Homology without simplices

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Winter 2013/2014 and Winter 2014/2015

CHAPTER 1

Homotopy

1.1. The homotopy relation

Let X and Y be topological spaces. (If you are not sufficiently familiar with topological spaces, you should assume that X and Y are metric spaces.) Let f and g be continuous maps from X to Y. Let $[0,1]$ be the unit interval with the standard topology, a subspace of $\mathbb{R}$.

Definition 1.1.1. A *homotopy* from f to g is a continuous map

$$h : X \times [0,1] \to Y$$

such that $h(x,0) = f(x)$ and $h(x,1) = g(x)$ for all $x \in X$. If such a homotopy exists, we say that f and g are *homotopic*, and write $f \simeq g$. We also sometimes write $h : f \simeq g$ to indicate that h is a homotopy from the map f to the map g.

Remark 1.1.2. If you made the assumption that X and Y are metric spaces, then you should use the product metric on $X \times [0,1]$ and $Y \times [0,1]$, so that for example

$$d((x_1,t_1),(x_2,t_2)) := \max\{d(x_1,x_2), |t_1 - t_2|\}$$

for $x_1, x_2 \in X$ and $t_1, t_2 \in [0,1]$. If you were happy with the assumption that X and Y are "just" topological spaces, then you need to know the definition of *product of two topological spaces* in order to make sense of $X \times [0,1]$ and $Y \times [0,1]$.

Remark 1.1.3. A homotopy $h : X \times [0,1] \to Y$ from $f : X \to Y$ to $g : X \to Y$ can be seen as a "family" of continuous maps

$$h_t : X \to Y\ ;\ h_t(x) = h(x,t)$$

such that $h_0 = f$ and $h_1 = g$. The important thing is that h_t depends continuously on $t \in [0,1]$.

Example 1.1.4. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the identity map. Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be the map such that $g(x) = 0 \in \mathbb{R}^n$ for all $x \in \mathbb{R}^n$. Then f and g are homotopic. The map $h : \mathbb{R}^n \times [0,1]$ defined by $h(x,t) = tx$ is a homotopy from f to g.

Example 1.1.5. Let $f : S^1 \to S^1$ be the identity map, so that $f(z) = z$. Let $g : S^1 \to S^1$ be the antipodal map, $g(z) = -z$. Then f and g are homotopic. Using complex number notation, we can define a homotopy by $h(z,t) = e^{\pi i t} z$.

Example 1.1.6. Let $f : S^2 \to S^2$ be the identity map, so that $f(z) = z$. Let $g : S^2 \to S^2$ be the antipodal map, $g(z) = -z$. Then f and g are *not* homotopic. We will prove this later in the course.

Example 1.1.7. Let $f : S^1 \to S^1$ be the identity map, so that $f(z) = z$. Let $g : S^1 \to S^1$ be the constant map with value 1. Then f and g are *not* homotopic. We will prove this quite soon.

PROPOSITION 1.1.8. *"Homotopic" is an equivalence relation on the set of continuous maps from* X *to* Y.

PROOF. Reflexive: For every continuous map $f : X \to Y$ define the *constant homotopy* $h : X \times [0,1] \to Y$ by $h(x,t) = f(x)$.
Symmetric: Given a homotopy $h : X \times [0,1] \to Y$ from a map $f : X \to Y$ to a map $g : X \to Y$, define the *reverse homotopy* $\bar{h} : X \times [0,1] \to Y$ by $\bar{h}(x,t) = h(x, 1-t)$. Then $\bar{h}$ is a homotopy from g to f.
Transitive: Given continuous maps $e, f, g : X \to Y$, a homotopy h from e to f and a homotopy k from f to g, define the *concatenation homotopy* $k * h$ as follows:

$$(x,t) \mapsto \begin{cases} h(x,2t) & \text{if } 0 \leqslant t \leqslant 1/2 \\ k(x,2t-1) & \text{if } 1/2 \leqslant t \leqslant 1 \ . \end{cases}$$

Then $k * h$ is a homotopy from e to g. □

DEFINITION 1.1.9. The equivalence classes of the above relation "homotopic" are called *homotopy classes*. The homotopy class of a map $f : X \to Y$ is often denoted by $[f]$. The set of homotopy classes of maps from X to Y is often denoted by $[X, Y]$.

PROPOSITION 1.1.10. *Let* X, Y *and* Z *be topological spaces. Let* $f : X \to Y$ *and* $g : X \to Y$ *and* $u : Y \to Z$ *and* $v : Y \to Z$ *be continuous maps. If* f *is homotopic to* g *and* u *is homotopic to* v*, then* $u \circ f : X \to Z$ *is homotopic to* $v \circ g : X \to Z$.

PROOF. Let $h : X \times [0,1] \to Y$ be a homotopy from f to g and let $w : Y \times [0,1] \to Z$ be a homotopy from u to v. Then $u \circ h$ is a homotopy from $u \circ f$ to $u \circ g$ and the map $X \times [0,1] \to Z$ given by $(x,t) \mapsto w(g(x),t)$ is a homotopy from $u \circ g$ to $v \circ g$. Because the homotopy relation is transitive, it follows that $u \circ f \simeq v \circ g$. □

DEFINITION 1.1.11. Let X and Y be topological spaces. A (continuous) map $f : X \to Y$ is a *homotopy equivalence* if there exists a map $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.
We say that X is *homotopy equivalent* to Y if there exists a map $f : X \to Y$ which is a homotopy equivalence.

DEFINITION 1.1.12. If a topological space X is homotopy equivalent to a point, then we say that X is *contractible*. This amounts to saying that the identity map $X \to X$ is homotopic to a constant map from X to X.

EXAMPLE 1.1.13. $\mathbb{R}^m$ is contractible, for any $m \geq 0$.

EXAMPLE 1.1.14. $\mathbb{R}^m \smallsetminus \{0\}$ is homotopy equivalent to S^{m-1}.

EXAMPLE 1.1.15. The general linear group of $\mathbb{R}^m$ is homotopy equivalent to the orthogonal group $O(m)$. The Gram-Schmidt orthonormalisation process leads to an easy proof of that.

1.2. Homotopy classes of maps from the circle to itself

Let $p : \mathbb{R} \to S^1$ be the (continuous) map given in complex notation by $p(t) = \exp(2\pi i t)$ and in real notation by $p(t) = (\cos(2\pi t), \sin(2\pi t))$. In the first formula we think of S^1 as a subset of $\mathbb{C}$ and in the second formula we think of S^1 as a subset of $\mathbb{R}^2$.
Note that p is surjective and $p(t+1) = p(t)$ for all $t \in \mathbb{R}$. We are going to use p to understand the homotopy classification of continuous maps from S^1 to S^1. The main lemma is as follows.

LEMMA 1.2.1. *Let* $\gamma\colon [0,1] \to S^1$ *be continuous, and* $a \in \mathbb{R}$ *such that* $p(a) = \gamma(0)$. *Then there exists a unique continuous map* $\tilde{\gamma}\colon [0,1] \to \mathbb{R}$ *such that* $\gamma = p \circ \tilde{\gamma}$ *and* $\tilde{\gamma}(0) = a$.

PROOF. The map γ is *uniformly continuous* since $[0,1]$ is compact. It follows that there exists a positive integer n such that $d(\gamma(x), \gamma(y)) < 1/100$ whenever $|x-y| \le 1/n$. Here d denotes the standard (euclidean) metric on S^1 as a subset of $\mathbb{R}^2$. We choose such an n and write

$$[0,1] = \bigcup_{k=1}^{n} [t_{k-1}, t_k]$$

where $t_k = k/n$. We try to define $\tilde{\gamma}$ on $[0, t_k]$ by induction on k. For the induction beginning we need to define $\tilde{\gamma}$ on $[0, t_1]$ where $t_1 = 1/n$. Let $U \subset S^1$ be the open ball of radius $1/100$ with center $\gamma(0)$. (Note that *open ball* is a metric space concept.) Then $\gamma([0, t_1]) \subset U$. Therefore, in defining $\tilde{\gamma}$ on $[0, t_1]$, we need to ensure that $\tilde{\gamma}([0, t_1])$ is contained in $p^{-1}(U)$. Now $p^{-1}(U) \subset \mathbb{R}$ is a disjoint union of open intervals which are mapped homeomorphically to U under p. One of these, call it V_a , contains a, since $p(a) = \gamma(0) \in U$. The others are translates of the form $\ell + V_a$ where $\ell \in \mathbb{Z}$. Since $[0, t_1]$ is connected, its image under $\tilde{\gamma}$ will also be connected, whatever $\tilde{\gamma}$ is, and so it must be contained entirely in exactly one of the intervals $\ell + V_a$. Since we want $\tilde{\gamma}(0) = a$, we must have $\ell = 0$, that is, image of $\tilde{\gamma}$ contained in V_a. Since the map p restricts to a homeomorphism from V_a to U, we must have $\tilde{\gamma} = q\gamma$ where q is the inverse of the homeomorphism from V_a to U. This formula determines the map $\tilde{\gamma}$ on $[0, t_1]$.
The induction steps are like the induction beginning. In the next step we define $\tilde{\gamma}$ on $[t_1, t_2]$, using a "new" a which is $\tilde{\gamma}(t_1)$ and a "new" U which is the open ball of radius $1/100$ with center $\gamma(t_1)$. □

Now let $g\colon S^1 \to S^1$ be any continuous map. We want to associate with it an integer, the degree of g. Choose $a \in \mathbb{R}$ such that $p(a) = g(1)$. Let $\gamma = g \circ p$ on $[0,1]$; this is a map from $[0,1]$ to S^1. Construct $\tilde{\gamma}$ as in the lemma. We have $p\tilde{\gamma}(1) = \gamma(1) = \gamma(0) = p\tilde{\gamma}(0)$, which implies $\tilde{\gamma}(1) = \tilde{\gamma}(0) + \ell$ for some $\ell \in \mathbb{Z}$.

DEFINITION 1.2.2. This ℓ is the degree of g, denoted $\deg(g)$.

It looks as if this might depend on our choice of a with $p(a) = g(1)$. But if we make another choice then we only replace a by $m + a$ for some $m \in \mathbb{Z}$, and we only replace $\tilde{\gamma}$ by $m + \tilde{\gamma}$. Therefore our calculation of $\deg(g)$ leads to the same result.

Remark. Suppose that $g\colon S^1 \to S^1$ is a continuous map which is close to the constant map $z \mapsto 1 \in S^1$ (complex notation). To be more precise, assume $d(g(z), 1) < 1/1000$ for all $z \in S^1$. Then $\deg(g) = 0$.
The verification is mechanical. Define $\gamma\colon [0,1] \to S^1$ by $\gamma(t) = g(p(t))$. Let $V \subset \mathbb{R}$ be the open interval from $-1/100$ to $1/100$. The map p restricts to a homeomorphism from V to $p(V) \subset S^1$, with inverse $q\colon p(V) \to V$. Put $\tilde{\gamma} = q \circ \gamma$, which makes sense because the image of γ is contained in $p(V)$ by our assumption. Then $p \circ \tilde{\gamma} = \gamma$ as required. Now the image of $\tilde{\gamma}$ is contained in V and therefore

$$|\deg(g)| = |\tilde{\gamma}(1) - \tilde{\gamma}(0)| \le 2/100$$

and so $\deg(g) = 0$.

Remark. Suppose that $f, g\colon S^1 \to S^1$ are continuous maps. Let $w\colon S^1 \to S^1$ be defined by $w(z) = f(z) \cdot g(z)$ (using the multiplication in $S^1 \subset \mathbb{C}$). Then $\deg(w) = \deg(f) + \deg(g)$. The verification is also mechanical. Define $\varphi, \gamma, \omega\colon [0,1] \to S^1$ by $\varphi(t) = f(p(t))$, $\gamma(t) =$

$g(p(t))$ and $\omega(t) = w(p(t))$. Construct $\tilde{\varphi}\colon [0,1] \to \mathbb{R}$ and $\tilde{\gamma}\colon [0,1] \to \mathbb{R}$ as in lemma 1.2.1. Put $\tilde{\omega} := \tilde{\varphi} + \tilde{\gamma}$. Then $p \circ \tilde{\omega} = \omega$, so

$$\deg(w) = \tilde{\omega}(1) - \tilde{\omega}(0) = \cdots = \deg(f) + \deg(g).$$

LEMMA 1.2.3. *If $f, g\colon S^1 \to S^1$ are continuous maps which are homotopic, $f \sim g$, then they have the same degree.*

PROOF. Let $h\colon S^1 \times [0,1] \to S^1$ be a homotopy from f to g. As usual let $h_t\colon S^1 \to S^1$ be the map defined by $h_t(z) = h(z,t)$, for fixed $t \in [0,1]$. For fixed $t \in [0,1]$ we can find $\delta > 0$ such that $d(h_t(z), h_s(z)) < 1/1000$ for all $z \in S^1$ and all s which satisfy $|s-t| < \delta$. Therefore $h_s(z) = g_s(z) \cdot h_t(z)$ for such s, where $g_s\colon S^1 \to S^1$ is a map which satisfies $d(g_s(z), 1) < 1/1000$ for all $z \in S^1$. Therefore $\deg(g_s) = 0$ by the remarks above and so $\deg(h_s) = \deg(g_s) + \deg(h_t) = \deg(h_t)$.
We have now shown that the the map $[0,1] \to \mathbb{Z}$ given by $t \mapsto \deg(h_t)$ is locally constant (equivalently, *continuous* as a map of metric spaces) and so it is constant (since $[0,1]$ is connected). In particular $\deg(f) = \deg(h_0) = \deg(h_1) = \deg(g)$. □

LEMMA 1.2.4. *If $f, g\colon S^1 \to S^1$ are continuous maps which have the same degree, then they are homotopic.*

PROOF. Certainly f is homotopic to a map which takes 1 to 1 and g is homotopic to a map which takes 1 to 1 (using complex notation, $1 \in S^1 \subset \mathbb{C}$). Therefore we can assume without loss of generality that $f(1) = 1$ and $g(1) = 1$.
Let $\varphi\colon [0,1] \to S^1$ and $\gamma\colon [0,1] \to S^1$ be defined by $\varphi(t) = f(p(t))$ and $\gamma(t) = g(p(t))$. Construct $\tilde{\varphi}$ and $\tilde{\gamma}$ as in the lemma, using $a = 0$ in both cases, so that $\tilde{\varphi}(0) = 0 = \tilde{\gamma}(0)$. Then

$$\tilde{\varphi}(1) = \deg(f) = \deg(g) = \tilde{\gamma}(1).$$

Note that f can be recovered from $\tilde{\varphi}$ as follows. For $z \in S^1$ choose $t \in [0,1]$ such that $p(t) = z$. Then $f(z) = f(p(t)) = \varphi(t) = p\tilde{\varphi}(t)$. If $z = 1 \in S^1$, we can choose $t = 0$ or $t = 1$, but this ambiguity does not matter since $p\tilde{\varphi}(1) = p\tilde{\varphi}(0)$. Similarly, g can be recovered from $\tilde{\gamma}$. Therefore we can show that f is homotopic to g by showing that $\tilde{\varphi}$ is homotopic to $\tilde{\gamma}$ *with endpoints fixed.* In other words we need a continuous

$$H\colon [0,1] \times [0,1] \to \mathbb{R}$$

where $H(s,0) = \tilde{\varphi}(s)$, $H(s,1) = \tilde{\gamma}(s)$ and $H(0,t) = 0$ for all $t \in [0,1]$ and $H(1,t) = \tilde{\varphi}(1) = \tilde{\gamma}(1)$ for all $t \in [0,1]$. This is easy to do: let $H(s,t) = (1-t)\tilde{\varphi}(s) + t\tilde{\gamma}(s)$. □

Summarizing, we have shown that the degree function gives us a well defined map from $[S^1, S^1]$ to $\mathbb{Z}$, and moreover, that this map is injective. It is not hard to show that this map is also surjective! Namely, for arbitrary $\ell \in \mathbb{Z}$ the map $f\colon S^1 \to S^1$ given by $f(z) = z^\ell$ (complex notation) has $\deg(f) = \ell$. (Verify this.)

COROLLARY 1.2.5. *The degree function is a bijection from $[S^1, S^1]$ to $\mathbb{Z}$.* □

CHAPTER 2

Fiber bundles and fibrations

2.1. Fiber bundles and bundle charts

Definition 2.1.1. Let $p\colon E \to B$ be a continuous map between topological spaces and let $x \in B$. The subspace $p^{-1}(\{x\})$ is sometimes called the *fiber* of p over x.

Definition 2.1.2. Let $p\colon E \to B$ be a continuous map between topological spaces. We say that p is a *fiber bundle* if for every $x \in B$ there exist an open neighborhood U of x in B, a topological space F and a homeomorphism $h\colon p^{-1}(U) \to U \times F$ such that h followed by projection to U agrees with p.

Note that h restricts to a homeomorphism from the fiber of f over x to $\{x\} \times F$. Therefore F must be homeomorphic to the fiber of p over x.

Terminology. Often E is called the *total space* of the fiber bundle and B is called the *base space*. A homeomorphism $h\colon p^{-1}(U) \to U \times F$ as in the definition is called a *bundle chart*. A fiber bundle $p\colon E \to B$ whose fibers are discrete spaces (intuitively, just sets) is also called a *covering space*. (A *discrete space* is a topological space $(X, \mathcal{O})$ in which $\mathcal{O}$ is the entire power set of X.)
Here is an easy way to make a fiber bundle with base space B. Choose a topological space F, put $E = B \times F$ and let $p\colon E \to B$ be the projection to the first factor. Such a fiber bundle is considered unexciting and is therefore called *trivial*. Slightly more generally, a fiber bundle $p\colon E \to B$ is *trivial* if there exist a topological space F and a homeomorphism $h\colon E \to B \times F$ such that h followed by the projection $B \times F \to B$ agrees with p. Equivalently, the bundle is trivial if it admits a bundle chart $h\colon p^{-1}(U) \to U \times F$ where U is all of B. Two fiber bundles $p_0\colon E_0 \to B$ and $p_1\colon E_1 \to B$ with the same base space B are considered *isomorphic* if there exists a homeomorphism $g\colon E_0 \to E_1$ such that $p_1 \circ g = p_0$. In that case g is an *isomorphism* of fiber bundles.

According to the definition above a fiber bundle is a *map*, but the expression is often used informally for a space rather than a map (the total space of the fiber bundle).

Proposition 2.1.3. *Let $p\colon E \to B$ be a fiber bundle where B is a connected space. Let $x_0, y_0 \in B$. Then the fibers of p over x_0 and y_0, respectively, are homeomorphic.*

Proof. For every $x \in B$ choose an open neighborhood U_x of x, a space F_x and a bundle chart $h_x\colon p^{-1}(U_x) \to U_x \times F_x$. The open sets U_x for all $x \in B$ form an open cover of B. We make an equivalence relation R on the set B in the following manner: xRy means that there exist elements

$$x_0, x_1, \ldots, x_k \in B$$

such that $x_0 = x$, $x_k = y$ and $U_{x_{j-1}} \cap U_{x_j} \neq \emptyset$ for $j = 1, \ldots, k$. Clearly xRy implies that F_x is homeomorphic to F_y. Therefore it suffices to show that R has only one equivalence class. Each equivalence class is open, for if $x \in B$ belongs to such an equivalence class,

then U_x is contained in the equivalence class. Each equivalence class is closed, since its complement is open, being the union of the other equivalence classes. Since B is connected, this means that there can only be one equivalence class. $\square$

Example 2.1.4. One example of a fiber bundle is $p\colon \mathbb{R} \to S^1$, where $p(t) = \exp(2\pi i t)$. We saw this in section 1. To show that it is a fiber bundle, select some $z \in S^1$ and some $t \in \mathbb{R}$ such that $p(t) = z$. Let $V =]t-\delta, t+\delta[$ where δ is a positive real number, not greater than $1/2$. Then p restricts to a homeomorphism from $V \subset \mathbb{R}$ to an open neighborhood $U = p(V)$ of z in S^1; let $q\colon U \to V$ be the inverse homeomorphism. Now $p^{-1}(U)$ is the disjoint union of the translates $\ell + V$, where $\ell \in \mathbb{Z}$. This amounts to saying that

$$g\colon U \times \mathbb{Z} \to p^{-1}(U)$$

given by $(y, m) \mapsto m + q(y)$ is a homeomorphism. The inverse h of g is then a bundle chart. Moreover $\mathbb{Z}$ plays the role of a discrete space. Therefore this fiber bundle is a covering space. It is not a trivial fiber bundle because the total space, $\mathbb{R}$, is not homeomorphic to $S^1 \times \mathbb{Z}$.

Example 2.1.5. The Möbius strip leads to another popular example of a fiber bundle. Let $E \subset S^1 \times \mathbb{C}$ consist of all pairs (z, w) where $w^2 = c^2 z$ for some $c \in \mathbb{R}$. This is a (non-compact) implementation of the Möbius strip. There is a projection

$$q\colon E \to S^1$$

given by $q(z, w) = z$. Let us look at the fibers of q. For fixed $z \in S^1$, the fiber of q over z is identified with the space of all $w \in \mathbb{C}$ such that $w^2 = c^2 z$ for some real c. This is equivalent to $w = c\sqrt{z}$ where $\sqrt{z}$ is one of the two roots of z in $\mathbb{C}$. In other words, w belongs to the one-dimensional linear *real* subspace of $\mathbb{C}$ spanned by the two square roots of z. In particular, each fiber of q is homeomorphic to $\mathbb{R}$. The fact that all fibers are homeomorphic to each other should be taken as an indication (though not a proof) that q is a fiber bundle. The full proof is left as an exercise, along with another exercise which is slightly harder: show that this fiber bundle is not trivial.

In preparation for the next example I would like to recall the concept of *one-point compactification*. Let $X = (X, \mathcal{O})$ be a locally compact topological space. (That is to say, X is a Hausdorff space in which every element $x \in X$ has a compact neighborhood.) Let $X^c = (X^c, \mathcal{U})$ be the topological space defined as follows. As a set, X^c is the disjoint union of X and a singleton (set with one element, which in this case we call ∞). The topology $\mathcal{U}$ on X^c is defined as follows. A subset V of X^c belongs to $\mathcal{U}$ if and only if

- either $\infty \notin V$ and $V \in \mathcal{O}$;
- or $\infty \in V$ and $X^c \smallsetminus V$ is a *compact* subset of X.

