}$.
By a very similar calculation, $H^0(X)$ is isomorphic to $[[X,\star]]$. Using proposition 6.5, we then obtain that $H^0(X)$ is isomorphic to the abelian group of continuous maps from X to $\mathbb{Z}$. For example, if X is connected, then $H^0(X)$ is isomorphic to $\mathbb{Z}$.

Example 6.13. Take n arbitrary and $X=\star$. Now $H_n(\star) = [[S^n,\star]]/[[\star,\star]]$. Using proposition 6.5, we find $[[S^n,\star]] \cong \mathbb{Z}$ when $n>0$ and $[[S^0,\star]] \cong \mathbb{Z}\oplus\mathbb{Z}$; also $[[\star,\star]] = \mathbb{Z}$. By an easy calculation, the quotient $[[S^n,\star]]/[[\star,\star]]$ is therefore 0 when $n>0$, and isomorphic to $\mathbb{Z}$ when $n=0$. So we have:

$$H_n(\star) \quad \cong \quad \begin{cases} \mathbb{Z} & \text{if } n=0 \\ 0 & \text{if } n>0 \end{cases}$$

Similarly, $H^n(\star) = [[\star,S^n]]/[[\star,\star]]$. Using corollary 6.8 this time, we find that $[[\star,S^n]] \cong \mathbb{Z}$ when $n>0$ and $[[\star,S^0]] \cong \mathbb{Z}\oplus\mathbb{Z}$. By an easy calculation, the quotient $[[\star,S^n]]/[[\star,\star]]$ is therefore 0 when $n>0$, and isomorphic to $\mathbb{Z}$ when $n=0$. Therefore:

$$H^n(\star) \quad \cong \quad \begin{cases} \mathbb{Z} & \text{if } n=0 \\ 0 & \text{if } n>0 \end{cases}$$