Then X^c is compact Hausdorff and the inclusion $u\colon X \to X^c$ determines a homeomorphism of X with $u(X) = X^c \smallsetminus \{\infty\}$. The space X^c is called the *one-point compactification* of X. The notation X^c is not standard; instead people often write $X \cup \infty$ and the like. The one-point compactification can be characterized by various good properties; see books on point set topology. For use later on let's note the following, which is clear from the definition of the topology on X^c. Let $Y = (Y, \mathcal{W})$ be any topological space. A map $g\colon Y \to X^c$ is continuous if and only if the following hold:

- $g^{-1}(X)$ is open in Y
- the map from $g^{-1}(X)$ to X obtained by restricting g is continuous

- for every compact subset K of X, the preimage $g^{-1}(K)$ is a closed subset of Y (that is, its complement is an element of $\mathcal{W}$).

EXAMPLE 2.1.6. A famous example of a fiber bundle which is also a crucial example in homotopy theory is the Hopf map from S^3 to S^2, so named after its inventor Heinz Hopf. (Date of invention: around 1930.) Let's begin with the observation that S^2 is homeomorphic to the one-point compactification $\mathbb{C} \cup \infty$ of $\mathbb{C}$. (The standard homeomorphism from S^2 to $\mathbb{C} \cup \infty$ is called *stereographic projection.*) We use this and therefore describe the Hopf map as a map

$$p\colon S^3 \to \mathbb{C} \cup \infty.$$

Also we like to think of S^3 as the unit sphere in $\mathbb{C}^2$. So elements of S^3 are pairs (z, w) where $z, w \in \mathbb{C}$ and $|z|^2 + |w|^2 = 1$. To such a pair we associate

$$p(z, w) = z/w$$

using complex division. This is the Hopf map. Note that in cases where $w = 0$, we must have $z \neq 0$ as $|z|^2 = |z|^2 + |w|^2 = 1$; therefore z/w can be understood and must be understood as $\infty \in \mathbb{C} \cup \infty$ in such cases. In the remaining cases, $z/w \in \mathbb{C}$.
Again, let us look at the fibers of p before we try anything more ambitious. Let $s \in \mathbb{C} \cup \infty$. If $s = \infty$, the preimage of $\{s\}$ under p consists of all $(z, w) \in S^3$ where $w = 0$. This is a circle. If $s \notin \{0, \infty\}$, the preimage of $\{s\}$ under p consists of all $(z, w) \in S^3$ where $w \neq 0$ and $z/w = s$. So this is the intersection of $S^3 \subset \mathbb{C}^2$ with the one-dimensional complex linear subspace $\{(z, w) \mid z = sw\} \subset \mathbb{C}^2$. It is also a circle! Therefore all the fibers of p are homeomorphic to the same thing, S^1. We take this as an indication (though not a proof) that p is a fiber bundle.
Now we *show* that p is a fiber bundle. First let $U = \mathbb{C}$, which we view as an open subset of $\mathbb{C} \cup \infty$. Then

$$p^{-1}(U) = \{(z, w) \in S^3 \subset \mathbb{C}^2 \mid w \neq 0\}.$$

A homeomorphism h from there to $U \times S^1 = \mathbb{C} \times S^1$ is given by

$$(z, w) \mapsto (z/w, w/|w|).$$

This has the properties that we require from a bundle chart: the first coordinate of $h(z, w)$ is $z/w = p(z, w)$. (The formula $g(y, z) = (yz, z)/\|(yz, z)\|$ defines a homeomorphism g inverse to h.) Next we try $V = (\mathbb{C} \cup \infty) \smallsetminus \{0\}$, again an open subset of $\mathbb{C} \cup \infty$. We have the following commutative diagram

$$\begin{tikzcd} S^3 \arrow[r, "\alpha"] \arrow[d, "p"] & S^3 \arrow[d, "p"] \\ \mathbb{C} \cup \infty \arrow[r, "\zeta"] & \mathbb{C} \cup \infty \end{tikzcd}$$

where $\alpha(z, w) = (w, z)$ and $\zeta(s) = s^{-1}$. (This amounts to saying that $p \circ \alpha = \zeta \circ p$.) Therefore the composition

$$p^{-1}(V) \xrightarrow{\alpha} p^{-1}(U) \xrightarrow{h} U \times S^1 \xrightarrow{(s,w) \mapsto (s^{-1}, w)} V \times S^1$$

has the properties required of a bundle chart. Since $U \cup V$ is all of $\mathbb{C} \cup \infty$, we have produced enough charts to know that p is a fiber bundle. $\square$

2.2. Restricting fiber bundles

Let $p\colon E \to B$ be a fiber bundle. Let A be a subset of B. Put $E_{|A} = p^{-1}(A)$. This is a subset of E. We want to regard A as a subspace of B (with the subspace topology) and $E_{|A}$ as a subspace of E.

PROPOSITION 2.2.1. *The map $p_A\colon E_{|A} \to A$ obtained by restricting p is also a fiber bundle.*

PROOF. Let $x \in A$. Choose a bundle chart $h\colon p^{-1}(U) \to U \times F$ for p such that $x \in U$. Let $V = U \cap A$, an open neighborhood of x in A. By restricting h we obtain a bundle chart $h_A\colon p^{-1}(V) \to V \times F$ for p_A. □

Remark. In this proof it is important to remember that a bundle chart as above is not just *any* homeomorphism $h\colon p^{-1}(U) \to U \times F$. There is a condition: for every $y \in p^{-1}(U)$ the U-coordinate of $h(y) \in U \times F$ must be equal to $p(y)$. The following informal point of view is recommended: A bundle chart $h\colon p^{-1}(U) \to U \times F$ for p is just a way to specify, simultaneously and continuously, homeomorphisms h_x from the fibers of p over elements $x \in U$ to F. Explicitly, h determines the h_x and the h_x determine h by means of the equation

$$h(y) = (x, h_x(y)) \in U \times F$$

when $y \in p^{-1}(x)$, that is, $x = p(y)$.

Let $p\colon E \to B$ be any fiber bundle. Then B can be covered by open subsets U_i such that $E_{|U_i}$ is a trivial fiber bundle. This is true by definition: choose the U_i together with bundle charts $h_i\colon p^{-1}(U_i) \to U_i \times F_i$. Rename $p^{-1}(U_i) = E_{|U_i}$ if you must. Then each h_i is a bundle isomorphism of $p_{|U_i}\colon E_{|U_i} \to U_i$ with a trivial fiber bundle $U_i \times F_i \to U_i$. There are cases where we can say more. One such case merits a detailed discussion because it takes us back to the concept of homotopy.

LEMMA 2.2.2. *Let B be any space and let $q\colon E \to B \times [0,1]$ be a fiber bundle. Then B admits a covering by open subsets U_i such that*

$$q_{|U_i \times [0,1]}\colon E_{|U_i \times [0,1]} \longrightarrow U_i \times [0,1]$$

is a trivial fiber bundle.

PROOF. We fix $x_0 \in B$ for this proof. We try to construct an open neighborhood U of $\{x_0\}$ in B such that $q_{|U \times [0,1]}\colon E_{|U \times [0,1]} \longrightarrow U \times [0,1]$ is a trivial fiber bundle. This is enough.

To minimize bureaucracy let us set it up as a proof by *analytic induction*. So let J be the set of all $t \in [0,1]$ for which there exist an open $U' \subset B$ and an open subset U'' of $[0,1]$ which is also an interval containing 0, such that $x_0 \in U'$ and $t \in U''$ and such that $q_{|U' \times U''}$ is a trivial fiber bundle. The following should be clear.

- J is an open subset of $[0,1]$.
- J is nonempty since $0 \in J$.
- If $t \in J$ then $[0,t] \subset J$; hence J is an interval.

If $1 \in J$, then we are happy. So we assume $1 \notin J$ for a contradiction. Then $J = [0,\sigma[$ for some σ where $0 < \sigma \leq 1$. Since q is a fiber bundle, the point (x_0, σ) admits an open neighborhood V in $B \times [0,1]$ with a bundle chart $g\colon q^{-1}(V) \to V \times F_V$. Without loss of generality V has the form $V' \times V''$ where $V' \subset B$ is an open neighborhood of x_0 in B and V'' is an interval which is also an open neighborhood of σ in $[0,1]$. There exists $r < \sigma$ such that $V'' \supset [r,\sigma]$. Then $r \in J$ and so there exists $W = W' \times W''$ open in $B \times [0,1]$

with a bundle chart $h\colon q^{-1}(W) \to U \times F_W$ such that $x_0 \in W'$ and $W'' = [0, \tau[$ where $\tau > r$. Without loss of generality, $W' = V'$. Now $W'' \cup V''$ is an open subset of $[0, 1]$ which is an interval (since $r \in W'' \cap V''$). It contains both 0 and σ. Now let $U' = V'$ and $U'' = W'' \cup V''$. If we can show that $q_{|U' \times U''}$ is a trivial fiber bundle, then the proof is complete because $U' \times U''$ contains $\{x_0\} \times [0, \sigma]$, which implies that $\sigma \in J$, which is the contradiction that we need. Indeed we can make a bundle chart

$$k\colon q^{-1}(U' \times U'') \to (U' \times U'') \times F_W$$

as follows. For $(x, t) \in U' \times U''$ with $t \leq r$ we take $k_{(x,t)} = h_{(x,t)}$. For $(x, t) \in U' \times U''$ with $t \geq r$ we take

$$k_{(x,t)} = h_{(x,r)} \circ g^{-1}_{(x,r)} \circ g_{(x,t)}\,.$$

Decoding: $h_{(x,t)}$ is a homeomorphism from the fiber of q over $(x, t) \in W \subset B \times [0, 1]$ to F_W. Similarly $g_{(x,t)}$ is a homeomorphism from the fiber of q over $(x, t) \in V \subset B \times [0, 1]$ to F_V. Also note that

$$h_{(x,r)} \circ g^{-1}_{(x,r)}$$

is a homeomorphism from F_V to F_W, depending on $x \in V_1 = W_1 \subset B$. □

2.3. Pullbacks of fiber bundles

Let $p\colon E \to B$ be a fiber bundle. Let $g\colon X \to B$ be any continuous map of topological spaces.

Definition 2.3.1. The pullback of $p\colon E \to B$ along g is the space

$$g^*E := \{\, (x, y) \in X \times E \mid g(x) = p(y)\}.$$

It is regarded as a subspace of $X \times E$ with the subspace topology.

Lemma 2.3.2. *The projection $g^*E \to X$ given by $(x, y) \mapsto x$ is a fiber bundle.*

Proof. First of all it is helpful to write down the obvious maps that we have in a commutative diagram:

$$\begin{tikzcd} g^*E \arrow[r, "r"] \arrow[d, "q"] & E \arrow[d, "p"] \\ X \arrow[r, "g"] & B \end{tikzcd}$$

Here q and r are the projections given by $(x, y) \mapsto x$ and $(x, y) \mapsto y$. *Commutative* means that the two compositions taking us from g^*E to B agree. Suppose that we have an open set $V \subset B$ and a bundle chart

$$h\colon p^{-1}(V) \xrightarrow{\cong} V \times F.$$

Now $U := g^{-1}(V)$ is open in X. Also $q^{-1}(U)$ is an open subset of g^*E and we describe elements of that as pairs (x, y) where $x \in U$ and $y \in E$, with $g(x) = p(y)$. We make a homeomorphism

$$q^{-1}(U) \to U \times F$$

by the formula $(x, y) \mapsto (x, h_{g(x)}(y)) = (x, h_{p(y)}(y))$. It is a homeomorphism because the inverse is given by

$$(x, z) \mapsto (x, (h_{g(x)})^{-1}(z))$$

for $x \in U$ and $z \in F$, so that $(g(x), z) \in V \times F$. Its is also clearly a bundle chart. In this way, every bundle chart

$$h\colon p^{-1}(V) \xrightarrow{\cong} V \times F$$

for $p\colon E \to B$ determines a bundle chart

$$q^{-1}(U) \xrightarrow{\cong} U \times F$$

with the same F, where U is the preimage of V under g. Since $p\colon E \to B$ is a fiber bundle, we have many such bundle charts $p^{-1}(V_j) \to V_j \times F_j$ such that the union of the V_j is all of B. Then the union of the corresponding U_j is all of X, and we have bundle charts $q^{-1}(U_j) \to U_j \times F_j$. This proves that q is a fiber bundle. □

This proof was too long and above all too formal. Reasoning in a less formal way, one should start by noticing that the fiber of q over $z \in X$ is essentially the same (and certainly homeomorphic) to the fiber of p over $g(z) \in B$. Namely,

$$q^{-1}(z) = \{(x, y) \in X \times E \mid g(x) = p(y),\ x = z\} \ = \ \{z\} \times p^{-1}(\{g(z)\}) \ .$$

Now recall once again that a bundle chart $h\colon p^{-1}(U) \to U \times F$ for p is just a way to specify, simultaneously and continuously, homeomorphisms h_x from the fibers of p over elements $x \in U$ to F. If we have such a bundle chart for p, then for any $z \in g^{-1}(U)$ we get a homeomorphism from the fiber of q over z, which "is" the fiber of p over $g(z)$, to F. And so, by letting z run through $g^{-1}(U)$, we get a bundle chart for q.

Example 2.3.3. Restriction of fiber bundles is a special case of pullback, up to isomorphism of fiber bundles. More precisely, suppose that $p\colon E \to B$ is a fiber bundle and let $A \subset B$ be a subspace, with inclusion $g\colon A \to B$. Then there is an isomorphism of fiber bundles from $p_A\colon E_{|A} \to A$ to the pullback $g^*E \to A$. This takes $y \in E_{|A}$ to the pair $(p(y), y) \in g^*E \subset A \times E$.

2.4. Homotopy invariance of pullbacks of fiber bundles

Theorem 2.4.1. *Let $p\colon E \to B$ be a fiber bundle. Let $f, g\colon X \to B$ be continuous maps, where X is a compact Hausdorff space. If f is homotopic to g, then the fiber bundles $f^*E \to X$ and $g^*E \to X$ are isomorphic.*

Remark 2.4.2. The compactness assumption on X is unnecessarily strong; *paracompact* is enough. But paracompactness is also a more difficult concept than compactness. Therefore we shall prove the theorem as stated, and leave a discussion of improvements for later.

Remark 2.4.3. Let X be a compact Hausdorff space and let $U_0, U_1, \ldots, U_n$ be open subsets of X such that the union of the U_i is all of X. *Then there exist continuous functions*

$$\varphi_0, \varphi_1, \ldots, \varphi_n\colon X \to [0, 1]$$

such that $\sum_{j=0}^{n} \varphi_j \equiv 1$ *and such that* $\mathrm{supp}(\varphi_j)$*, the support of* φ_j *, is contained in* U_j *for* $j = 0, 1, \ldots, n$. Here $\mathrm{supp}(\varphi_j)$ is the closure in X of the open set

$$\{x \in X \mid \varphi_j(x) > 0\}.$$

A collection of functions $\varphi_0, \varphi_1, \ldots, \varphi_n$ with the stated properties is called a *partition of unity subordinate to the open cover of* X *given by* $U_0, \ldots, U_n$. For readers who are not aware of this existence statement, here is a reduction (by induction) to something which they might be aware of.

First of all, if X is a compact Hausdorff space, then it is a *normal* space. This means, in addition to the Hausdorff property, that any two disjoint closed subsets of X admit disjoint open neighborhoods. Next, for any normal space X we have the *Tietze-Urysohn extension lemma*. This says that if A_0 and A_1 are disjoint closed subsets of X, then there

is a continuous function $\psi\colon X \to [0,1]$ such that $\psi(x) = 1$ for all $x \in A_1$ and $\psi(x) = 0$ for all $x \in A_0$. Now suppose that a normal space X is the union of two open subsets U_0 and U_1. Because X is normal, we can find an open subset $V_0 \subset U_0$ such that the closure of V_0 in X is contained in U_0 and the union of V_0 and U_1 is still X. Repeating this, we can also find an open subset $V_1 \subset U_1$ such that the closure of V_1 in X is contained in U_1 and the union of V_1 and V_0 is still X. Let $A_0 = X \smallsetminus V_0$ and $A_1 = X \smallsetminus V_1$. Then A_0 and A_1 are disjoint closed subsets of X, and so by Tietze-Urysohn there is a continuous function $\psi\colon X \to [0,1]$ such that $\psi(x) = 1$ for all $x \in A_1$ and $\psi(x) = 0$ for all $x \in A_0$. This means that $\operatorname{supp}(\psi)$ is contained in the closure of $X \smallsetminus A_0 = V_0$, which is contained in U_0. We take $\varphi_1 = \psi$ and $\varphi_0 = 1 - \psi$. Since $1 - \psi$ is zero on A_1, its support is contained in the closure of V_1, which is contained in U_1. This establishes the induction beginning (case $n = 1$).

For the induction step, suppose that we have an open cover of X given by $U_0, \dots, U_n$ where $n \geq 2$. By inductive assumption we can find a partition of unity subordinate to the cover $U_0 \cup U_1, U_2, \dots, U_n$ and by the induction beginning, another partition of unity subordinate to $U_0, U_1 \cup U_2 \cup \cdots U_n$. Call the functions in the first partition of unity $\varphi_{01}, \varphi_2, \dots, \varphi_n$ and those in the second ψ_0, ψ_1, we see that the functions $\psi_0\varphi_{01}, \psi_1\varphi_{01}, \varphi_2, \dots, \varphi_n$ form a partition of unity subordinate to the cover by $U_0, \dots, U_n$. $\square$

Proof of theorem 2.4.1. Let $h\colon X \times [0,1] \to B$ be a homotopy from f to g, so that $h_0 = f$ and $h_1 = g$. Then $h^*E \to X \times [0,1]$ is a fiber bundle. We give this a new name, say $q\colon L \to X \times [0,1]$. Let ι_0 and ι_1 be the maps from X to $X \times [0,1]$ given by $\iota_0(x) = (x,0)$ and $\iota_1(x) = (x,1)$. It is not hard to verify that the fiber bundle $f^*E \to X$ is isomorphic to $\iota_0^*L \to X$ and $g^*E \to X$ is isomorphic to $\iota_1^*L \to X$. Therefore all we need to prove is the following.

Let $q\colon L \to X \times [0,1]$ be a fiber bundle, where X is compact Hausdorff. Then the fiber bundles $\iota_0^*L \to X$ and $\iota_1^*L \to X$ obtained from q by pullback along ι_0 and ι_1 are isomorphic. To make this even more explicit: given the fiber bundle $q\colon L \to X \times [0,1]$, we need to produce a homeomorphism from $L_{|X\times\{0\}}$ to $L_{|X\times\{1\}}$ which fits into a commutative diagram

$$\begin{tikzcd}
L_{|X\times\{0\}} \arrow[r, "\text{our homeom.}"] \arrow[d, "\text{res. of } q"'] & L_{|X\times\{1\}} \arrow[d, "\text{res. of } q"] \\
X \times \{0\} \arrow[r, "(x,0)\mapsto(x,1)"] & X \times \{1\}
\end{tikzcd}$$

Here $L_{|K}$ means $q^{-1}(K)$, for any $K \subset X \times [0,1]$.

By a lemma proved last week (lecture notes week 2), we can find a covering of X by open subsets U_i such that that $q_{U_i \times [0,1]}\colon L_{|U_i \times [0,1]} \to U_i \times [0,1]$ is a trivial bundle, for each i. Since X is compact, finitely many of these U_i suffice, and we can assume that their names are $U_1, \dots, U_n$. Let $\varphi_1, \dots, \varphi_n$ be continuous functions from X to $[0,1]$ making up a partition of unity subordinate to the open covering of X by $U_1, \dots, U_n$. For $j = 0, 1, 2, \dots, n$ let $\nu_j = \sum_{k=1}^{j} \varphi_k$ and let $\Gamma_j \subset X \times [0,1]$ be the graph of ν_j. Note that Γ_0 is $X \times \{0\}$ and Γ_n is $X \times \{1\}$. It suffices therefore to produce a homeomorphism

$e_j\colon L_{|\Gamma_{j-1}} \to L_{|\Gamma_j}$ which fits into a commutative diagram

$$\begin{tikzcd} L_{|\Gamma_{j-1}} \arrow[r, "e_j"] \arrow[d, "\text{res. of } q"'] & L_{|\Gamma_j} \arrow[d, "\text{res. of } q"] \\ \Gamma_{j-1} \arrow[r, "(x,\nu_{j-1}(x)) \mapsto (x,\nu_j(x))"] & \Gamma_j \end{tikzcd}$$

(for $j = 1, 2, \ldots, n$). Since $q_{U_j \times [0,1]}\colon L_{|U_j \times [0,1]} \to U_j \times [0,1]$ is a trivial fiber bundle, we have a single bundle chart for it, a homeomorphism

$$g\colon L_{|U_j \times [0,1]} \longrightarrow (U_i \times [0,1]) \times F$$

with the additional good property that we require of bundle charts. Fix j now and write $L = L' \cup L''$ where L' consists of the $y \in L$ for which $q(y) = (x,t)$ with $x \notin \operatorname{supp}(\varphi_j)$, and L'' consists of the $y \in L$ for which $q(y) = (x,t)$ with $x \in U_j$. Both L' and L'' are open subsets of L. Now we make our homeomorphism $e = e_j$ as follows. By inspection, $L_{|\Gamma_{j-1}} \cap L' = L_{|\Gamma_j} \cap L'$, and we take e to be the identity on $L_{|\Gamma_{j-1}} \cap L'$. By restricting the bundle chart g, we have a homeomorphism $L_{|\Gamma_{j-1}} \cap L'' \to U_j \times F$; more precisely, a homeomorphism from $L_{|\Gamma_{j-1}} \cap L''$ to $(\Gamma_{j-1} \cap U_j \times [0,1]) \times F$. By the same reasoning, we have a homeomorphism $L_{|\Gamma_j} \cap L'' \to U_j \times F$; more precisely, a homeomorphism from $L_{|\Gamma_j} \cap L''$ to $(\Gamma_j \cap U_j \times [0,1]) \times F$. Therefore we have a preferred homeomorphism from $L_{|\Gamma_{j-1}} \cap L''$ to $L_{|\Gamma_j} \cap L''$, and we use that as the definition of e on $L_{|\Gamma_{j-1}} \cap L''$. By inspection, the two definitions of e which we have on the overlap $L_{|\Gamma_{j-1}} \cap L' \cap L''$ agree, so e is well defined. □

Corollary 2.4.4. *Let $p\colon E \to B$ be a fiber bundle where B is compact Hausdorff and contractible. Then p is a trivial fiber bundle.*

Proof. By the contractibility assumption, the identity map $f\colon B \to B$ is homotopic to a constant map $g\colon B \to B$. By the theorem, the fiber bundles $f^*E \to B$ and $g^*E \to B$ are isomorphic. But clearly $f^*E \to B$ is isomorphic to the original fiber bundle $p\colon E \to B$. And clearly $g^*E \to B$ is a trivial fiber bundle. □

Corollary 2.4.5. *Let $q\colon E \to B \times [0,1]$ be a fiber bundle, where B is compact Hausdorff. Suppose that the restricted bundle*

$$q_{B\times\{0\}}\colon E_{|B\times\{0\}} \to B \times \{0\}$$

admits a section, i.e., there exists a continuous map $s\colon B \times \{0\} \to E_{|B\times\{0\}}$ such that $q \circ s$ is the identity on $B \times \{0\}$. Then $q\colon E \to B \times [0,1]$ admits a section $\bar{s}\colon B \times [0,1] \to E$ which agrees with s on $B \times \{0\}$.

Proof. Let $f, g\colon B \times [0,1] \to B \times [0,1]$ be defined by $f(x,t) = (x,t)$ and $g(x,t) = (x,0)$. These maps are clearly homotopic. Therefore the fiber bundles $f^*E \to B \times [0,1]$ and $g^*E \to B \times [0,1]$ are isomorphic fiber bundles. Now $f^*E \to B \times [0,1]$ is clearly isomorphic to the original fiber bundle

$$q\colon E \to B \times \{0,1\}$$

and $g^*E \to B \times [0,1]$ is clearly isomorphic to the fiber bundle

$$E_{|B\times\{0\}} \times [0,1] \to B \times [0,1]$$

given by $(y,t) \mapsto (q(y),t)$ for $y \in E_{|B\times\{0\}}$, that is, $y \in E$ with $q(y) = (x,0)$ for some $x \in B$. Therefore we may say that there is a homeomorphism $h\colon E_{|B\times\{0\}} \times [0,1] \to E$

which is over $B \times [0,1]$, in other words, which satisfies

$$(q \circ h)(y,t) = (q(y),t)$$

for all $y \in E_{|B\times\{0\}}$ and $t \in [0,1]$. Without loss of generality, h satisfies the additional condition $h(y,0) = y$ for all $y \in E_{|B\times\{0\}}$. (In any case we have a homeomorphism $u\colon E_{|B\times\{0\}} \to E_{|B\times\{0\}}$ defined by $u(y) = h(y,0)$. If it is not the identity, use the homeomorphism $(y,t) \mapsto h(u^{-1}(y),t)$ instead of $(y,t) \mapsto h(y,t)$.) Now define $\bar{s}$ by $\bar{s}(x,t) = h(s(x),t)$ for $x \in B$ and $t \in [0,1]$. □

2.5. The homotopy lifting property

Definition 2.5.1. A continuous map $p\colon E \to B$ between topological spaces is said to have the *homotopy lifting property* (HLP) if the following holds. Given any space X and continuous maps $f\colon X \to E$ and $h\colon X\times[0,1] \to B$ such that $h(x,0) = p(f(x))$ for all $x \in X$, there exists a continuous map $H\colon X\times[0,1] \to E$ such that $p\circ H = h$ and $H(x,0) = f(x)$ for all $x \in X$. A map with the HLP can be called a *fibration* (sometimes *Hurewicz fibration*).

It is customary to summarize the HLP in a commutative diagram with a dotted arrow:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "x\mapsto(x,0)"'] & E \arrow[d, "p"] \\
X\times[0,1] \arrow[r, "h"] \arrow[ur, dotted, "H"] & B
\end{tikzcd}$$

Indeed, the HLP for the map p means that once we have the data in the outer commutative square, then the dotted arrow labeled H can be found, making both triangles commutative. More associated customs: we think of h as a homotopy between maps h_0 and h_1 from X to B, and we think of $f\colon X \to E$ as a *lift* of the map h_0 , which is just a way of saying that $p \circ f = h_0$.

More generally, or less generally depending on point of view, we say that $p\colon E \to B$ satisfies the HLP for a class of spaces $\mathcal{Q}$ if the dotted arrow in the above diagram can always be supplied when the space X belongs to that class $\mathcal{Q}$.

Proposition 2.5.2. *Let $p\colon E \to B$ be a fiber bundle. Then p has the HLP for compact Hausdorff spaces.*

Proof. Suppose that we have the data X, f and h as in the above diagram, but we are still trying to construct or find the diagonal arrow H. We are assuming that X is compact Hausdorff. The pullback of p along h is a fiber bundle $h^*E \to X\times[0,1]$. The restricted fiber bundle

$$(h^*E)_{|X\times\{0\}} \to X\times\{0\}$$

has a continuous section s given essentially by f, and if we say it very carefully, by the formula

$$(x,0) \mapsto ((x,0),f(x)) \in h^*E \subset (X\times[0,1]\,)\times E\ .$$

The section s extends to a continuous section $\bar{s}$ of $h^*E \to X\times[0,1]$ by corollary 2.4.5. Now we can define $H := r\circ\bar{s}$, where r is the standard projection from h^*E to E. □

EXAMPLE 2.5.3. Let $p\colon S^3 \to S^2$ be the Hopf fiber bundle. Assume if possible that p is nullhomotopic; we shall try to deduce something absurd from that. So let

$$h\colon S^3 \times [0,1] \to S^2$$

be a nullhomotopy for p. Then $h_0 = p$ and h_1 is a constant map. Applying the HLP in the situation

$$\begin{tikzcd} S^3 \arrow[r, "\mathrm{id}"] \arrow[d, "x \mapsto (x,0)"'] & S^3 \arrow[d, "p"] \\ S^3 \times [0,1] \arrow[r, "h"] \arrow[ur, dotted, "H"] & S^2 \end{tikzcd}$$

we deduce the existence of $H\colon S^3 \times [0,1] \to S^3$, a homotopy from the identity map $H_0 = \mathrm{id}\colon S^3 \to S^3$ to a map $H_1\colon S^3 \to S^3$ with the property that $p \circ H_1$ is constant. Since p itself is certainly not constant, this means that H_1 is not surjective. If H_1 is not surjective, it is nullhomotopic. (A non-surjective map from any space to a sphere is nullhomotopic; that's an exercise.) Consequently $\mathrm{id}\colon S^3 \to S^3$ is also nullhomotopic, being homotopic to H_1. This means that S^3 is contractible.
Is that absurd enough? We shall prove later in the course that S^3 is not contractible. Until then, what we have just shown can safely be stated like this: *if S^3 is not contractible, then the Hopf map $p\colon S^3 \to S^2$ is not nullhomotopic.* (I found this argument in Dugundji's book on topology. Hopf used rather different ideas to show that p is not nullhomotopic.)

Let $p\colon E \to B$ be a fibration (for a class of spaces $\mathcal{Q}$) and let $f\colon X \to B$ be any continous map between topological spaces. We define the pullback f^*E by the usual formula,

$$f^*E = \{(x,y) \in X \times E \mid f(x) = p(y)\ \}.$$

LEMMA 2.5.4. *The projection $f^*E \to X$ is also a fibration for the class of spaces $\mathcal{Q}$.*

The proof is an exercise. □

In example 2.5.3, the HLP was used for something resembling a computation with homotopy classes of maps. Let us try to formalize this, as an attempt to get hold of some algebra in homotopy theory. So let $p\colon E \to B$ be a continuous map which has the HLP for a class of topological spaces $\mathcal{Q}$. Let $f\colon X \to B$ be any continuous map of topological spaces. Now we have a commutative square

$$\begin{tikzcd} f^*E \arrow[r, "q_2"] \arrow[d, "q_1"] & E \arrow[d, "p"] \\ X \arrow[r, "f"] & B \end{tikzcd}$$

where q_1 and q_2 are the projections. Take any space W in the class $\mathcal{Q}$. There is then a commutative diagram of sets and maps

$$\begin{tikzcd} {[W, f^*E]} \arrow[r] \arrow[d] & {[W, E]} \arrow[d] \\ {[W, X]} \arrow[r] & {[W, B]} \end{tikzcd}$$

PROPOSITION 2.5.5. *The above diagram of sets of homotopy classes is "half exact" in the following sense: given* $a \in [W, X]$ *and* $b \in [W, E]$ *with the same image in* $[W, B]$*, there exists* $c \in [W, f^*E]$ *which is taken to* a *and* b *by the appropriate maps in the diagram.*

PROOF. Represent a by a map $\alpha\colon W \to X$, and b by some map $\beta\colon W \to E$. By assumption, $f \circ \alpha$ is homotopic to $p \circ \beta$. Let $h = (h_t)_{t\in[0,1]}$ be a homotopy, so that $h_0 = p \circ \beta$ and $h_1 = f \circ \alpha$, and $h_t\colon W \to B$ for $t \in [0, 1]$. By the HLP for p, there exists a homotopy $H\colon W \times [0, 1] \to E$ such that $p \circ H = h$ and $H_0 = \beta$. Then H_1 is homotopic to $H_0 = \beta$, and $p \circ H_1 = f \circ \alpha$. Therefore the formula $w \mapsto (\alpha(w), H_1(w))$ defines a map $W \to f^*E$. The homotopy class c of that is the solution to our problem. □

Looking back, we can say that example 2.5.3 is an application of proposition 2.5.5 with $p\colon E \to B$ equal to the Hopf fibration and f equal to the inclusion of a point (and $\mathcal{Q}$ equal to the class of compact Hausdorff spaces, say). We made some unusual choices: $W = E$ and $b = [\mathrm{id}] \in [W, E]$.

2.6. Remarks on paracompactness and fiber bundles

Quoting from many books on point set topology: a topological space $X = (X, \mathcal{O})$ is *paracompact* if it is Hausdorff and every open cover $(U_i)_{i\in\Lambda}$ of X admits a locally finite refinement $(V_j)_{j\in\Psi}$.

There is a fair amount of open cover terminology in that definition. In this formulation, we take the view that an open cover of X is a *family*, i.e., a map from a set to $\mathcal{O}$ (with a special property). This is slightly different from the equally reasonable view that an open cover of X is a subset of $\mathcal{O}$ (with a special property), and it justifies the use of round brackets as in $(U_i)_{i\in\Lambda}$, as opposed to curly brackets. Here the map in question is from Λ to $\mathcal{O}$. There is an understanding that $(V_j)_{j\in\Psi}$ is also an open cover of X, but Ψ need not coincide with Λ. *Refinement* means that for every $j \in \Psi$ there exists $i \in \Lambda$ such that $V_j \subset U_i$. *Locally finite* means that every $x \in X$ admits an open neighborhood W in X such that the set $\{j \in \Psi \mid W \cap V_j \neq \emptyset\}$ is a finite subset of Ψ.

It is wonderfully easy to get confused about the meaning of paracompactness. There is a strong similarity with the concept of compactness, and it is obvious that *compact* (together with Hausdorff) implies paracompact, but it is worth emphasizing the differences. Namely, where compactness has something to do with open covers and *sub*-covers, the definition of paracompactness uses the notion of *refinement* of one open cover by another open cover. We require that every V_j is *contained* in some U_i ; we do not require that every V_j is *equal* to some U_i. And *locally finite* does not just mean that for every $x \in X$ the set $\{j \in \Psi \mid x \in V_j\}$ is a finite subset of Ψ. It means more.

For some people, the Hausdorff condition is not part of *paracompact*, but for me, it is.

An important theorem: every metrizable space is paracompact. This is due to A.H. Stone who, as a Wikipedia page reminds me, is not identical with Marshall Stone of the Stone-Weierstrass theorem and the Stone-Čech compactification. The proof is not very complicated, but you should look it up in a book on point-set topology which is not too ancient, because it was complicated in the A.H. Stone version.

Another theorem which is very important for us: in a paracompact space X, every open cover $(U_i)_{i\in\Lambda}$ admits a subordinate partition of unity. In other words there exist continuous functions $\varphi_i\colon X \to [0, 1]$, for $i \in \Lambda$, such that

- every $x \in X$ admits an open neighborhood W in X for which the set
$$\{i \in \Lambda \mid W \cap \operatorname{supp}(\varphi_i) \neq \emptyset\}$$
is finite;
- $\sum_{i \in \Lambda} \varphi_i \equiv 1$;
- $\operatorname{supp}(\varphi_i) \subset U_i$.

The second condition is meaningful if we assume that the first condition holds. (Then, for every $x \in X$, there are only finitely many nonzero summands in $\sum_{i \in \Lambda} \varphi_i(x)$. The first condition also ensures that for any subset $\Xi \subset \Lambda$, the sum $\sum_{i \in \Xi} \varphi_i$ is a continuous function on X.)
The proof of this theorem (existence of subordinate partition of unity for any open cover of a paracompact space) is again not very difficult, and boils down mostly to showing that paracompact spaces are *normal*. Namely, in a normal space, locally finite open covers admit subordinate partitions of unity, and this is easy.

Many of the results about fiber bundles in this chapter rely on partitions of unity, and to ensure their existence, we typically assumed compactness here and there. But now it emerges that paracompactness is enough.
Specifically, in theorem 2.4.1 it is enough to assume that X is paracompact. In corollary 2.4.4 it is enough to assume that B is paracompact (and contractible). In corollary 2.4.5 it is enough to assume that B is paracompact. In proposition 2.5.2 we have the stronger conclusion that p has the HLP for paracompact spaces.

Proof of variant of thm. 2.4.1. Here we assume only that X is paracompact (previously we assumed that it was compact). By analogy with the case of compact X, we can easily reduce to the following statement. *Let* $q\colon L \to X \times [0,1]$ *be a fiber bundle, where* X *is paracompact. Then the fiber bundles* $\iota_0^* L \to X$ *and* $\iota_1^* L \to X$ *obtained from* q *by pullback along* ι_0 *and* ι_1 *are isomorphic.* And to make this more explicit: given the fiber bundle $q\colon L \to X \times [0,1]$, we need to produce a homeomorphism h from $L_{|X \times \{0\}}$ to $L_{|X \times \{1\}}$ which fits into a commutative diagram

$$\begin{tikzcd}
L_{|X \times \{0\}} \arrow[r, "h"] \arrow[d, "\text{res. of } q"'] & L_{|X \times \{1\}} \arrow[d, "\text{res. of } q"] \\
X \times \{0\} \arrow[r, "(x,0) \mapsto (x,1)"] & X \times \{1\}
\end{tikzcd}$$

By a lemma proved in lecture notes week 2, we can find an open cover $(U_i)_{i \in \Lambda}$ of X such that that $q_{U_i \times [0,1]}\colon L_{|U_i \times [0,1]} \to U_i \times [0,1]$ is a trivial bundle, for each $i \in \Lambda$. Let $(\varphi_i)_{i \in \Lambda}$ be a partition of unity subordinate to $(U_i)_{i \in \Lambda}$. So $\varphi_i\colon X \to [0,1]$ is a continuous function with $\operatorname{supp}(\varphi_i) \subset U_i$, and $\sum_i \varphi_i \equiv 1$. Every $x \in X$ admits a neighborhood W in X such that the set
$$\{i \in \Lambda \mid \operatorname{supp}(\varphi_i) \cap W \neq \emptyset\}$$
is finite.
Now choose a total ordering on the set Λ. (A total ordering on Λ is a relation $\leq$ on Λ which is transitive and reflexive, and has the additional property that for any distinct $i, j \in \Lambda$, precisely one of $i \leq j$ or $j \leq i$ holds. We need to assume something here to get such an ordering: for example the Axiom of Choice in set theory is equivalent to the

Well-Ordering Principle, which states that every set can be well-ordered. A well-ordering is also a total ordering.) Given $x \in X$, choose an open neighborhood W of x such that the set of $i \in \Lambda$ having $\mathrm{supp}(\varphi_i) \cap W \neq \emptyset$ is finite; say it has n elements. We list these elements in their order (provided by the total ordering on Λ which we selected):

$$i_1 \leq i_2 \leq i_3 \leq \cdots i_n \ .$$

The functions $\varphi_{i_1}, \varphi_{i_2}, \ldots, \varphi_{i_n}$ (restricted to W) make up a partition of unity on W which is subordinate to the covering by open subsets $W \cap U_{i_1}, W \cap U_{i_2}, \ldots W \cap U_{i_n}$. Now we can proceed exactly as in the proof of theorem 2.4.1 to produce (in n steps) a homeomorphism h_W which makes the following diagram commute:

$$\begin{tikzcd}
L_{|W\times\{0\}} \arrow[r, "h_W"] \arrow[d, "\text{res. of } q"'] & L_{|W\times\{1\}} \arrow[d, "\text{res. of } q"] \\
W \times \{0\} \arrow[r, "(x,0)\mapsto(x,1)"] & W \times \{1\}
\end{tikzcd}$$

Finally we can regard W or x as variables. If we choose, for every $x \in X$, an open neighborhood W_x with properties like W above, then the W_x for all $x \in X$ constitute an open cover of X. For each W_x we get a homeomorphism h_{W_x} as above. These homeomorphisms agree with each other wherever this is meaningful, and so define together a homeomorphism $h\colon L_{|X\times\{0\}} \to L_{|X\times\{1\}}$ with the property that we require. $\square$

CHAPTER 3

Presheaves and sheaves on topological spaces

3.1. Presheaves and sheaves

Definition 3.1.1. A *presheaf* on a topological space X is a rule $\mathcal{F}$ which to every open subset U of X assigns a set $\mathcal{F}(U)$, and to every pair of nested open sets $U \subset V \subset X$ a map

$$\mathrm{res}_{V,U} \colon \mathcal{F}(V) \to \mathcal{F}(U)$$

which satisfies the following conditions.

- For open sets $U \subset V \subset W$ in X we have $\mathrm{res}_{V,U} \circ \mathrm{res}_{W,V} = \mathrm{res}_{W,U}$ (an equality of maps from $\mathcal{F}(W)$ to $\mathcal{F}(U)$).
- $\mathrm{res}_{V,V} = \mathrm{id} \colon \mathcal{F}(V) \to \mathcal{F}(V)$ for every open V in X.

Example 3.1.2. An important and obvious example for us is the following. Fix X as above and let Y be another topological space. For open U in X let $\mathcal{F}(U)$ be the *set* of all continuous maps from U to Y. Note that we make no attempt here to define a topology on $\mathcal{F}(U)$; we just take it as a set. For open sets $U \subset V \subset X$ there is an obvious restriction map $\mathcal{F}(V) \to \mathcal{F}(U)$. That is, a continuous map from V to Y determines by restriction a continuous map from U to Y. The conditions for a presheaf are clearly satisfied.

Example 3.1.3. Let $p \colon Y \to X$ be any continuous map. We can use this to make a presheaf $\mathcal{F}$ on X as follows. For an open set U in X, let $\mathcal{F}(U)$ be the set of continuous maps $g \colon U \to Y$ such that $p \circ g = \mathrm{id}_U$. For open sets $U \subset V \subset X$ let $\mathrm{res}_{V,U} \colon \mathcal{F}(V) \to \mathcal{F}(U)$ be given by restriction in the usual sense. Namely, if $f \in \mathcal{F}(V)$, then $f \colon V \to Y$ is a continuous map which satisfies $p \circ f = \mathrm{id}_V$, and so the restriction $f_{|U}$ is a continuous map $U \to Y$ which satisfies $p \circ f_{|U} = \mathrm{id}_U$.

Example 3.1.4. Suppose that X happens to be a differentiable (smooth) manifold (in which case it is also a topological space). For open U in X, let $\mathcal{F}(U)$ be the set of smooth functions from U to $\mathbb{R}$. For open subsets $U \subset V \subset X$, let $\mathrm{res}_{V,U} \colon \mathcal{F}(V) \to \mathcal{F}(U)$ be given by restriction in the usual sense. The conditions for a presheaf are clearly satisfied by $\mathcal{F}$.

Example 3.1.5. Given a topological space X and a set S, define $\mathcal{F}(U) = S$ for every open U in X. For open sets $U \subset V \subset X$, let $\mathrm{res}_{V,U} \colon \mathcal{F}(V) \to \mathcal{F}(U)$ be the identity map of S. The conditions for a presheaf are clearly satisfied.

Example 3.1.6. Fix X as above and let Y be another topological space. For open U in X put $\mathcal{F}(U) = [U, Y]$, the set of homotopy classes of continuous maps from U to Y. For open sets $U \subset V \subset X$ there is an obvious restriction map $\mathcal{F}(V) \to \mathcal{F}(U)$. That is, a homotopy class of continuous maps from V to Y determines by restriction a homotopy class of continuous maps from U to Y. The conditions for a presheaf are clearly satisfied. This example looks as if it might become very important in this course, since it connects presheaves and the concept of homotopy. But it will not become very important except as a source of homework problems and counterexamples.

EXAMPLE 3.1.7. Fix X as above and let Y be another topological space. For an open subset U of X let $\mathcal{F}(U)$ be the set of *formal linear combinations* (with integer coefficients) of continuous maps from U to Y. So an element of $\mathcal{F}(U)$ might look like $5f - 3g + 9h$ where f, g and h are continuous maps from U to Y. We do not insist that f, g, h in this expression are distinct, but if for example f and g are equal, then we take the view that $5f - 3g + 9h$ and $2f + 9h$ define the same element of $\mathcal{F}(U)$. This remark is important when we define the restriction map

$$\mathrm{res}_{V,U} \colon \mathcal{F}(V) \to \mathcal{F}(U)$$

This is of course determined by restriction of continuous maps. So for example, if

$$3a - 6b + 10c - d$$

is an element of $\mathcal{F}(V)$, and here we may as well assume that the continuous maps $a, b, c, d \colon V \to Y$ are distinct (because we can simplify the expression if not), then $\mathrm{res}_{V,U}$ takes that element to $3(a_{|U}) - 6(b_{|U}) + 10(c_{|U}) - d_{|U} \in \mathcal{F}(U)$. And here we can not assume that the continuous maps $a_{|U}, b_{|U}, c_{|U}, d_{|U} \colon U \to Y$ are all distinct. In any case the conditions for a presheaf are clearly satisfied.
This example looks silly and unimportant, but it is not silly and it will become very important in this course. Let's also note that there are more grown-up ways to describe $\mathcal{F}(U)$ for this presheaf $\mathcal{F}$. Instead of saying *the set of formal linear combinations with integer coefficients of continuous maps from* U *to* Y, we can say: the free abelian group generated by the set of continuous maps from U to Y. Or we can say: the free $\mathbb{Z}$-module generated by the set of continuous maps from U to Y. (See also section 3.4 for some clarifications.)

With a view to the next definition we introduce some practical notation. Let X be a space, let $\mathcal{F}$ be a presheaf on X, and suppose that U, V are open subsets of X such that $U \subset V$. Then we have the restriction map $\mathrm{res}_{V,U} \colon \mathcal{F}(V) \to \mathcal{F}(U)$. Let $s \in \mathcal{F}(V)$. Instead of writing $\mathrm{res}_{V,U}(s) \in \mathcal{F}(U)$, we sometimes write $s_{|U} \in \mathcal{F}(U)$.

DEFINITION 3.1.8. A presheaf $\mathcal{F}$ on a topological space X is called a *sheaf* on X if it has the following additional properties. For every collection of open subsets $(W_i)_{i \in \Lambda}$ of X, and every collection

$$(s_i \in \mathcal{F}(W_i))_{i \in \Lambda}$$

with the property $s_{i|W_i \cap W_j} = s_{j|W_i \cap W_j} \in \mathcal{F}(W_i \cap W_j)$, there exists a unique

$$s \in \mathcal{F}(\bigcup_{i \in \Lambda} W_i)$$

such that $s_{|W_i} = s_i$ for all $i \in \Lambda$. In particular, $\mathcal{F}(\emptyset)$ has exactly one element.

In a slightly more wordy formulation: if we have elements $s_i \in \mathcal{F}(W_i)$ for all $i \in \Lambda$, and we have agreement of s_i and s_j on $W_i \cap W_j$ for all $i, j \in \Lambda$, then there is a unique $s \in \mathcal{F}(\bigcup_i W_i)$ which agrees with s_i on each W_i.
To silence a particularly nagging and persistent type of critic, including the critic within myself, let me explain in detail why this implies that $\mathcal{F}(\emptyset)$ has exactly one element. Put $\Lambda = \emptyset$. For each $i \in \Lambda$, select an open subset W_i. (Easy, because there is no $i \in \Lambda$.) For each $i \in \Lambda$, select an element $s_i \in \mathcal{F}(W_i)$. (Easy.) Verify that, for each i and j in Λ, we have $s_{i|U_i \cap U_j} = s_{j|U_i \cap U_j}$. (Easy.) Conclude that there exists a *unique*

$$s \in \mathcal{F}(\bigcup_{i \in \Lambda} W_i)$$

such that $s_{|W_i} = s_i$ for every $i \in \Lambda$. Now note that $\bigcup_{i\in\Lambda} W_i = \emptyset$ and verify that *every* $t \in \mathcal{F}(\emptyset)$ satisfies the condition $t_{|W_i} = s_i$ for every $i \in \Lambda$. (Easy.) Therefore *every* element t of $\mathcal{F}(\emptyset)$ must be equal to that distinguished element s which we have already spotted.

Obviously it is now our duty to scan the list of the examples above and decide for each of these presheaves $\mathcal{F}$ whether it is a sheaf. It is a good idea to ask first in each case whether $\mathcal{F}(\emptyset)$ has exactly one element. If that is not the case, then it is not a sheaf. It looks like a mean reason to refuse sheaf status to a presheaf. But often when $\mathcal{F}(\emptyset)$ does not have exactly one element, the presheaf $\mathcal{F}$ turns out to have other properties which prevent us from promoting it to sheaf status by simply redefining $\mathcal{F}(\emptyset)$. — The following lemma is also a good tool in testing for the sheaf property.

LEMMA 3.1.9. *Let $\mathcal{F}$ be a sheaf on X and let $(W_i)_{i\in\Lambda}$ be a collection of pairwise disjoint open subsets of X. Then the formula $s \mapsto (s_{|W_i})_{i\in\Lambda}$ determines a bijection*

$$\mathcal{F}(\bigcup_{i\in\Lambda} W_i) \longrightarrow \prod_{i\in\Lambda} \mathcal{F}(W_i).$$

PROOF. Take an element in $\prod_{i\in\Lambda} \mathcal{F}(W_i)$ and denote it by $(s_i)_{i\in\Lambda}$, so that s_i is an element of $\mathcal{F}(W_i)$. Since $W_i \cap W_j = \emptyset$ and $\mathcal{F}(\emptyset)$ has exactly one element, the matching condition

$$s_{i|W_i\cap W_j} = s_{j|W_i\cap W_j}$$

is vacuously satisfied for all $i, j \in \Lambda$. Hence by the sheaf property, there is a unique element s in $\mathcal{F}(\bigcup_{i\in\Lambda} W_i)$ such that $s_{|W_i} = s_i$ for all $i \in \Lambda$. This means precisely that $s \mapsto (s_{|W_i})_{i\in\Lambda}$ is a bijection. (The surjectivity is expressed in *there is* and the injectivity in the word *unique*.) □

Discussion of example 3.1.2. This is a sheaf. What is being said here is that if we have open $W_i \subset X$ for each $i \in \Lambda$, and continuous maps $f_i\colon W_i \to Y$ for each i such that f_i and f_j agree on $W_i \cap W_j$ for all $i, j \in \Lambda$, then we have a unique continuous map f from $\bigcup W_i$ to Y which agrees with f_i on W_i for each $i \in \Lambda$.

Discussion of example 3.1.3. This is a sheaf. We can reason as in the case of example 3.1.2.

Discussion of example 3.1.4. This is a sheaf. What is being said here is that if X is a smooth manifold, and we have open $W_i \subset X$ for each $i \in \Lambda$, and smooth functions $f_i\colon W_i \to \mathbb{R}$ for each i such that f_i and f_j agree on $W_i \cap W_j$ for all $i, j \in \Lambda$, then we have a unique smooth $f\colon \bigcup W_i \to Y$ which agrees with f_i on W_i for each $i \in \Lambda$. An interesting aspect of this example is that, in contrast to examples 3.1.2 and 3.1.3, it seems to express something which is not part of the world of topological spaces, something "differentiable". So I am suggesting that the notion of *smooth manifold* could be redefined along the following lines: a smooth manifold is a topological Hausdorff space X together with a sheaf $\mathcal{F}$ ... which we would call the sheaf of *smooth* functions (on open subsets of X) and which would presumably have to be a subsheaf (notion yet to be defined) of the sheaf of *continuous* functions on open subsets of X. That would be an alternative to defining smooth manifolds using charts and atlases. Of course this has been noticed and has been done by the ancients, but I am digressing.

Discussion of example 3.1.5. Here we have to make a case distinction. If S has exactly one element, then this presheaf $\mathcal{F}$ is a sheaf, and the verification is easy. If S has more than one element, or is empty, then $\mathcal{F}$ is not a sheaf because $\mathcal{F}(\emptyset)$ does not have exactly one element.

Can we fix this by redefining $\mathcal{F}(\emptyset)$ to have exactly one element? Let us try. So let $\mathcal{G}$ be the presheaf on X defined by $\mathcal{G}(U) = S$ when U is nonempty, and $\mathcal{G}(\emptyset) = \{*\}$, a set with a single element $*$. It is a presheaf as follows: for open subsets $U \subset V$ of X we let $\mathrm{res}_{V,U} \colon \mathcal{G}(V) \to \mathcal{G}(U)$ be the identity map of S if $U \neq \emptyset$; otherwise it is the unique map of sets from $\mathcal{G}(V)$ to $\{*\}$.
Is this presheaf $\mathcal{G}$ a sheaf? The answer depends a little on X, and on S. Suppose that X has disjoint open nonempty subsets U_1 and U_2. By lemma 3.1.9, the diagonal map from $S = \mathcal{G}(U_1 \cup U_2)$ to $S \times S = \mathcal{G}(U_1) \times \mathcal{G}(U_2)$ is bijective. We have a problem with that if S has more than one element. The case where S has exactly one element was excluded, so only the possibility $S = \emptyset$ remains. And indeed, if S is empty, we don't have a problem: $\mathcal{G}$ is a sheaf. Also, if X does not have any disjoint nonempty open subsets U_1 and U_2, we don't have a problem: $\mathcal{G}$ is a sheaf, no matter what S is.

Discussion of example 3.1.6. In general, this is not a sheaf, although it responds nicely to the two standard tests. (One standard test is to ask: what is $\mathcal{F}(\emptyset)$? Here we get the set of homotopy classes of maps from $\emptyset$ to Y, and that set has exactly one element, as it should have if $\mathcal{F}$ were a sheaf. The other standard test comes from lemma 3.1.9. If $(W_i)_{i\in\Lambda}$ is a collection of disjoint open subsets of X, then

$$\mathcal{F}(\textstyle\bigcup_i W_i) = [\bigcup_i W_i, Y]$$

which is in bijection with $\prod_{i\in\Lambda}[W_i, Y]$ by composition with the inclusions $W_j \to \bigcup_{i\in\Lambda} W_i$ for each $j \in \Lambda$.) For a counterexample, let $X = Y = S^1$. In X we have the open sets U_1 and U_2 where $U_1 = S^1 - \{1\}$ and $U_2 = S^1 \smallsetminus \{-1\}$, using complex number notation. Since U_1 and U_2 are contractible and Y is path connected, both $\mathcal{F}(U_1)$ and $\mathcal{F}(U_2)$ have exactly one element. Since $U_1 \cap U_2$ is the disjoint union of two contractible open sets V_1 and V_2, we get

$$\mathcal{F}(U_1 \cap U_2) = \mathcal{F}(V_1 \cup V_2)$$

which is in bijection with $\mathcal{F}(V_1) \times \mathcal{F}(V_2)$, which again has exactly one element. If $\mathcal{F}$ were a sheaf, it would follow from these little calculations that $\mathcal{F}(U_1 \cup U_2)$ has exactly one element. But $\mathcal{F}(U_1 \cup U_2) = \mathcal{F}(X) = [X, Y] = [S^1, S^1]$, and we know that this has infinitely many elements.

Discussion of example 3.1.7. This is obviously not a sheaf because $\mathcal{F}(\emptyset)$ has more than one element. Indeed, there is exactly one continuous map from $\emptyset$ to Y. So $\mathcal{F}(\emptyset)$ is the free $\mathbb{Z}$-module one one generator, which means that it is isomorphic to $\mathbb{Z}$.
It might seem pointless to look for further reasons to deny sheaf status to $\mathcal{F}$. It is like kicking somebody who is already down. Nevertheless, because this is an important example, it will be instructive for us to know more about it, and we could argue that by showing interest we are showing some patience and kindness. Also, there is a new aspect here: the sets $\mathcal{F}(U)$ always always carry the structure of abelian groups alias $\mathbb{Z}$-modules, and the maps $\mathrm{res}_{V,U}$ are always homomorphisms.
Suppose that $X = \{1, 2, 3, 4, 5, 6\}$ with the discrete topology (every subset of X is declared to be open). Let $Y = \{a, b\}$, a set with two elements, also with the discrete topology. We note that X is the disjoint union of six open subsets U_i , where $i = 1, 2, 3, 4, 5, 6$ and $U_i = \{i\}$. We have $\mathcal{F}(U_i) = \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^2$ (free $\mathbb{Z}$-module on two generators) because each U_i has exactly two continuous maps to Y. We have $\mathcal{F}(\bigcup_i U_i) = \mathcal{F}(X) = \mathbb{Z}^{64}$ (free $\mathbb{Z}$-module on 64 generators) because there are 64 continuous maps from X to Y. It follows that the map

$$\mathcal{F}(\textstyle\bigcup_i U_i) \longrightarrow \prod_{i=1}^{6} \mathcal{F}(U_i)$$

of lemma 3.1.9 (which in the present circumstances is a $\mathbb{Z}$-module homomorphism) cannot be bijective, because that would make it a $\mathbb{Z}$-module isomorphism between $\mathbb{Z}^{64}$ and $\mathbb{Z}^{12}$. (For an abstract interpretation of what is happening, the notion of *tensor product* is useful. Namely, $\mathcal{F}(\bigcup_i U_i) \cong \mathbb{Z}^{64}$ is isomorphic to the tensor product

$$\mathcal{F}(U_1) \otimes \mathcal{F}(U_2) \otimes \cdots \otimes \mathcal{F}(U_6).$$

It is unsurprising that this is *not* isomorphic to the product $\prod_{i=1}^{6} \mathcal{F}(U_i)$. So it emerges that $\mathcal{F}$ fails to have the sheaf property because it has another respectable property.)
Next, re-define X and Y in such a way that X and Y are two topological spaces related by a covering map $p\colon Y \to X$ with finite fibers. In other words, p is a fiber bundle whose fibers are finite sets (viewed as topological spaces with the discrete topology). For simplicity, suppose also that X is connected. Choose an open covering $(W_j)_{j\in\Lambda}$ of X such that p admits a bundle chart over W_j for each j:

$$h_j\colon p^{-1}(W_j) \to W_j \times F$$

where F is a finite set (with the discrete topology). There is no loss of generality in asking for the same F in all cases, independent of j, because X is connected (see proposition 2.1.3). For $j \in \Lambda$ and $z \in F$ there is a continuous map $\sigma_{j,z}\colon W_j \to Y$ given by $\sigma_{j,z}(x) = h_j^{-1}(x, z)$ for $x \in W_j$. Define

$$s_j = \sum_{z\in F} \sigma_{j,z}\,.$$

This is a formal linear combination of continuous maps from W_j to Y which has meaning as an element $\mathcal{F}(W_j)$. So we can write $s_j \in \mathcal{F}(W_j)$. The matching condition

$$s_{i|W_i\cap W_j} = s_{j|W_i\cap W_j}$$

is satisfied. However it seems to be hard or impossible to produce $s \in \mathcal{F}(X) = \mathcal{F}(\bigcup_j W_j)$ such that $s_{|W_i} = s_i$ for all $i \in \Lambda$. This indicates another violation of the sheaf property. (Unfortunately, showing that in many cases such an s does not exist is also hard; we may return to this when we are wiser.)

3.2. Categories, functors and natural transformations

The concept of a *category* and the related notions *functor* and *natural transformation* emerged in the middle of the 20th century (Eilenberg-MacLane, 1945) and were immediately used to re-organize algebraic topology (Eilenberg-Steenrod, 1952). Later these notions became very important in many other branches of mathematics, especially algebraic geometry. Category theory has many definitions of great depth, I think, but in the foundations very few theorems and fewer proofs of any depth. Among those who love difficult proofs, it has a reputation of shallowness, boring-ness; for many of the theorizers who appreciate good definitions, it is an ever-ongoing revelation. Young mathematicians tend to like it better than old mathematicians ... probably because it helps them to see some order in a multitude of mathematical facts.

Definition 3.2.1. A *category* $\mathcal{C}$ consists of a class $\mathrm{Ob}(\mathcal{C})$ whose elements are called the *objects of* $\mathcal{C}$ and the following additional data.

- For any two objects c and d of $\mathcal{C}$, a set $\mathrm{mor}_{\mathcal{C}}(c, d)$ whose elements are called the *morphisms* from c to d.
- For any object c in $\mathcal{C}$, a distinguished element $\mathrm{id}_c \in \mathrm{mor}_{\mathcal{C}}(c, c)$, called the *identity morphism* of c.

- For any three objects b, c, d of $\mathcal{C}$, a map from $\mathrm{mor}_{\mathcal{C}}(c, d) \times \mathrm{mor}_{\mathcal{C}}(b, c)$ to $\mathrm{mor}_{\mathcal{C}}(b, d)$ called *composition* and denoted by $(f, g) \mapsto f \circ g$.

These data are subject to certain conditions, namely:

- Composition of morphisms is associative.
- The identity morphisms act as two-sided neutral elements for the composition.

The associativity condition, written out in detail, means that

$$(f \circ g) \circ h = f \circ (g \circ h)$$

whenever a, b, c, d are objects of $\mathcal{C}$ and $f \in \mathrm{mor}_{\mathcal{C}}(c, d)$, $g \in \mathrm{mor}_{\mathcal{C}}(b, c)$, $h \in \mathrm{mor}_{\mathcal{C}}(a, b)$. The condition on identity morphisms means that $f \circ \mathrm{id}_c = f = \mathrm{id}_d \circ f$ whenever c and d are objects in $\mathcal{C}$ and $f \in \mathrm{mor}_{\mathcal{C}}(c, d)$. Saying that $\mathrm{Ob}(\mathcal{C})$ is a *class*, rather than a *set*, is a subterfuge to avoid problems which are likely to arise if, for example, we talk about *the set of all sets* (Russell's paradox). If the object class is a set, which sometimes happens, we speak of a *small category*.

Notation: we shall often write $\mathrm{mor}(c, d)$ instead of $\mathrm{mor}_{\mathcal{C}}(c, d)$ if it is obvious that the category in question is $\mathcal{C}$. Morphisms are often denoted by arrows, as in $f\colon c \to d$ when $f \in \mathrm{mor}(c, d)$. It is customary to say in such a case that c is the *source* or *domain* of f, and d is the *target* or *codomain* of f.

A morphism $f\colon c \to d$ in a category $\mathcal{C}$ is said to be an *isomorphism* if there exists a morphism $g\colon d \to c$ in $\mathcal{C}$ such that $g \circ f = \mathrm{id}_c \in \mathrm{mor}_{\mathcal{C}}(c, c)$ and $f \circ g = \mathrm{id}_d \in \mathrm{mor}_{\mathcal{C}}(d, d)$.

Example 3.2.2. The prototype is $\mathcal{S}\mathrm{ets}$, the category of sets. The objects of that are the sets. For two sets S and T, the set of morphisms $\mathrm{mor}(S, T)$ is the set of all maps from S to T. Composition is composition of maps as we know it and the identity morphisms are the identity maps as we know them.

Another very important example for us is $\mathcal{T}\mathrm{op}$, the category of topological spaces. The objects are the topological spaces. For topological spaces $X = (X, \mathcal{O}_X)$ and $Y = (Y, \mathcal{O}_Y)$, the set of morphisms $\mathrm{mor}(X, Y)$ is the set of continuous maps from X to Y. Composition is composition of continuous maps as we know it and the identity morphisms are the identity maps as we know them.

Another very important example for us is $\mathcal{H}\mathrm{o}\mathcal{T}\mathrm{op}$, the homotopy category of topological spaces. The objects are the topological spaces, as in $\mathcal{T}\mathrm{op}$. But the set of morphisms from $X = (X, \mathcal{O}_X)$ to $Y = (Y, \mathcal{O}_Y)$ is $[X, Y]$, the set of *homotopy classes* of continuous maps from X to Y. Composition $\circ$ is defined by the formula

$$[f] \circ [g] = [f \circ g]$$

for $[f] \in [Y, Z]$ and $[g] \in [X, Y]$. Here $f\colon Y \to Z$ and $g\colon X \to Y$ are continuous maps representing certain elements of $[Y, Z]$ and $[X, Y]$, and $f \circ g\colon X \to Z$ is their composition. There is an issue of well-defined-ness here, but fortunately we settled this long ago in chapter 1. As a result, associativity of composition is not in doubt because it is a consequence of associativity of composition in $\mathcal{T}\mathrm{op}$. The identity morphisms in $\mathcal{H}\mathrm{o}\mathcal{T}\mathrm{op}$ are the homotopy classes of the identity maps.

Another popular example is $\mathcal{G}\mathrm{roups}$, the category of groups. The objects are the groups. For groups G and H, the set of morphisms $\mathrm{mor}(G, H)$ is the set of group homomorphisms from G to H. Composition of morphisms is composition of group homomorphisms.

The definition of a category as above permits some examples which are rather strange. One type of strange example which will be important for us soon is as follows. Let $(P, \leq)$ be a partially ordered set, alias poset. That is to say, P is a set and $\leq$ is a relation on P

which is transitive ($x \leq y$ and $y \leq z$ forces $x \leq z$), reflexive ($x \leq x$ holds for all x) and antisymmetric (in the sense that $x \leq y$ and $y \leq x$ together implies $x = y$). We turn this setup into a small category (nameless) such that the object set is P. We decree that, for $x, y \in P$, the set $\mathrm{mor}(x, y)$ shall be empty if x is not $\leq y$, and shall contain exactly one element, denoted $*$, if $x \leq y$. Composition

$$\circ : \mathrm{mor}(y, z) \times \mathrm{mor}(x, y) \longrightarrow \mathrm{mor}(x, z)$$

is defined as follows. If y is not $\leq z$, then $\mathrm{mor}(y, z)$ is empty and so $\mathrm{mor}(y, z) \times \mathrm{mor}(x, y)$ is empty, too. There is exactly one map from the empty set to $\mathrm{mor}(x, z)$ and we take that. If x is not $\leq y$, then $\mathrm{mor}(y, z) \times \mathrm{mor}(x, y)$ is empty, and we have only one choice for our composition map, and we take that. The remaining case is the one where $x \leq y$ and $y \leq z$. Then $x \leq z$ by transitivity. Therefore $\mathrm{mor}(y, z) \times \mathrm{mor}(x, y)$ has exactly one element, but more importantly, $\mathrm{mor}(x, z)$ has also exactly one element. Therefore there is exactly one map from $\mathrm{mor}(y, z) \times \mathrm{mor}(x, y)$ to $\mathrm{mor}(x, z)$ and we take that.
Another type of strange example (less important for us but still instructive) can be constructed by starting with a specific group G, with multiplication map $\mu\colon G \times G \to G$. From that we construct a small category (nameless) whose object set has exactly one element, denoted $*$. We let $\mathrm{mor}(*, *) = G$. The composition map

$$\mathrm{mor}(*, *) \times \mathrm{mor}(*, *) \to \mathrm{mor}(*, *)$$

now has to be a map from $G \times G$ to G, and for that we choose μ, the multiplication of G. Since μ has an associativity property, composition of morphisms is associative. For the identity morphism $\mathrm{id}_* \in \mathrm{mor}(*, *)$ we take the neutral element of G.
There are also some easy ways to make new categories out of old ones. One important example: let $\mathcal{C}$ be any category. We make a new category $\mathcal{C}^{\mathrm{op}}$, the *opposite* category of $\mathcal{C}$. It has the same objects as $\mathcal{C}$, but we let

$$\mathrm{mor}_{\mathcal{C}^{\mathrm{op}}}(c, d) := \mathrm{mor}_{\mathcal{C}}(d, c)$$

when c and d are objects of $\mathcal{C}$, or equivalently, objects of $\mathcal{C}^{\mathrm{op}}$. The identity morphism of an object c in $\mathcal{C}^{\mathrm{op}}$ is the identity morphism of c in $\mathcal{C}$. Composition

$$\mathrm{mor}_{\mathcal{C}^{\mathrm{op}}}(c, d) \times \mathrm{mor}_{\mathcal{C}^{\mathrm{op}}}(b, c) \longrightarrow \mathrm{mor}_{\mathcal{C}^{\mathrm{op}}}(b, d)$$

is defined by noting $\mathrm{mor}_{\mathcal{C}^{\mathrm{op}}}(c, d) \times \mathrm{mor}_{\mathcal{C}^{\mathrm{op}}}(b, c) = \mathrm{mor}_{\mathcal{C}}(d, c) \times \mathrm{mor}_{\mathcal{C}}(c, b)$ and going from there to $\mathrm{mor}_{\mathcal{C}}(c, b) \times \mathrm{mor}_{\mathcal{C}}(d, c)$ by an obvious bijection, and from there to $\mathrm{mor}_{\mathcal{C}}(d, b) = \mathrm{mor}_{\mathcal{C}^{\mathrm{op}}}(b, d)$ using composition of morphisms in the category $\mathcal{C}$.

It turns out that there is something like a *category of all categories*. Let us not try to make that very precise because there are some small difficulties and complications in that. In any case there is a concept of morphism between categories, and the name of that is *functor*.

Definition 3.2.3. A *functor* from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a rule F which to every object c of $\mathcal{C}$ assigns an object $F(c)$ of $\mathcal{D}$, and to every morphism $g\colon b \to c$ in $\mathcal{C}$ a morphism $F(g)\colon F(b) \to F(c)$ in $\mathcal{D}$, subject to the following conditions.

- For any object c in $\mathcal{C}$ with identity morphism id_c, we have $F(\mathrm{id}_c) = \mathrm{id}_{F(c)}$.
- Whenever a, b, c are objects in $\mathcal{C}$ and $h \in \mathrm{mor}_{\mathcal{C}}(a, b)$, $g \in \mathrm{mor}_{\mathcal{C}}(b, c)$, we have $F(g \circ h) = F(g) \circ F(h)$ in $\mathrm{mor}_{\mathcal{D}}(F(a), F(c))$.

EXAMPLE 3.2.4. A functor F from the category $\mathcal{T}op$ to the category $\mathcal{S}ets$ can be defined as follows. For a topological space X let $F(X)$ be the set of path components of X. A continuous map $g\colon X \to Y$ determines a map $F(g)\colon F(X) \to F(Y)$ like this: $F(g)$ applied to a path component C of X is the unique path component of Y which contains $g(C)$.
Fix a positive integer n. Let $\mathcal{R}ings$ be the category of rings and ring homomorphisms. (For me, a ring does not have to be commutative, but it should have distinguished elements 0 and 1 and in this example I require $0 \neq 1$.) A functor F from $\mathcal{R}ings$ to $\mathcal{G}roups$ can be defined by $F(R) = \mathrm{GL}_n(R)$, where $\mathrm{GL}_n(R)$ is the group of invertible $n \times n$ matrices with entries in R. A ring homomorphism $g\colon R_1 \to R_2$ determines a group homomorphism $F(g)$ from $F(R_1)$ to $F(R_2)$. Namely, in an invertible $n \times n$-matrix with entries in R_1 , apply g to each entry to obtain an invertible $n \times n$-matrix with entries in R_2.
Let G be a group which comes with an action on a set S. In example 3.2.2 we constructed from G a category with one object $*$ and $\mathrm{mor}(*,*) = G$. A functor F from that category to $\mathcal{S}ets$ can now be defined by $F(*) = S$, and $F(g) =$ translation by g, for $g \in \mathrm{mor}(*,*) = G$. More precisely, to $g \in G = \mathrm{mor}(*,*)$ we associate the map $F(g)$ from $S = F(*)$ to $S = F(*)$ given by $x \mapsto g \cdot x$ (which has a meaning because we are assuming an action of G on S).
Let $\mathcal{C}$ be any category and let x be any object of $\mathcal{C}$. A functor F_x from $\mathcal{C}$ to $\mathcal{S}ets$ can be defined as follows. Let $F_x(c) = \mathrm{mor}_{\mathcal{C}}(x, c)$. For a morphism $g\colon c \to d$ in $\mathcal{C}$ define $F_x(g)\colon F_x(c) \to F_x(d)$ by $F_x(g)(h) = g \circ h$. In more detail, we are assuming $h \in F_x(c) = \mathrm{mor}_{\mathcal{C}}(x, c)$ and $g \in \mathrm{mor}_{\mathcal{C}}(c, d)$, so that $g \circ h \in \mathrm{mor}_{\mathcal{C}}(x, d) = F_x(d)$.

The functors of definition 3.2.3 are also called *covariant functors* for more precision. There is a related concept of *contravariant functor*. A contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is simply a (covariant) functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$ (see example 3.2.2). If we write this out, it looks like this. A contravariant functor F from $\mathcal{C}$ to $\mathcal{D}$ is a rule which to every object c of $\mathcal{C}$ assigns an object $F(c)$ of $\mathcal{D}$, and to every morphism $g\colon c \to d$ in $\mathcal{C}$ a morphism $F(g)\colon F(d) \to F(c)$; note that the source of $F(g)$ is $F(d)$, and the target is $F(c)$. And so on.

EXAMPLE 3.2.5. Let $\mathcal{C}$ be any category and let x be any object of $\mathcal{C}$. A contravariant functor F^x from $\mathcal{C}$ to $\mathcal{S}ets$ can be defined as follows. Let $F^x(c) = \mathrm{mor}_{\mathcal{C}}(c, x)$. For a morphism $g\colon c \to d$ in $\mathcal{C}$ define

$$F^x(g)\colon F^x(d) \to F^x(c)$$

by $F^x(g)(h) = h \circ g$. In more detail, we are assuming $h \in F^x(d) = \mathrm{mor}_{\mathcal{C}}(d, x)$ and $g \in \mathrm{mor}_{\mathcal{C}}(c, d)$, so that $h \circ g \in \mathrm{mor}_{\mathcal{C}}(c, x) = F^x(c)$.
There is a contravariant functor P from $\mathcal{S}ets$ to $\mathcal{S}ets$ given by $P(S) =$ power set of S, for a set S. In more detail, a morphism $g\colon S \to T$ in $\mathcal{S}ets$ determines a map $P(g)\colon P(T) \to P(S)$ by "preimage". That is, $P(g)$ applied to a subset U of T is $g^{-1}(U)$, a subset of S. (You may have noticed that this example of a contravariant functor is not very different from a special case of the preceding one; we will return to this point later.)
Next, let $\mathcal{M}an$ be the category of smooth manifolds. The objects are the smooth manifolds (of any dimension). The morphisms from a smooth manifold M to a smooth manifold N are the smooth maps from M to N. For any fixed integer $k \geq 0$ the rule which assigns to a smooth manifold M the real vector space $\Omega^k(M)$ of smooth differential k-forms is a contravariant functor from $\mathcal{M}an$ to the category $\mathcal{V}ect$ of real vector spaces (with linear maps as morphisms). Namely, a smooth map $f\colon M \to N$ determines a linear map $f^*\colon \Omega^k(N) \to \Omega^k(M)$. (You must have seen the details if you know anything about differential forms.)

A presheaf $\mathcal{F}$ on a topological space X is nothing but a contravariant functor from the poset of open subsets of X to $\mathcal{S}ets$. In more detail, write $\mathcal{O}$ for the topology on X, the set of open subsets of X. We can regard $\mathcal{O}$ as a partially ordered set (poset) in the following way: for $U, V \in \mathcal{O}$ we decree that $U \leq V$ if and only if $U \subset V$. A partially ordered set is a small category, as explained in example 3.2.2; therefore $\mathcal{O}$ is (the object set of) a small category. For $U, V \in \mathcal{O}$, there is exactly one morphism from U to V if $U \subset V$, and none if U is not contained in V. To that one morphism (if $U \subset V$) the presheaf $\mathcal{F}$ assigns a map $\mathrm{res}_{V,U} \colon \mathcal{F}(V) \to \mathcal{F}(U)$. The conditions on $\mathcal{F}$ in definition 3.1.1 are special cases of the conditions on a contravariant functor.

The story does not end there. The functors from a category $\mathcal{C}$ to a category $\mathcal{D}$ also form something like a category. There is a concept of morphism between functors from $\mathcal{C}$ to $\mathcal{D}$, and the name of that is *natural transformation*.

Definition 3.2.6. Let F and G be functors, both from a category $\mathcal{C}$ to a category $\mathcal{D}$. A *natural transformation* from F to G is a rule ν which for every object c in $\mathcal{C}$ selects a morphism $\nu_c \colon F(c) \to G(c)$ in $\mathcal{D}$, subject to the following condition. Whenever $u \colon c \to d$ is a morphism in $\mathcal{C}$, the square of morphisms

$$\begin{tikzcd} F(c) \arrow[r, "\nu_c"] \arrow[d, "F(u)"] & G(c) \arrow[d, "G(u)"] \\ F(d) \arrow[r, "\nu_d"] & G(d) \end{tikzcd}$$

in $\mathcal{D}$ commutes; that is, the equation $G(u) \circ \nu_c = \nu_d \circ F(u)$ holds in $\mathrm{mor}_{\mathcal{D}}(F(c), G(d))$.

Example 3.2.7. MacLane (in his book *Categories for the working mathematician*) gives the following pretty example. For a fixed integer $n \geq 1$ the rule which to a ring R assigns the group $\mathrm{GL}_n(R)$ can be viewed as a functor GL_n from the category of rings to the category of groups, as was shown earlier. There we allowed non-commutative rings, but here we need commutative rings, so we shall view GL_n as a functor from the category $c\mathcal{R}ings$ of commutative rings to $\mathcal{G}roups$. Note that $\mathrm{GL}_1(R)$ is essentially the group of units of the ring R. The group homomorphisms

$$\det \colon \mathrm{GL}_n(R) \to \mathrm{GL}_1(R)$$

(one for every commutative ring R) make up a natural transformation from the functor $\mathrm{GL}_n \colon c\mathcal{R}ings \to \mathcal{G}roups$ to the functor $\mathrm{GL}_1 \colon c\mathcal{R}ings \to \mathcal{G}roups$.
Returning to smooth manifolds and differential forms: we saw that for any fixed $k \geq 0$ the assignment $M \mapsto \Omega^k(M)$ can be viewed as a contravariant functor from $\mathcal{M}an$ to $\mathcal{V}ect$. The exterior derivative maps

$$d \colon \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$$

(one for each object M of $\mathcal{M}an$) make up a natural transformation from the contravariant functor Ω^k to the contravariant functor Ω^{k+1}.

Notation: let F and G be functors from $\mathcal{C}$ to $\mathcal{D}$. Sometimes we describe a natural transformation ν from F to G by a strong arrow, as in $\nu \colon F \Rightarrow G$.

Remark: one reason for being a little cautious in saying *category of categories* etc. is that the functors from one big category (such as $\mathcal{T}op$ for example) to another big category (such as $\mathcal{S}ets$ for example) do not obviously form a set. Of course, some people would not exercise that kind of caution and would instead say that the definition of category as

given in 3.2.1 is not bold enough. In any case, it must be permitted to say *the category of small categories.*

3.3. The category of presheaves on a space

Let $X = (X, \mathcal{O})$ be a topological space. We have seen that a presheaf $\mathcal{F}$ on X is the same thing as a contravariant functor from the poset $\mathcal{O}$ (partially ordered by inclusion, and then viewed as a category) to $\mathcal{S}$ets. Therefore it is not surprising that we define a *morphism* from a presheaf $\mathcal{F}$ on X to a presheaf $\mathcal{G}$ on X to be a natural transformation between contravariant functors from $\mathcal{O}$ to $\mathcal{S}$ets. Writing this out in detail, we obtain the following definition.

Definition 3.3.1. Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on the topological space X. A *morphism* or *map* of presheaves from $\mathcal{F}$ to $\mathcal{G}$ is a rule which for every open set U in X selects a map $\lambda_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$, subject to the following condition. Whenever U and V are open subsets of X and $U \subset V$, the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\lambda_U"] & \mathcal{G}(U) \\ \mathcal{F}(V) \arrow[u, "\mathrm{res}_{V,U}"] \arrow[r, "\lambda_V"] & \mathcal{G}(V) \arrow[u, "\mathrm{res}_{V,U}"'] \end{tikzcd}$$

in $\mathcal{S}$ets commutes; that is, the maps $\mathrm{res}_{V,U} \circ \lambda_V$ and $\lambda_U \circ \mathrm{res}_{V,U}$ from $\mathcal{F}(V)$ to $\mathcal{G}(U)$ agree.

With this definition of morphism, it is clear that there is a category of presheaves on X. It is a small category.

Example 3.3.2. Let X be a topological space. Let $\mathcal{F}$ be the presheaf on X such that $\mathcal{F}(U)$, for open $U \subset X$, is the set of continuous maps from U to $\mathbb{R}$, and such that $\mathrm{res}_{V,U} \colon \mathcal{F}(V) \to \mathcal{F}(U)$ is given by restriction of functions. Let $\mathcal{G}$ be the presheaf on X such that $\mathcal{G}(U)$, for open $U \subset X$, is the set of all open subsets of X which are contained in U. More precisely $\mathcal{G}$ is a presheaf because in the situation $U \subset V$ we define

$$\mathrm{res}_{V,U} \colon \mathcal{G}(V) \to \mathcal{G}(U)$$

by $W \mapsto W \cap U$ for an open subset W of X contained in V. (Then $W \cap U$ is an open subset of X contained in U.) A morphism α from presheaf $\mathcal{F}$ to presheaf $\mathcal{G}$ is defined by

$$\alpha_U(g) = g^{-1}(\,]0, \infty[\,)$$

for $g \in \mathcal{F}(U)$. In a more wordy formulation: to an element g of $\mathcal{F}(U)$, alias continuous function $g \colon U \to \mathbb{R}$, the morphism $\alpha \colon \mathcal{F} \to \mathcal{G}$ assigns an element of $\mathcal{G}(U)$, alias open set of X contained in U, by taking the preimage of $]0, \infty[$ under g.

3.4. (Appendix): Abelian group vocabulary

It is customary to describe the binary operation in an abelian group by a $+$ sign, if there is no danger of confusion. Thus, if A is an abelian group and $a, b \in A$, we like to write $a + b$ instead of ab or $a \cdot b$; also $-b$ instead of b^{-1} and 0 instead of 1 for the neutral element.

The expression *abelian group* is synonymous with $\mathbb{Z}$*-module.* The name $\mathbb{Z}$*-module* is a reminder that there is some interaction between the ring $\mathbb{Z}$ and the elements of any abelian group A. This looks a lot like the multiplication of vectors by scalars in a vector

space. Namely, let A be an abelian group (written with $+$ etc.), let a be an element of A and $z \in \mathbb{Z}$. Then we can define

$$z \cdot a \in A$$

as follows: if $z \geq 0$ we mean $a + a + \cdots + a$ (there are z summands in the sum); if $z \leq 0$ then we know already what $(-z)\cdot a$ means and $z\cdot a$ should be the inverse, $z\cdot a = -((-z)\cdot a)$. This "scalar multiplication" has an associativity property:

$$(wz) \cdot a = w \cdot (z \cdot a)$$

and also two distributivity properties, $(w+z)\cdot a = w\cdot a + z\cdot a$ as well as $z\cdot(a+b) = z\cdot a + z\cdot b$. Furthermore, $1 \cdot a = a$ for all $a \in A$ and $z \cdot 0 = 0$ for all $z \in \mathbb{Z}$. We might feel tempted to say that A is a vector space over the field $\mathbb{Z}$, but there is the objection that $\mathbb{Z}$ is not a field.

(Of course there is a more general concept of R-*module*, where R can be any ring. An R-module is an abelian group A with a map $R \times A \to A$ which we write in the form $(r, a) \mapsto r \cdot a$. That map is subject to many conditions, such as $(rs) \cdot a = r \cdot (s \cdot a)$ and $r \cdot (a + b) = r \cdot a + r \cdot b$, for all $r \in R$ and $a, b \in A$, and a few more. Look it up in any algebra book.)

Definition 3.4.1. Let S be a set. The *free abelian group* generated by S is the set A_S of all functions $f\colon S \to \mathbb{Z}$ such that $\{s \in S \mid f(s) \neq 0\}$ is a *finite* subset of S. It is an abelian group by pointwise addition; that is, for $f, g \in A_S$ we define $f + g \in A_S$ by $(f + g)(s) = f(s) + g(s) \in \mathbb{Z}$.

Notation. Elements of the free abelian group A_S generated by S can also be thought of as *formal linear combinations*, with integer coefficients, of elements of S. In other words, we may write

$$\sum_{s \in S} a_s \cdot s$$

where $a_s \in \mathbb{Z}$ for all $s \in \mathbb{Z}$, and we mean the function $f \in A_S$ such that $f(s) = a_s$ for all $s \in S$. Now it is important to insist that the sum have only finitely many (nonzero) summands, $a_s \neq 0$ for only finitely many $s \in S$.
My notation A_S for the free abelian group generated by S is meant to be temporary. I can't think of any convincing standard notation for it.

An important property of the free abelian group generated by S. The group A_S comes with a distinguished map $u\colon S \to A_S$ so that $u(s)$ is the function from S to $\mathbb{Z}$ taking s to 1 and all other elements of S to 0. Together, the abelian group A_S and the map (of sets) $u\colon S \to A_S$ have the following property. *Given any abelian group* B *and map* $v\colon S \to B$, *there exists a unique homomorphism of abelian groups* $q_v\colon A_S \to B$ *such that* $q_v \circ u = v$. Diagrammatic statement:

$$\begin{tikzcd} S \arrow[r, "u"] \arrow[dr, "v"'] & A_S \arrow[d, dotted, "q_v"] \\ & B \end{tikzcd}$$

The proof is easy. Every element a of A_S can be written uniquely in the form

$$\sum_{s \in S} a_s \cdot u(s)$$

with $a_s \in \mathbb{Z}$, with only finitely many nonzero a_s. Therefore

$$q_\nu(a) = q_\nu\Big(\sum_{s\in S} a_s \cdot u(s)\Big) = \sum_{s\in S} q_\nu\big(a_s \cdot u(s)\big) = \sum_{s\in S} a_s \cdot q_\nu(u(s)) = \sum_{s\in S} a_s \cdot \nu(s)\ .$$

(The following complaint can be made: *Just a minute ago you said that we can write elements* a *of* A_S *in the form* $\sum_{s\in S} a_s \cdot s$ *, but now it is* $\sum_{s\in S} a_s \cdot u(s)$*, or what?* The complaint is justified: $\sum_{s\in S} a_s \cdot s$ is a short and imprecise form of $\sum_{s\in S} a_s \cdot u(s)$.)

3.5. Preview

If our main interest is in understanding notions like homotopy and classifying topological spaces up to homotopy equivalence, why should we learn something about presheaves and sheaves? In this section I try to give an answer, very much from the point of view of category theory.

Summarizing the experience of the first few weeks in category language, we might agree on the following. In the category $\mathcal{T}op$ of topological spaces (and continuous maps), we introduced the homotopy relation $\simeq$ on morphisms. This led to a new category $\mathcal{H}o\mathcal{T}op$ with the same objects as $\mathcal{T}op$, where a morphism from X to Y is a homotopy class of continuous maps from X to Y. We made some attempts to understand sets of homotopy classes $[X, Y] = \mathrm{mor}_{\mathcal{H}o\mathcal{T}op}(X, Y)$ in some cases; for example we understood $[S^1, S^1]$ and we showed that $[S^3, S^2]$ has more than one element. A vague impression of computability may have taken hold, but nothing very systematic emerged.

Here is a very simple-minded attempt to make things easier by introducing some algebra into topology. We can make a new category $\mathbb{Z}\mathcal{T}op$ where the objects are still the topological spaces and where the set of morphisms from X to Y is the *free abelian group* generated by the set of continuous maps from X to Y. In other words, a morphism from X to Y in $\mathbb{Z}\mathcal{T}op$ is a formal linear combination (with integer coefficents) of continuous maps from X to Y, such as $4f - 3g + 7u + 1v$, where $f, g, u, v\colon X \to Y$ are continuous maps. Note that *formal* is *formal*; we make no attempt to simplify such expressions, except by allowing $4f - 3g + 7u + 1v = 4f + 4u + 1v$ if we happen to know that $g = u$, and the like. How do we compose morphisms in $\mathbb{Z}\mathcal{T}op$? We use the composition of morphisms in $\mathcal{T}op$ and enforce a distributive law, so we say for example that the composition of the morphism $4f - 3g + 7u$ from X to Y with the morphism $-2p + 5q$ from Y to Z is

$$-8(p \circ f) + 6(p \circ g) - 14(p \circ u) + 20(q \circ f) - 15(q \circ g) + 35(q \circ u),$$

a morphism from X to Z. In many ways $\mathbb{Z}\mathcal{T}op$ is a fine category, and perhaps better than $\mathcal{T}op$; the morphism sets are abelian groups and composition of morphisms

$$\mathrm{mor}_{\mathbb{Z}\mathcal{T}op}(Y, Z) \times \mathrm{mor}_{\mathbb{Z}\mathcal{T}op}(X, Y) \longrightarrow \mathrm{mor}_{\mathbb{Z}\mathcal{T}op}(X, Z)$$

is bilinear. That is, post-composition with a fixed element of $\mathrm{mor}_{\mathbb{Z}\mathcal{T}op}(Y, Z)$ gives a homomorphism of abelian groups, from $\mathrm{mor}_{\mathbb{Z}\mathcal{T}op}(X, Y)$ to $\mathrm{mor}_{\mathbb{Z}\mathcal{T}op}(X, Z)$, and pre-composition with a fixed element of $\mathrm{mor}_{\mathbb{Z}\mathcal{T}op}(X, Y)$ gives a homomorphism of abelian groups from $\mathrm{mor}_{\mathbb{Z}\mathcal{T}op}(Y, Z)$ to $\mathrm{mor}_{\mathbb{Z}\mathcal{T}op}(X, Z)$. We can relate $\mathcal{T}op$ to $\mathbb{Z}\mathcal{T}op$ by a functor

$$\mathcal{T}op \to \mathbb{Z}\mathcal{T}op$$

which takes any object to the same object, and a continuous map $f\colon X \to Y$ to the formal linear combination $1f$. And yet, it is hard to believe that any of this will give us new insights into anything.

But let us try to raise a well-formulated objection. We have lost something in replacing $\mathcal{T}op$ by $\mathbb{Z}\mathcal{T}op$: the sheaf property. More precisely, we know that we can construct a continuous map $f\colon X \to Y$ by specifying an open cover $(U_i)_{i\in\Lambda}$ of X, and for each i a continuous map $f_i\colon U_1 \to Y$, in such a way that

$$f_{i|U_i\cap U_j} = f_{j|U_i\cap U_j}$$

for all $i, j \in \Lambda$. (Then there is a unique continuous map $f\colon X \to Y$ such that $f_{|U_i} = f_i$ for all $i \in \Lambda$.) We could take the view that this is a property of $\mathcal{T}op$ which is important to us, one that we don't want to sacrifice when we experiment with modifications of $\mathcal{T}op$. But as we have seen, the sheaf property fails in so many ways in $\mathbb{Z}\mathcal{T}op$; see example 3.1.7 and the elaborate discussion of that example. I propose that we regard that as the one great weakness of $\mathbb{Z}\mathcal{T}op$.

Fortunately, in sheaf theory there is a fundamental construction called *sheafification* by which the sheaf property is enforced. In the following chapters we will apply that construction to $\mathbb{Z}\mathcal{T}op$ to restore the sheaf property. When that is done, we can once again speak of homotopies and homotopy classes, and it will turn out that we have a very manageable situation.

CHAPTER 4

Sheafification

4.1. The stalks of a presheaf

Let $\mathcal{F}$ a presheaf on a topological space X. Fix $z \in X$. There are situations where we want to understand the behavior of $\mathcal{F}$ near z, that is to say, in small neighborhoods of z. Then it is a good idea to work with pairs (U, s) where U is an open neighborhood of z and s is an element of $\mathcal{F}(U)$. Two such pairs (U, s) and (V, t) are considered to be *germ-equivalent* if there exists an open neighborhood W of z such that $W \subset U \cap V$ and $s_{|W} = t_{|W}$ in $\mathcal{F}(W)$. It is easy to show that germ equivalence is indeed an equivalence relation.

Definition 4.1.1. The set of equivalence classes is called the *stalk* of $\mathcal{F}$ at z and denoted by $\mathcal{F}_z$. The elements of $\mathcal{F}_z$ are often called *germs* (at z, of something ... depending on the meaning of $\mathcal{F}$).

Example 4.1.2. Let X and Y be topological spaces. Let $\mathcal{F}$ be the sheaf on X where $\mathcal{F}(U)$, for open $U \subset X$, is the set of continuous maps from U to Y. For $z \in X$, an element of the stalk $\mathcal{F}_z$ is called a *germ of continuous maps from* (X, z) *to* Y.

Example 4.1.3. Fix a continuous map $p\colon Y \to X$. Let $\mathcal{F}$ be the sheaf on X where $\mathcal{F}(U)$ is the set of continuous maps $s\colon U \to Y$ such that $p \circ s$ is the inclusion $U \to X$. An element of $\mathcal{F}(U)$ can be called a *continuous section of* p *over* U. For $z \in X$, an element of $\mathcal{F}_z$ can be called a *germ at* z *of continuous sections of* $p\colon X \to Y$.

Example 4.1.4. Let X be the union of the two coordinate axes in $\mathbb{R}^2$. For open U in X, let $\mathcal{G}(U)$ be the set of connected components of $X \smallsetminus U$. For open subsets U, V of X such that $U \subset V$, define

$$\mathrm{res}_{V,U}\colon \mathcal{G}(V) \to \mathcal{G}(U)$$

by saying that $\mathrm{res}_{V,U}(C)$ is the unique connected component of $X \smallsetminus U$ which contains C (where C can be any connected component of $X \smallsetminus V$). These definitions make $\mathcal{G}$ into a presheaf on X. For $z \in X$, what can we say about the stalk $\mathcal{G}_z$? If z is the origin, $z = (0, 0)$, then $\mathcal{G}_z$ has four elements. In all other cases $\mathcal{G}_z$ has two elements. (Despite that, for any $z \in X$ and any open neighborhood V of z in X, there exists an open neighborhood W of z in X such that $W \subset V$ and $\mathcal{G}(W)$ has more than 1000 elements.)

Now let $\alpha\colon \mathcal{F} \to \mathcal{G}$ be a map (morphism) of sheaves on X. Again fix $z \in X$. Then every pair (U, s), where U is an open neighborhood of z and $s \in \mathcal{F}(s)$, determines another pair $(U, \alpha(s))$ where U is still an open neighborhood of z and $\alpha(s) \in \mathcal{G}(U)$. The assignment $(U, s) \mapsto (U, \alpha(s))$ is compatible with germ equivalence. That is, if V is another open neighborhood of z in X, and $t \in \mathcal{F}(V)$, and (U, s) is germ equivalent to (V, t), then $(U, \alpha(s))$ is germ equivalent to $(V, \alpha(t))$. In short, α determines a map of sets from $\mathcal{F}_z$ to $\mathcal{G}_z$ which takes the equivalence class (the germ) of (U, s) to the equivalence class (the

germ) of $(U, \alpha(s))$. In category jargon: the assignment

$$\mathcal{F} \mapsto \mathcal{F}_z$$

is a *functor* from $\mathcal{P}\mathrm{reSh}(X)$, the category of presheaves on X, to $\mathcal{S}\mathrm{ets}$.

When a presheaf $\mathcal{F}$ on X is a sheaf, the stalks $\mathcal{F}_z$ carry a lot of information about $\mathcal{F}$. The following theorem illustrates that.

THEOREM 4.1.5. *Let $\beta\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Suppose that for every element z of X, the map of stalks $\mathcal{F}_z \to \mathcal{G}_z$ determined by β is a bijection. Then β is an isomorphism.*

PROOF. The claim that β is an isomorphism means, abstractly speaking, that there exists a morphism $\gamma\colon \mathcal{G} \to \mathcal{F}$ of sheaves such that $\beta \circ \gamma$ is the identity on $\mathcal{G}$ and $\gamma \circ \beta$ is the identity on $\mathcal{F}$. In more down-to-earth language it means simply that $\beta_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is a bijection for every open U in X, so this is what we have to show. To ease notation, we write $\beta\colon \mathcal{F}(U) \to \mathcal{G}(U)$.
We fix U, an open subset of X. First we want to show that $\beta\colon \mathcal{F}(U) \to \mathcal{F}(G)$ is *injective*. For that we set up a commutative square of sets and maps:

$$\begin{tikzcd}
\prod_{z\in U} \mathcal{F}_z \arrow[r, "\beta"] & \prod_{z\in U} \mathcal{G}_z \\
\mathcal{F}(U) \arrow[u] \arrow[r, "\beta"] & \mathcal{G}(U) \arrow[u]
\end{tikzcd}$$

The left-hand vertical arrow is obtained by noting that each $s \in \mathcal{F}(U)$ determines a pair (U, s) representing an element of $\mathcal{F}_z$, for each $z \in U$. The right-hand vertical arrow is similar. We show that the left-hand vertical arrow is injective. Suppose that $s, t \in \mathcal{F}(U)$ have the same image in $\prod_{z\in U} \mathcal{F}_z$. It follows that every $z \in U$ admits a neighborhood W_z in U such that $s_{|W_z} = t_{|W_z}$. Selecting such a W_z for every $z \in U$, we have an open cover

$$(W_z)_{z\in U}$$

of U. Since $s_{|W_z} = t_{|W_z}$ for each of the open sets W_z in the cover, the sheaf property for $\mathcal{F}$ implies that $s = t$. Hence the left-hand vertical arrow in our square is injective, and so is the right-hand arrow by the same argument. But the top horizontal arrow is bijective by our assumption. Therefore $\beta\colon \mathcal{F}(U) \to \mathcal{F}(G)$ is injective.
Next we show that $\beta\colon \mathcal{F}(U) \to \mathcal{F}(G)$ is *surjective*. We can use the same commutative square that we used to prove injectivity. An element $s \in \mathcal{G}(U)$ determines an element of $\prod_{z\in U} \mathcal{G}_z$ (right-hand vertical arrow) which comes from an element of $\prod_{z\in U} \mathcal{F}_z$ because the top horizontal arrow is bijective. So for each $z \in U$ we can find an element of $\mathcal{F}_z$ which under β is mapped to the germ of s at z (an element of $\mathcal{G}_z$). In terms of representatives of germs, this means that for each $z \in U$ we can find an open neighborhood V_z of z in U and an element $t_z \in \mathcal{F}(V_z)$ such that $\beta(t_z) = s_{|V_z} \in \mathcal{G}(V_z)$. Selecting such a V_z for every $z \in U$, we have an open cover

$$(V_z)_{z\in U}$$

of U and we have $t_z \in \mathcal{F}(V_z)$. Can we use the sheaf property of $\mathcal{F}$ to produce $t \in \mathcal{F}(U)$ such that $t_{|V_z} = t_z$ for all $z \in U$? We need to verify the matching condition,

$$t_{z|V_z\cap V_y} = t_{y|V_z\cap V_y} \in \mathcal{F}(V_z \cap V_y)$$

whenever $y, z \in U$. By the injectivity of $\beta\colon \mathcal{F}(V_z \cap V_y) \to \mathcal{G}(V_z \cap V_y)$, which we have established, it is enough to show

$$\beta(t_z)_{|V_z \cap V_y} = \beta(t_y)_{|V_z \cap V_y} \in \mathcal{G}(V_z \cap V_y).$$

This clearly holds as $\beta(t_z) = s_{|V_z}$ by construction, so that both sides of the equation agree with $s_{|V_z \cap V_y}$. So we obtain $t \in \mathcal{F}(U)$ such that $t_{|V_z} = t_z$ for all $z \in U$. Now it is easy to show that $\beta(t) = s$. Indeed we have $\beta(t)_{|V_z} = s_{|V_z}$ by construction, for all open sets V_z in the covering $(V_z)_{z \in U}$ of U, so the sheaf property of $\mathcal{F}$ implies $\beta(t) = s$. Since $s \in \mathcal{G}(U)$ was arbitrary, this means that $\beta\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective. □

4.2. Sheafification of a presheaf

PROPOSITION 4.2.1. *Let $\mathcal{F}$ be a presheaf on a topological space X. There is a sheaf $\Phi\mathcal{F}$ on X and there is a morphism $\eta\colon \mathcal{F} \to \Phi\mathcal{F}$ of presheaves such that, for every $z \in X$, the map of stalks $\mathcal{F}_z \to (\Phi\mathcal{F})_z$ determined by η is bijective.*

PROOF. Let U be an open subset of X. We are going to define $(\Phi\mathcal{F})(U)$ as a subset of the product

$$\prod_{z \in U} \mathcal{F}_z \ .$$

Think of an element of that product as a function s which for every $z \in U$ selects an element $s(z) \in \mathcal{F}_z$. The function s qualifies as an element of $(\Phi\mathcal{F})(U)$ if and only if it satisfies the following *coherence condition*. For every $y \in U$ there is an open neighborhood W of y in U and there is $t \in \mathcal{F}(W)$ such that the pair (W, t) simultaneously represents the germs $s(z) \in \mathcal{F}_z$ for all $z \in W$.
From the definition, it is clear that there are restriction maps

$$\mathrm{res}_{V,U}\colon (\Phi\mathcal{F})(V) \to (\Phi\mathcal{F})(U)$$

whenever U, V are open in X and $U \subset V$. Namely, a function s which selects an element $s(z) \in \mathcal{F}_z$ for every $z \in V$ determines by restriction a function $s_{|U}$ which selects an element $s(z) \in \mathcal{F}_z$ for every $z \in U$. The coherence condition is satisfied by $s_{|U}$ if it is satisfied by s. With these restriction maps, $\Phi\mathcal{F}$ is a presheaf. Furthermore, it is straightforward to see that $\Phi\mathcal{F}$ satisfies the sheaf condition. Indeed, suppose that $(V_i)_{i \in \Lambda}$ is a collection of open subsets of X, and suppose that elements $s_i \in (\Phi\mathcal{F})(V_i)$ have been selected, one for each $i \in \Lambda$, such that the matching condition

$$s_{i|V_i \cap V_j} = s_{j|V_i \cap V_j}$$

is satisfied for all $i, j \in \Lambda$. Then clearly we get a function s on $V = \bigcup_i V_i$ which for every $z \in V$ selects $s(z) \in \mathcal{F}_z$ by declaring, unambiguously,

$$s(z) := s_i(z)$$

for any i such that $z \in V_i$. The coherence condition is satisfied because it is satisfied by each s_i .
The morphism of presheaves $\eta\colon \mathcal{F} \to \Phi\mathcal{F}$ is defined in the following mechanical way. Given $t \in \mathcal{F}(U)$, we need to say what $\eta(t) \in (\Phi\mathcal{F})(U)$ should be. It is the function which to $z \in U$ assigns the element of $\mathcal{F}_z$ represented by the pair (U, t), that is to say, the germ of (U, t) at z.
Last not least, we need to show that for any $z \in X$ the map $\mathcal{F}_z \to (\Phi\mathcal{F})_z$ determined by η is a bijection. We fix z. *Injectivity*: we consider elements a and b of $\mathcal{F}_z$ represented by pairs (U_a, s_a) and (U_b, s_b) respectively, where U_a, U_b are neighborhoods of z and

$s_a \in \mathcal{F}(U_a)$, $s_b \in \mathcal{F}(U_b)$. Suppose that a and b are taken to the same element $t \in (\Phi\mathcal{F})_z$ by η. Then in particular $t(z) \in \mathcal{F}_z$ is the germ at z of s_a , and also the germ at z of s_b , so the germs of s_a and s_b (elements of $\mathcal{F}_z$) are equal. *Surjectivity*: let an element of $(\Phi\mathcal{F})_z$ be represented by a pair (U, t) where U is an open neighborhood of z in X and $t \in (\Phi\mathcal{F})(U)$. By the coherence condition, there exists an open neighborhood W of z in U and there exists $s \in \mathcal{F}(W)$ such that $t_{|W}$ is the function which to $y \in W$ assigns the germ at y of (W, s), an element of $\mathcal{F}_y$. But this means that the map of stalks $\mathcal{F}_z \to (\Phi\mathcal{F})_z$ determined by the morphism η takes the element of $\mathcal{F}_z$ represented by (W, s) to the element of $(\Phi\mathcal{F})_z$ represented by (U, t). □

Example 4.2.2. Let T be any set. Let $\mathcal{F}$ be the constant presheaf on X given by $\mathcal{F}(U) = T$ for all open subsets U of X (and $\mathrm{res}_{V,U} \colon \mathcal{F}(V) \to \mathcal{F}(U)$ is id_T). What does the sheaf $\Phi\mathcal{F}$ look like? This question has quite an interesting answer. Let's keep a cool head and approach it mechanically. For any $z \in X$, the stalk $\mathcal{F}_z$ can be identified with T. This is easy. Let U be an open subset of X. The elements of $(\Phi\mathcal{F})(U)$ are functions s which for every $z \in U$ select an element $s(z) \in \mathcal{F}_z = T$, subject to a coherence condition. So the elements of $(\Phi\mathcal{F})(U)$ are maps s from U to T subject to a coherence condition. What is the coherence condition? The condition is that s must be locally constant, i.e., every $z \in U$ admits an open neighborhood W in U such that $s_{|W}$ is constant. So the elements of $(\Phi\mathcal{F})(U)$ are the locally constant maps s from U to T. A locally constant map s from U to T is the same thing as a continuous map s from U to T, if we agree that T is equipped with the discrete topology (every subset of T is declared to be open). Summing up, $(\Phi\mathcal{F})(U)$ is the set of continuous functions from U to T. We can say that $\Phi\mathcal{F}$ is the sheaf of continuous functions (from open subsets of X) to T.
To appreciate the beauty of this answer, take a space X which is a little out of the ordinary; for example, $\mathbb{Q}$ with the standard topology inherited from $\mathbb{R}$, or the Cantor set (a subset of $\mathbb{R}$). For T, any set with more than one element is an interesting choice. (What happens if T has exactly one element? What happens if $T = \emptyset$?)

There are a few things of a general nature to be said about proposition 4.2.1 — not difficult, not surprising, but important. The construction Φ is a functor; we can view it as a functor from the category $\mathcal{P}\mathrm{re}\mathcal{S}\mathrm{h}(X)$ to itself. This means in particular that any morphism of presheaves $\alpha \colon \mathcal{F} \to \mathcal{G}$ on X determines a morphism

$$\Phi\alpha \colon \Phi\mathcal{F} \to \Phi\mathcal{G} \ .$$

Namely, for $s \in \Phi\mathcal{F}(V)$ we define $t = (\Phi\alpha)(s) \in \Phi\mathcal{G}(V)$ in such a way that $t(z) \in \mathcal{G}_z$ is the image of $s(z) \in \mathcal{F}_z$ under the map $\mathcal{F}_z \to \mathcal{G}_z$ induced by α. (It is easy to verify that t satisfies the coherence condition.)
Furthermore η is a natural transformation from the identity functor id on $\mathcal{P}\mathrm{re}\mathcal{S}\mathrm{h}(X)$ to the functor $\Phi \colon \mathcal{P}\mathrm{re}\mathcal{S}\mathrm{h}(X) \to \mathcal{P}\mathrm{re}\mathcal{S}\mathrm{h}(X)$. This means that, for a morphism of presheaves $\alpha \colon \mathcal{F} \to \mathcal{G}$ on X as above, the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\alpha"] \arrow[d, "\eta"] & \mathcal{G} \arrow[d, "\eta"] \\ \Phi\mathcal{F} \arrow[r, "\Phi\alpha"] & \Phi\mathcal{G} \end{tikzcd}$$

in $\mathcal{P}\mathrm{re}\mathcal{S}\mathrm{h}(X)$ is commutative. That is also easily verified.
There is one more thing of a general nature which must be mentioned. Let $\mathcal{F}$ be any

presheaf on X. What happens if we apply the functor Φ to the morphism $\eta_{\mathcal{F}}\colon \mathcal{F} \to \Phi\mathcal{F}$? The result is obviously a morphism of sheaves

$$\Phi(\eta_{\mathcal{F}})\colon \Phi\mathcal{F} \to \Phi(\Phi\mathcal{F}).$$

It is an *isomorphism* of sheaves. The verification is easy using theorem 4.1.5.

The sheaf $\Phi\mathcal{F}$ is *the sheafification* (or the *associated sheaf*) of the presheaf $\mathcal{F}$; also Φ may be called the sheafification functor, or the associated sheaf functor.

Corollary 4.2.3. *Let $\beta\colon \mathcal{F} \to \mathcal{G}$ be any morphism of presheaves on X. If $\mathcal{G}$ is a sheaf, then β has a unique factorization $\beta = \beta_1 \circ \eta_{\mathcal{F}}$ where $\eta_{\mathcal{F}}\colon \mathcal{F} \to \Phi\mathcal{F}$ is the morphism of proposition 4.2.1:*

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\beta"] \arrow[d, "\eta_{\mathcal{F}}"] & \mathcal{G} \\
\Phi\mathcal{F} \arrow[ur, dotted, "\beta_1"'] &
\end{tikzcd}$$

Proof. Apply Φ and η to $\mathcal{F}$, $\mathcal{G}$ and β to obtain a commutative diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\beta"] \arrow[d, "\eta_{\mathcal{F}}"] & \mathcal{G} \arrow[d, "\eta_{\mathcal{G}}"] \\
\Phi\mathcal{F} \arrow[r, "\Phi\beta"] & \Phi\mathcal{G}
\end{tikzcd}$$

By proposition 4.2.1, the vertical arrows determine bijections from $\mathcal{F}_z$ to $(\Phi\mathcal{F})_z$ and from $\mathcal{G}_z$ to $(\Phi\mathcal{G})_z$ for every $z \in X$. Both $\mathcal{G}$ and $\Phi\mathcal{G}$ are sheaves, so theorem 4.1.5 applies and we may deduce that the right-hand vertical arrow is an isomorphism of sheaves on X. Let $\lambda\colon \Phi\mathcal{G} \to \mathcal{G}$ be an inverse for that isomorphism. The factorization problem has a solution, $\beta_1 = \lambda \circ \Phi\beta$.
To see that the solution is unique, apply Φ and η to the commutative diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\beta"] \arrow[d, "\eta_{\mathcal{F}}"] & \mathcal{G} \\
\Phi\mathcal{F} \arrow[ur, dotted] &
\end{tikzcd}$$

in $\mathcal{P}\mathrm{re}\mathcal{S}\mathrm{h}(X)$. The result is a commutative diagram in $\mathcal{P}\mathrm{re}\mathcal{S}\mathrm{h}(X)$ in the shape of a prism:

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\beta"] \arrow[d, "\eta_{\mathcal{F}}"] \arrow[dr] & \mathcal{G} \arrow[dr, "\eta_{\mathcal{G}}"] & \\
\Phi\mathcal{F} \arrow[ur, dotted] \arrow[dr] & \Phi\mathcal{F} \arrow[r, "\Phi\beta"] \arrow[d, "\Phi(\eta_{\mathcal{F}})"] & \Phi\mathcal{G} \\
& \Phi(\Phi\mathcal{F}) \arrow[ur, dotted] &
\end{tikzcd}$$

Here the arrow labeled $\Phi(\eta_{\mathcal{F}})$ is an isomorphism of sheaves, as noted above under things of a general nature. This makes the lower dotted arrow unique. But the arrow labeled $\eta_{\mathcal{G}}$

is also an isomorphism by theorem 4.1.5 and the property of $\eta_{\mathcal{G}}$ stated in proposition 4.2.1. This ensures that the upper dotted arrow is determined by the lower dotted arrow. $\square$

4.3. Mapping cycles

Let X and Y be topological spaces. One of the first examples of a sheaf that we saw was the sheaf $\mathcal{F}$ on X such that

$$\mathcal{F}(U) = \text{set of continuous maps from } U \text{ to } Y$$

etc., for open U in X. From that we constructed a presheaf $\mathcal{G}$ on X such that that

$$\mathcal{G}(U) = \text{free abelian group generated by } \mathcal{F}(U)$$

etc., for open U in X. In other words, $\mathcal{G}(U)$ is the set of formal linear combinations (with coefficients in $\mathbb{Z}$) of continuous functions from X to Y. It turned out that $\mathcal{G}$ is never a sheaf, and for many reasons. The stalk $\mathcal{G}_z$ at $z \in X$ can be described (after some unraveling) as the set of formal linear combinations, with integer coefficients, of germs of continuous maps from (X, z) to Y. (Recall that *germ of continuous maps from* (X, z) *to* Y means an equivalence class of pairs (U, f) where U is an open neighborhood of z in X and $f\colon U \to Y$ is continuous.) Of course, we ask what $\mathcal{G}_z$ is because it feeds into the construction of $\Phi\mathcal{G}$, the sheafification of $\mathcal{G}$. It is permitted and even exciting to evaluate $\Phi\mathcal{G}$ on X, since X is an open subset of X.

Definition 4.3.1. An element of $(\Phi\mathcal{G})(X)$ will be called a *mapping cycle* from X to Y.

So what is a mapping cycle from X to Y ?

First answer. A mapping cycle from X to Y is a function s which for every $z \in X$ selects $s(z)$, a formal linear combination with integer coefficients of germs[1] of continuous maps from (X, z) to Y. There is a coherence condition to be satisfied: it must be possible to cover X by open sets W_i such that all values $s(z)$, where $z \in W_i$, can be simultaneously represented by one formal linear combination

$$\sum_j b_{ij} f_{ij}$$

where $f_{ij}\colon W_i \to Y$ are continuous maps and the b_{ij} are integers.

Second answer. A mapping cycle from X to Y can be specified (described, constructed) by choosing an open cover $(U_i)_{i\in\Lambda}$ of X and for every $i \in \Lambda$ a formal linear combination s_i with integer coefficients of continuous maps[2] from U_i to Y. There is a matching condition to be satisfied[3]: for any $i, j \in \Lambda$ and any $x \in U_i \cap U_j$, there should exist an open neighborhood W of x in $U_i \cap U_j$ such that $s_{i|W} = s_{j|W}$.

(The second answer is in some ways less satisfactory than the first because it does not say explicitly what a mapping cycle *is*, only how we can construct mapping cycles. But it can indeed be useful when we need to construct mapping cycles.)

Some of the "counter"examples which we saw previously now serve as illustrations of the concept of mapping cycle.

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[1] Grown-up formulation: selects an element in the free abelian group generated by the set of germs ...

[2] Grown-up formulation: for every $i \in \Lambda$ an element s_i in the free abelian group generated by the set of continuous maps ...

[3] Did you expect to see the condition $s_{i|U_i\cap U_j} = s_{j|U_i\cap U_j}$? Sheaf theory dictates a weaker condition!

EXAMPLE 4.3.2. If S is a set with 6 elements and T is a set with 2 elements, both to be viewed as topological spaces with the discrete topology, then the abelian group of mapping cycles from S to T is isomorphic to $\mathbb{Z}^{12} \cong \prod_{i=1}^{6}(\mathbb{Z} \oplus \mathbb{Z})$. Do not confuse with $\mathbb{Z}/12$.

EXAMPLE 4.3.3. Let X and Y be two topological spaces related by a covering map

$$p\colon Y \to X$$

with finite fibers. In other words, p is a fiber bundle whose fibers are finite sets (viewed as topological spaces with the discrete topology). For simplicity, suppose also that X is connected. Choose an open covering $(W_j)_{j\in\Lambda}$ of X such that p admits a bundle chart over W_j for each j:

$$h_j\colon p^{-1}(W_j) \to W_j \times F$$

where F is a finite set (with the discrete topology). For $j \in \Lambda$ and $z \in F$ there is a continuous map $\sigma_{j,z}\colon W_j \to Y$ given by $\sigma_{j,z}(x) = h_j^{-1}(x,z)$ for $x \in W_j$. Define

$$s_j = \sum_{z\in F} \sigma_{j,z}\,.$$

This is a formal linear combination of continuous maps from W_j to Y. Clearly

$$s_{i|W_i\cap W_j} = s_{j|W_i\cap W_j}$$

(yes, this is more than we require). Therefore, by *"second answer"*, we have specified a mapping cycle from X to Y (which agrees with s_j on W_j).

EXAMPLE 4.3.4. Let X and Y be topological spaces. Suppose that $X = V_1 \cup V_2$ where V_1 and V_2 are open subsets of X. Let continuous maps $f, g\colon V_1 \to Y$ be given such that

$$f_{|V_1\cap V_2} = g_{|V_1\cap V_2}\,.$$

Then it makes (some) sense to view the formal linear combination $f - g = 1 \cdot f + (-1) \cdot g$ as a mapping cycle from X to Y. How? We have the open cover of X consisting of V_1 and V_2 , and we specify $s_1 = f - g$ (a mapping cycle from V_1 to Y), and $s_2 = 0$ (a mapping cycle from V_2 to Y). Then $s_{1|V_1\cap V_2} = 0 = s_{2|V_1\cap V_2}$. So the matching condition is satisfied, and so by *"second answer"* we have specified a mapping cycle from X to Y.

Mapping cycles are complicated entities, but I hope that readers having survived the excursion into sheaf theory remain sufficiently intoxicated to find the definition obvious and unavoidable. With that, the excursion into sheaf theory is over (for now). Next we shall try to develop a comfortable relationship with mapping cycles. Here is a list of some of their good uses and properties.

(1) Every continuous map from X to Y determines a mapping cycle from X to Y.
(2) The mapping cycles from X to Y form an abelian group.
(3) A mapping cycle from X to Y can be composed with a (continuous) map from Y to Z to give a mapping cycle from X to Z. A mapping cycle from Y to Z can be composed with a (continuous) map from X to Y to give a mapping cycle from X to Z. But more remarkably, a mapping cycle from X to Y can be composed with a mapping cycle from Y to Z to give a mapping cycle from X to Z.
(4) Composition of mapping cycles is bilinear.
(5) Mapping cycles satisfy a sheaf property: if $(U_i)_{i\in\Lambda}$ is an open covering of X and $s_i\colon U_i \to Y$ is a mapping cycle, for each $i \in \Lambda$, such that

$$s_{i|U_i\cap U_j} = s_{j|U_i\cap U_j}$$

for all $i, j \in \Lambda$, then there is a unique mapping cycle s from X to Y such that $s_{|U_i} = s_i$ for all $i \in \Lambda$.

(6) There is exactly one mapping cycle from X to $\emptyset$. And there is exactly one mapping cycle from $\emptyset$ to Y, for any space Y.

(7) Mapping cycles from a topological disjoint union $X_1 \coprod X_2$ to Y are in bijection with pairs (s_1, s_2) where s_i is a mapping cycle from X_i to Y for $i = 1, 2$. Mapping cycles from X to a topological disjoint union $Y_1 \coprod Y_2$ are in bijection with pairs (s_1, s_2) where s_i is a mapping cycle from X to Y_i for $i = 1, 2$.

Some comments on that.
(1) A continuous map $f\colon X \to Y$ determines a mapping cycle $s = s_f$ where $s(z)$ is the germ of f at z. Interesting observation: the map $f \mapsto s_f$ from the set of continuous maps from X to Y to the set of mapping cycles from X to Y is injective.
(2) Obvious.
(3) Given a mapping cycle s from X to Y and a continuous map $g\colon Y \to Z$ we get a mapping cycle $g \circ s$ from X to Z by $x \mapsto \sum b_j(g \circ f_j)$ when $x \in X$ and $s(x) = \sum b_j f_j$. Given a mapping cycle s from Y to Z and a continuous map $g\colon X \to Y$ we get a mapping cycle $s \circ g$ from X to Z by $x \mapsto \sum b_j(f_j \circ g)$ when $x \in X$ and $s(x) = \sum b_j f_j$. Given a mapping cycle s from X to Y and a mapping cycle t from Y to Z we get a mapping cycle $t \circ s$ from X to Z by the formula

$$x \mapsto \sum (b_j c_{ij})(f_{ij} \circ g_j)$$

when $x \in X$ and $s(x) = \sum_j b_j g_j$ and $t(g_j(x)) = \sum_i c_{ij} f_{ij}$. (The notation is not fantastically precise or logical; in any case b_j, c_{ij} etc. are meant to be integers while f_{ij} , g_j etc. are meant to be germs of continuous functions. Note that f_{ij} in the displayed formula is a germ at $g_j(x)$, while g_j is a germ at x.)
(4) Should be clear from the last formula in the comment on (3).
(5) By construction.
(6) Mapping cycles from $\emptyset$ to Y: there is exactly one by construction. A mapping cycle s from X to $\emptyset$ is a function which for each $x \in X$ selects a formal linear combination of germs of continuous maps from (X, x) to $\emptyset$, etc.; since there no such germs, the only possible formal linear combination is the zero linear combination. This does satisfy the coherence condition.
(7) By construction and by inspection.

In category language, we can say that there is a category $\mathcal{A}\mathcal{T}\mathrm{op}$ whose objects are the topological spaces and where a morphism from space X to space Y is a mapping cycle from X to Y. There is an "inclusion" functor

$$\mathcal{T}\mathrm{op} \to \mathcal{A}\mathcal{T}\mathrm{op}$$

taking every object X to the same object X, and taking a morphism $f\colon X \to Y$ (continuous map) to the corresponding mapping cycle as explained in (1). For each X and Y, the set $\mathrm{mor}_{\mathcal{A}\mathcal{T}\mathrm{op}}(X, Y)$ is equipped with the structure of an abelian group. Composition of morphisms is bilinear. There is a zero object X in $\mathcal{A}\mathcal{T}\mathrm{op}$, i.e., an object with the property that $\mathrm{mor}_{\mathcal{A}\mathcal{T}\mathrm{op}}(X, Y)$ has exactly one element and $\mathrm{mor}_{\mathcal{A}\mathcal{T}\mathrm{op}}(Y, X)$ has exactly one element for arbitrary Y. Indeed, $X = \emptyset$ is a zero object in $\mathcal{A}\mathcal{T}\mathrm{op}$. The property expressed in (7) can also be formulated in category language, but we must postpone it because the vocabulary for that has not been introduced so far. In all, we can say that $\mathcal{A}\mathcal{T}\mathrm{op}$ is an *additive category*.

Finally, let me mention a good property of continuous maps which does not carry over to mapping cycles. Let X and Y are topological spaces. Suppose that we have a covering of X by finitely many *closed* subsets $A_1, A_2, \ldots, A_r$, and continuous maps $f_i \colon A_i \to Y$ for $i = 1, 2, \ldots, r$ such that f_i agrees with f_j on $A_i \cap A_j$, for all $i, j \in \{1, 2, \ldots, r\}$. Then there exists a unique continuous $f \colon X \to Y$ which agrees with f_i on A_i for each $i \in \{0, 1, \ldots, r\}$. This principle, which we often use subconsciously to construct continuous maps, is unsafe (to say the least) when used with mapping cycles.

CHAPTER 5

Homotopies in $\mathcal{AT}op$

5.1. The homotopy relation

Definition 5.1.1. Let X and Y be topological spaces. We call two mapping cycles f and g from X to Y *homotopic* if there exists a mapping cycle h from $X \times [0,1]$ to Y such that $f = h \circ \iota_0$ and $g = h \circ \iota_0$. Here $\iota_0, \iota_1 \colon X \to X \times [0,1]$ are defined by $\iota_0(x) = (x,0)$ and $\iota_1(x) = (x,1)$ as usual. Such a mapping cycle h is a *homotopy* from f to g.

Remark. In that definition, $X \times [0,1]$ still means the product of X and $[0,1]$ in $\mathcal{T}op$. We saw some evidence suggesting that in $\mathcal{AT}op$ this does not have the properties that we might expect from a product (in a category sense).

Lemma 5.1.2. *"Homotopic" is an equivalence relation on the set of mapping cycles from* X *to* Y. The set of equivalence classes will be denoted by $[[X,Y]]$ and the equivalence class of a mapping cycle f will be denoted by $[[f]]$.

Proof. Reflexivity and symmetry are fairly obvious. Transitivity is more interesting. (I am indebted to S. Mahanta for the following pretty argument.) Let h be a homotopy from e to f and k a homotopy from f to g, where e, f and g are mapping cycles from X to Y. We can agree that it suffices to produce a mapping cycle ℓ from $X \times [0,2]$ to Y such that ℓ restricted to $X \times \{0\}$ agrees with e and ℓ restricted to $X \times \{1\}$ agrees with g. Let

$$u \colon X \times [0,2] \longrightarrow X \times [0,1], \qquad v \colon X \times [0,2] \longrightarrow X \times [0,1], \qquad p \colon X \times [0,2] \to X$$

be the continuous maps given by $u(x,t) \mapsto (x, \min\{t,1\})$, $v(x,t) = (x, \max\{t,1\})$ and $p(x,t) = x$. Put

$$\ell := u^*h + v^*k - p^*f.$$

For that we can also write $\ell = (h \circ u) + (k \circ v) - (f \circ p)$. □

Proposition 5.1.3. *The set* $[[X,Y]]$ *is an abelian group.*

Proof. This amounts to observing that the homotopy relation is compatible with addition of mapping cycles. In other words, if f is homotopic to g and u is homotopic to v, where f, g, u, v are mapping cycles from X to Y, then $f + u$ is homotopic to $g + v$. Indeed, if h is a homotopy from f to g and k is a homotopy from u to v, then $h + k$ is a homotopy from $f + u$ to $g + v$. □

Lemma 5.1.4. *A composition map* $[[Y,Z]] \times [[X,Y]] \to [[X,Z]]$ *can be defined by* $([[f]],[[g]]) \mapsto [[f \circ g]]$. *Composition is bilinear, i.e., for fixed* $[[g]]$ *the map* $[[f]] \mapsto [[f \circ g]]$ *is a homomorphism of abelian groups and for fixed* $[[f]]$ *the map* $[[g]] \mapsto [[f \circ g]]$ *is a homomorphism of abelian groups.* □

As a result there is a homotopy category $\mathcal{H}o\mathcal{AT}op$ whose objects are (still) the topological spaces, while the set of morphisms from X to Y is $[[X,Y]]$.

5.2. First calculations

Write $\star$ for a singleton, alias one-point space.

PROPOSITION 5.2.1. *For any space* X *the abelian group* $[[X, \star]]$ *is isomorphic to the set of continuous (=locally constant) functions from* X *to* $\mathbb{Z}$*, where* $\mathbb{Z}$ *has the discrete topology.*

PROOF. We learned in example 4.2.2 that the set of mapping cycles from X to $\star$ is identified with the set of continuous functions from X to $\mathbb{Z}$. (It is $(\Phi\mathcal{G})(X)$ where $\Phi\mathcal{G}$ is the sheaf associated to the constant presheaf $\mathcal{G}$ which has $\mathcal{G}(U) = \mathbb{Z}$ for all open $U \subset X$.) Similarly, the set of mapping cycles from $X \times [0,1]$ to $\star$ is identified with the set of continuous functions from $X \times [0,1]$ to $\mathbb{Z}$. But a continuous function h from $X \times [0,1]$ to $\mathbb{Z}$ is constant on $\{x\} \times [0,1]$ for each $x \in X$, and so will have the form $h(x,t) = g(x)$ for a unique continuous $g\colon X \to \mathbb{Z}$. It follows that the homotopy relation on the set of mapping cycles from X to $\star$ is trivial, i.e., two mapping cycles from X to $*$ are homotopic only if they are equal. □

EXAMPLE 5.2.2. Take $X = \mathbb{Q}$, a subspace of $\mathbb{R}$ with the standard topology. The group $[[\mathbb{Q}, \star]]$ is uncountable because the set of continuous maps from $\mathbb{Q}$ to $\mathbb{Z}$ is uncountable.

LEMMA 5.2.3. *For a path-connected (non-empty) space* Y *the abelian group* $[[\star, Y]]$ *is isomorphic to* $\mathbb{Z}$*.*

PROOF. Fix some point $z \in Y$. A mapping cycle from $\star$ to Y is the same thing as a formal linear combination of points in Y, say $\sum_j b_j y_j$ where $b_j \in \mathbb{Z}$ and $y_j \in Y$. In the abelian group $[[\star, Y]]$ we have

$$[[\Sigma_j b_j y_j]] = \Sigma_j b_j [[y_j]] = (\Sigma_j b_j)[[z]].$$

(Here $[[y_j]]$ for example denotes the homotopy class of the mapping cycle determined by the continuous map $\star \to Y$ which has image $\{y_j\}$. As that continuous map is homotopic to the map $\star \to Y$ which has image $\{z\}$, we obtain $[[y_j]] = [[z]]$.) Therefore $[[\star, Y]]$ is cyclic, generated by the element $[[z]]$. To see that it is infinite cyclic we use the homomorphism

$$[[\star, Y]] \to [[\star, \star]]$$

given by composition with the continuous map $Y \to \star$. Now $[[\star, \star]]$ is infinite cyclic by proposition 5.2.1. It is also clear that the homomorphism just above takes $[[z]]$ to the generator of $[[\star, \star]]$, the class of the identity mapping cycle. Hence it must be an isomorphism and so $[[\star, Y]]$ is infinite cyclic. □

COROLLARY 5.2.4. *For any space* Y *the abelian group* $[[\star, Y]]$ *is isomorphic to the free abelian group generated by the set of path components of* Y*.*

PROOF. The abelian group of mapping cycles from $\star$ to Y is simply the free abelian group A generated by the underlying set of Y. Write this as a direct sum $\bigoplus_{\lambda \in \Lambda} A_\lambda$ where Λ is an indexing set for the path components Y_λ of Y and A_λ is the free abelian group generated by the underlying set of Y_λ. Now fix some λ. *Claim*: If $f \in A$ is homotopic to $g \in A$, by a mapping cycle $h\colon [0,1] \to Y$, then the coordinate of f in A_λ is homotopic to the coordinate of g in A_λ , by a mapping cycle $[0,1] \to Y_\lambda$. To see this, cover the interval $[0,1]$ by finitely many open subsets U_i such that $h_{|U_i}$ can be represented by a formal linear combination of continuous maps from U_i to Y. This is possible by the coherence condition on h. Choose a subdivision

$$0 = t_0 < t_1 < \cdots t_{N-1} < t_N = 1$$

of $[0,1]$ such that for each of the the subintervals $[t_r, t_{r+1}]$, where $r = 0, 1, \ldots, N-1$, there exists U_i containing it. Let $h_{t_r} \in A$ be obtained by restricting h to t_r. Then $h_{t_0} = f$ and $h_{t_N} = g$, so it suffices to show that the λ-coordinate of h_{t_r} is homotopic to the λ-coordinate of $h_{t_{r+1}}$, for $r = 0, 1, \ldots, N-1$. But $[t_r, t_{r+1}]$ is contained in some U_i and so there is a formal linear combination

$$\sum_j b_j u_j$$

where $b_j \in \mathbb{Z}$ and the u_j are continuous maps from $[t_r, t_{r+1}]$ to Y, and $\sum_j b_j u_j$ restricts to h_{t_r} on t_r and to $h_{t_{r+1}}$ on t_{r+1}. Each u_j maps to only one path component of Y; in the formal linear combination $\sum_j b_j u_j$, select the terms $b_j u_j$ where u_j is a map to Y_λ and discard the others. Then the selected linear sub-combination is a homotopy from the λ-component of h_{t_r} to the λ-component of $h_{t_{r+1}}$. This proves the claim.
Therefore $[[\star, Y]]$ is the direct sum of the $[[\star, Y_\lambda]]$. By the lemma above, each $[[\star, Y_\lambda]]$ is isomorphic to $\mathbb{Z}$. □

PROPOSITION 5.2.5. *For topological spaces X and Y where X is a topological disjoint union $X_1 \amalg X_2$, there is an isomorphism*

$$[[X, Y]] \longrightarrow [[X_1, Y]] \times [[X_2, Y]] \ ; \ [[f]] \mapsto ([[f_{|X_1}]], [[f_{|X_2}]]) \ .$$

For topological spaces X and Y where Y is a topological disjoint union $Y_1 \amalg Y_2$, there is an isomorphism

$$[[X, Y_1]] \oplus [[X, Y_2]] \longrightarrow [[X, Y]] \ ; [[f]] \oplus [[g]] \mapsto [[j_1 \circ f + j_2 \circ g]]$$

where $j_1 \colon Y_1 \to Y$ and $j_2 \colon Y_2 \to Y$ are the inclusions.

PROOF. First statement: the set $\mathrm{mor}_{\mathcal{AT}op}(X, Y)$ of mapping cycles breaks up as a product $\mathrm{mor}_{\mathcal{AT}op}(X_1, Y) \times \mathrm{mor}_{\mathcal{AT}op}(X_2, Y)$ by restriction to X_1 and X_2, and a similar statement holds for the set $\mathrm{mor}_{\mathcal{AT}op}(X \times [0,1], Y)$. Second statement: the set $\mathrm{mor}_{\mathcal{AT}op}(X, Y)$ of mapping cycles breaks up as a direct sum $\mathrm{mor}_{\mathcal{AT}op}(X, Y_1) \times \mathrm{mor}_{\mathcal{AT}op}(X, Y_2)$, and a similar statement holds for $\mathrm{mor}_{\mathcal{AT}op}(X \times [0,1], Y)$. □

PROPOSITION 5.2.6. *For any topological space X we have*

$$[[\emptyset, X]] = 0 = [[X, \emptyset]] \, .$$

PROOF. The abelian group of mapping cycles from X to $\emptyset$ is a trivial group and the abelian group of mapping cycles from $\emptyset$ to X is a trivial group. □

5.3. Homology and cohomology: the definitions

DEFINITION 5.3.1. For $n \geq 0$, the n-th *homology group* of a topological space X is the abelian group

$$H_n(X) := [[S^n, X]]/[[\star, X]] \ .$$

The n-th *cohomology group* of X is the abelian group

$$H^n(X) := [[X, S^n]]/[[X, \star]].$$

Comments. There is an understanding here that $[[\star, X]]$ is a subgroup of $[[S^n, X]]$. How? By pre-composing mapping cycles from $\star$ to X with the unique continuous map from S^n to $\star$, we obtain a (well defined) homomorphism $[[\star, X]] \to [[S^n, X]]$. Conversely, by pre-composing mapping cycles from S^n to X with a selected continuous map from $\star$ to S^n, inclusion of the base point, we obtain a homomorphism $[[S^n, X]] \to [[\star, X]]$. The composition $[[\star, X]] \to [[S^n, X]] \to [[\star, X]]$ is the identity on $[[\star, X]]$. So we can say that $[[\star, X]]$

is a direct summand of $[[S^n, X]]$. We remove it, suppress it etc., when we form $H_n(X)$. Similarly, by post-composing mapping cycles from X to S^n with the unique continuous map $S^n \to \star$, we obtain a homomorphism $[[X, S^n]] \to [[X, \star]]$. Conversely, by post-composing mapping cycles from X to $\star$ with a selected continuous map $\star \to S^n$, inclusion of the base point, we obtain a homomorphism $[[X, \star]] \to [[X, S^n]]$. The composition $[[X, \star]] \to [[X, S^n]] \to [[X, \star]]$ is the identity on $[[X, \star]]$. Therefore $[[X, \star]]$ is a direct summand of $[[X, S^n]]$. We remove it, suppress it etc., when we form $H^n(X)$.

You will be unsurprised to hear that H_n is a functor from $\mathcal{T}op$ to the category of abelian groups. We can also say that it is a functor from $\mathcal{AT}op$ to abelian groups. Both statements are obvious from the definition. Equally clear from the definition, but important to keep in mind: if $f, g\colon X \to Y$ are homotopic maps, then the induced homomorphisms f_* and g_* from $H_n(X)$ to $H_n(Y)$ are the same. (Therefore we might say that H_n is a functor from $\mathcal{H}o\mathcal{T}op$ to the category of abelian groups. Indeed it is a functor from $\mathcal{H}o\mathcal{AT}op$ to abelian groups ...)
Similarly H^n is a contravariant functor from $\mathcal{T}op$ (or from $\mathcal{AT}op$, or from $\mathcal{H}o\mathcal{T}op$, or from $\mathcal{H}o\mathcal{AT}op$) to the category of abelian groups.

So far we have few tools available for computing $H_n(X)$ and $H^n(X)$ in general. But in the cases $n = 0$, arbitrary X, we are ready for it, and in the case where n is arbitrary and $X = \star$ we are also ready for it.

Example 5.3.2. Take $n = 0$ and X arbitrary. Then $H_0(X) = [[S^0, X]]/[[\star, X]]$. For S^0 we can write $\star \amalg \star$ (disjoint union of two copies of $\star$), and using the first part of proposition 5.2.5, we get $[[S^0, X]] \cong [[\star, X]] \times [[\star, X]]$. Therefore $H_0(X) \cong [[\star, X]]$. Using corollary 5.2.4, it follows that $H_0(X)$ is identified with the free abelian group generated by the set of path components of X. For example, if X is path connected, then $H_0(X)$ is isomorphic to $\mathbb{Z}$.
By a very similar calculation, $H^0(X)$ is isomorphic to $[[X, \star]]$. Using proposition 5.2.1, we then obtain that $H^0(X)$ is isomorphic to the abelian group of continuous maps from X to $\mathbb{Z}$. For example, if X is connected, then $H^0(X)$ is isomorphic to $\mathbb{Z}$.

Example 5.3.3. Take n arbitrary and $X = \star$. Now $H_n(\star) = [[S^n, \star]]/[[\star, \star]]$. Using proposition 5.2.1, we find $[[S^n, \star]] \cong \mathbb{Z}$ when $n > 0$ and $[[S^0, \star]] \cong \mathbb{Z} \oplus \mathbb{Z}$; also $[[\star, \star]] = \mathbb{Z}$. By an easy calculation, the quotient $[[S^n, \star]]/[[\star, \star]]$ is therefore 0 when $n > 0$, and isomorphic to $\mathbb{Z}$ when $n = 0$. So we have:

$$H_n(\star) \quad \cong \quad \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

Similarly, $H^n(\star) = [[\star, S^n]]/[[\star, \star]]$. Using corollary 5.2.4 this time, we find that $[[\star, S^n]] \cong \mathbb{Z}$ when $n > 0$ and $[[\star, S^0]] \cong \mathbb{Z} \oplus \mathbb{Z}$. By an easy calculation, the quotient $[[\star, S^n]]/[[\star, \star]]$ is therefore 0 when $n > 0$, and isomorphic to $\mathbb{Z}$ when $n = 0$. Therefore:

$$H^n(\star) \quad \cong \quad \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

Remark 5.3.4. For inductive arguments, it is often convenient to identify the sphere S^n in the definition of $H_n(X)$ or $H^n(X)$ with the one-point compactification $\mathbb{R}^n \cup \{\infty\}$ of $\mathbb{R}^n$. For me the preferred identification is a homeomorphism from $\mathbb{R}^n \cup \{\infty\}$ to S^n given by a form of stereographic projection which takes the origin $(0, 0, \ldots, 0)$ to $(1, 0, \ldots, 0)$ and which takes ∞ to $(-1, 0, \ldots, 0)$. In somewhat more detail, this takes $(x_1, \ldots, x_n) \in \mathbb{R}^n$ to the (other) point of S^n where the unique straight line through $(-1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$

and $(1, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ meets $S^n \subset \mathbb{R}^{n+1}$. In $\mathbb{R}^n \cup \{\infty\}$, the preferred choice of base point is the point ∞. *Amazing corollary*: our preferred choice of base point in S^n is $(-1, 0, 0, \ldots, 0)$.
Some important special cases: $\mathbb{R}^0 \cup \{\infty\} = \{0, \infty\}$ is identified with $S^0 = \{-1, 1\}$ by $0 \mapsto 1$ and $\infty \mapsto -1$. And $\mathbb{R}^1 \cup \{\infty\}$ is identified with S^1 by

$$x \mapsto \left(\frac{4 - x^2}{4 + x^2}, \frac{4x}{4 + x^2}\right)$$

for $x \in \mathbb{R}^1$. Note that this last identification is differentiable on $\mathbb{R}^1$ and respects the standard orientations.

CHAPTER 6

The homotopy decomposition theorem and the Mayer-Vietoris sequence

6.1. The homotopy decomposition theorem

Notation for the following theorem and the corollary: X and Y are topological spaces, V and W are open subsets of Y such that $V \cup W = Y$, and C is a closed subset of X. We assume that X is *paracompact*.

THEOREM 6.1.1. *Let $\gamma : X \times [0,1] \to Y$ be a mapping cycle which restricts to zero on an open neighborhood of $X \times \{0\}$. Then there exists a decomposition*

$$\gamma = \gamma^V + \gamma^W \ ,$$

where $\gamma^V : X \times [0,1] \to V$ and $\gamma^W : X \times [0,1] \to W$ are mapping cycles, both zero on an open neighborhood of $X \times \{0\}$. If γ is zero on some neighborhood of $C \times [0,1]$, then it can be arranged that γ^V and γ^W are zero on a neighborhood of $C \times [0,1]$.

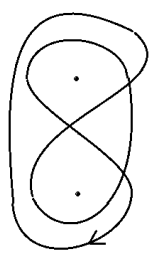
The proof of this is hard. We postpone it.

COROLLARY 6.1.2. *Let $a \in [[X,V]]$ and $b \in [[X,W]]$ be such that the images of a and b in $[[X,Y]]$ agree. Then there exists $c \in [[X, V \cap W]]$ whose image in $[[X,V]]$ is a and whose image in $[[X,W]]$ is b.*

PROOF. Let α be a mapping cycle which represents a and let β be a mapping cycle which represents b. Choose a mapping cycle $\gamma : X \times [0,1] \to Y$ which is a homotopy from 0 to $\beta - \alpha$. It is easy to arrange this in such a way that γ is zero on a neighborhood of $X \times \{0\}$. Use the theorem to obtain a decomposition $\gamma = \gamma^V + \gamma^W$. Let γ_1^V and γ_1^W be the restrictions of γ^V and γ^W to $X \times \{1\}$. Then α and $\alpha + \gamma_1^V$ are homotopic as mapping cycles $X \to V$, by the homotopy $\alpha \circ p + \gamma^V$, where p is the projection $X \times [0,1] \to X$. Similarly $\beta = \alpha + \gamma_1^V + \gamma_1^W$ and $\alpha + \gamma_1^V$ are homotopic as mapping cycles $X \to W$. Finally, $\alpha + \gamma_1^V = \beta - \gamma_1^W$ lands in $V \cap W$ by construction. So $c = [[\alpha + \gamma_1^V]]$ is a solution. $\square$

REMARK 6.1.3. The corollary is in a formal way very reminiscent of proposition 2.5.5. However the assumptions there were somewhat different. Instead of a union-intersection square of spaces serving as targets, we had a pullback square and a fibration condition. We can ask whether that was necessary or appropriate. Does corollary 6.1.2 have a more direct analogue in $\mathcal{H}o\mathcal{T}op$? In other words, given spaces X and $Y = V \cup W$ as in corollary 6.1.2, and elements $a \in [X,V]$ and $b \in [X,W]$ such that the images of a and b in $[X,Y]$ agree, does there exist $c \in [X, V \cap W]$ whose image in $[X,V]$ is a and whose image in $[X,W]$ is b ? Interestingly the answer is no in general. A relatively easy counterexample (easier for you if you know the concept *fundamental group*) can be constructed as follows. Let $p, q \in \mathbb{R}^2$, $p = (0,1)$ and $q = (0,-1)$. Let $Y = \mathbb{R}^2 \smallsetminus \{q\}$, $V = \mathbb{R}^2 \smallsetminus \{p,q\}$ and W the open upper half-plane. Then $V \cap W = W \smallsetminus \{p\}$. For X take S^1. It is rather easy to

invent $a \in [X, V]$ which maps to the zero element in $[X, Y]$, but which does not come from $[X, V \cap W]$. Therefore if we take $b \in [X, W]$ to be the class of the constant map, we have a "situation". Picture of a map in the homotopy class a:



There are also deeper counterexamples where $X = S^n$ for some $n > 1$. For those we need to work harder.

6.2. The Mayer-Vietoris sequence in homology

A sequence of abelian groups $(A_n)_{n\in\mathbb{Z}}$ together with homomorphisms

$$f_n \colon A_n \to A_{n-1}$$

for all $n \in \mathbb{Z}$ is called an *exact sequence of abelian groups* if the kernel of f_n is equal to the image of f_{n+1} , for all $n \in \mathbb{Z}$. More generally, we sometimes have to deal with diagrams of abelian groups and homomorphisms in the shape of a string

$$A_n \to A_{n-1} \to A_{n-2} \to \cdots \to A_{n-k} \ .$$

Such a diagram is *exact* if the kernel of each homomorphism in the string is equal to the image of the preceding one, if there is a preceding one.

Definition 6.2.1. (*Alternative definition of homology.*) For a space Y, and $n \geq 0$, redefine $H_n(Y)$ as the abelian group of homotopy classes of mapping cycles $\mathbb{R}^n \to Y$ with compact support (i.e., mapping cycles which are zero on the complement of a compact subset of $\mathbb{R}^n$).

Comment. Quite generally, the *support* of a mapping cycle $f \colon X \to Y$ is a closed subset of X, the complement of the largest subset U of X such that $f|_U$ is zero. — In the above definition, we regard two mapping cycles $\mathbb{R}^n \to Y$ with compact support as *homotopic* if they are related by a homotopy $\mathbb{R}^n \times [0, 1] \to Y$ which has compact support.
To relate the old definition of $H_n(Y)$ to the new one, we make a few observations. Given a mapping cycle $\alpha \colon \mathbb{R}^n \to Y$ which has compact support we immediately obtain a mapping cycle (of the same name) from $\mathbb{R}^n \cup \{\infty\}$ to Y by extending trivially to ∞. To view this as a mapping cycle $S^n \to Y$, we need to use our preferred identification of S^n with $\mathbb{R}^n \cup \{\infty\}$. See remark 5.3.4. Conversely, given a mapping cycle $\beta \colon S^n \to Y$ representing an element of $H_n(Y)$ according to the old definition, we may subtract a suitable constant to arrange that β is zero when restricted to the base point of S^n. We can also assume that β is zero on a neighborhood of the base point; if not, compose with a continuous map $S^n \to S^n$ which is homotopic to the identity and takes a neighborhood of the base point to the base

point. Using the standard identification $S^n \cong \mathbb{R}^n \cup \{\infty\}$, we can view $\beta \circ u$ as a mapping cycle $\mathbb{R}^n \cup \{\infty\} \to Y$ and also as a mapping cycle $\mathbb{R}^n \to Y$ with compact support.

Definition 6.2.2. Suppose that Y comes with two open subspaces V and W such that $V \cup W = Y$. The *boundary homomorphism*

$$\partial : H_n(Y) \to H_{n-1}(V \cap W)$$

is defined as follows, using the alternative definition of H_n. Let $x \in H_n(Y)$ be represented by a mapping cycle $\gamma : \mathbb{R}^n \to Y$ with compact support. Without loss of generality (see remark 6.2.3), the support of γ is contained in $]0,1[\times\mathbb{R}^n$. Then we can think of γ as a homotopy with compact support, $\gamma : [0,1] \times \mathbb{R}^n \to Y$. (Here I want the $[0,1]$ factor on the left for bureaucratic reasons; for now let's regard this as unimportant.) Choose a decomposition $\gamma = \gamma^V + \gamma^W$ as in theorem 6.1.1. The theorem guarantees that γ^V and γ^W can be arranged to have compact support as well. Let $\partial(x)$ be the class of the mapping cycle

$$\gamma_1^V : \mathbb{R}^{n-1} \to V \cap W\,,$$

composition of γ^V with the map $(z_1, \ldots, z_{n-1}) \mapsto (1, z_1, \ldots, z_{n-1})$. Note that γ_1^V has again compact support.

We must show that this is well defined. There were two choices involved: the choice of representative γ, with compact support in $]0,1[\times\mathbb{R}^n$, and the choice of decomposition $\gamma = \gamma^V + \gamma^W$. For the moment, keep γ fixed, and let us see what happens if we try another decomposition of γ. Any other decomposition will have the form

$$(\gamma^V + \eta) + (\gamma^W - \eta)$$

where $\eta : [0,1] \times \mathbb{R}^{n-1} \longrightarrow V \cap W$ is a mapping cycle with compact support, and the support has empty intersection with $\{0\} \times \mathbb{R}^{n-1}$. We need to show that $\gamma_1^V + \eta_1$ is homotopic (with compact support) to γ_1^V. But this is clear since η_1 is homotopic to zero by the homotopy η.

Next we worry about the choice of representative γ. Let φ be another representative of the same class x, also with compact support in $]0,1[\times\mathbb{R}^n$. Let $\lambda : \mathbb{R}^n \times [0,1] \to Y$ be a homotopy from φ to γ with compact support. (Writing the factor $[0,1]$ on the right might help us to avoid confusion.) Without loss of generality the support of λ is contained in $]0,1[\times\mathbb{R}^n \times [0,1]$. We can therefore think of λ as a homotopy in a different way:

$$[0,1] \times \big(\mathbb{R}^n \times [0,1]\big) \longrightarrow Y\,.$$

Then we can apply the homotopy decomposition theorem and choose a decomposition $\lambda = \lambda^V + \lambda^W$ where λ^V and λ^W have compact support. We then find that λ_1^V is a mapping cycle from $\mathbb{R}^{n-1} \times [0,1]$ to $V \cap W$ which we may regard as a homotopy (now with parameters written on the right). The homotopy is between γ_1^V and φ_1^V, provided the decompositions $\gamma = \gamma^V + \gamma^W$ and $\varphi = \varphi^V + \varphi^W$ are the ones obtained by restricting the decomposition $\lambda = \lambda^V + \lambda^W$. □

Remark 6.2.3. Let K be a compact subset of $\mathbb{R}^n$. Then it is easy to construct a homotopy

$$(h_t : \mathbb{R}^n \to \mathbb{R}^n)_{t\in[0,1]}$$

such that $h_0 = \mathrm{id}$ and $h_1^{-1}(K)$ is contained in $]0,1[\times\mathbb{R}^{n-1}$, and $h_t(z) = z$ for all $t \in [0,1]$ and all z outside a compact subset of $\mathbb{R}^n$. So if K is the support of a mapping cycle γ: $\mathbb{R}^n \to Y$, then $\gamma \circ h_1$ has compact support contained in $]0,1[\times\mathbb{R}^n$. Moreover there is a homotopy with compact support relating γ to $\gamma \circ h_1$.

The boundary homomorphisms ∂ can be used to make a sequence of abelian groups and homomorphisms

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(Y) \arrow[d, "\partial"] & & \\
& H_n(V \cap W) \arrow[r] & H_n(V) \oplus H_n(W) \arrow[r] & H_n(Y) \arrow[d, "\partial"] \\
& & & H_{n-1}(V \cap W) \arrow[r] & \cdots
\end{tikzcd}$$

where $n \in \mathbb{Z}$. (Set $H_n(X) = 0$ for $n < 0$ and any space X. The unlabelled homomorphisms in the sequence are as follows: $H_n(V) \oplus H_n(W) \to H_n(Y)$ is $j_{V*} + j_{W*}$, the sum of the two maps given by composition with the inclusions $j_V \colon V \to Y$ and $j_W \colon W \to Y$, and $H_n(V \cap W) \to H_n(V) \oplus H_n(W)$ is $(e_{V*}, -e_{W*})$, where e_{V*} and e_{W*} are given by composition with the inclusions $e_V \colon V \cap W \to V$ and $e_W \colon V \cap W \to W$.) The sequence is called the homology *Mayer-Vietoris* sequence of Y and V, W.

THEOREM 6.2.4. *The homology Mayer-Vietoris sequence of* Y *and* V, W *is exact.*[1]

Terminology for the proof. Write $I = [0, 1]$. Let X and Q be topological spaces and let $h \colon I \times X \to Q$ be a map or mapping cycle (which we think of as a homotopy). Let $p \colon I \times X \to X$ be the projection and let $\iota_0, \iota_1 \colon X \to I \times X$ be the maps given by $x \mapsto (0, x)$ and $x \mapsto (1, x)$, respectively. We say that h is *stationary* near $\{0, 1\} \times X$ if there exist open neighborhoods U_0 and U_1 of $\{0\} \times X$ and $\{1\} \times X$, respectively, in $I \times X$ such that h agrees with $h \circ \iota_0 \circ p$ on U_0 and with $h \circ \iota_1 \circ p$ on U_1.

PROOF. (i) Exactness of the pieces $H_n(V \cap W) \to H_n(V) \oplus H_n(W) \to H_n(Y)$ follows from corollary 6.1.2, for all $n \in \mathbb{Z}$. (It is more convenient to use the standard definition of H_n at this point.) More precisely, we have exactness of

$$[[S^n, V \cap W]] \to [[S^n, V]] \oplus [[S^n, W]] \to [[S^n, Y]]$$

by corollary 6.1.2, and we have exactness of

$$[[\star, V \cap W]] \to [[\star, V]] \oplus [[\star, W]] \to [[\star, Y]]$$

by corollary 6.1.2. Note also that $[[\star, V]] \oplus [[\star, W]] \to [[\star, Y]]$ is surjective. Then it follows easily that

$$\frac{[[S^n, V \cap W]]}{[[\star, V \cap W]]} \to \frac{[[S^n, V]] \oplus [[S^n, W]]}{[[\star, V]] \oplus [[\star, W]]} \to \frac{[[S^n, Y]]}{[[\star, Y]]}$$

is exact.
(ii) Next we look at pieces of the form

$$H_n(V) \oplus H_n(W) \longrightarrow H_n(Y) \xrightarrow{\partial} H_{n-1}(V \cap W) \ .$$

The cases $n < 0$ are trivial. In the case $n = 0$, the claim is that the homomorphism $H_0(V) \oplus H_0(W) \to H_0(Y)$ is surjective. This is a pleasant exercise. Now assume $n > 0$. It is clear from the definition of ∂ that the composition of the two homomorphisms is zero. Suppose then that $[[\gamma]] \in H_n(Y)$ is in the kernel of ∂. Here $\gamma : \mathbb{R}^n \to Y$ is a mapping cycle

[1] If you wish, view this as a sequence of abelian groups and homomorphisms indexed by the integers, by setting for example $A_{3n} = H_n(Y)$ for $n \geq 0$, $A_{3n+1} = H_n(V) \oplus H_n(W)$ for $n \geq 0$, $A_{3n+2} = H_n(V \cap W)$ for $n \geq 0$, and $A_m = 0$ for all $m \leq 0$.

with compact support contained in $]0,1[\times\mathbb{R}^{n-1}$. We must show that $[[\gamma]]$ is in the image of $H_n(V)\oplus H_n(W)\to H_n(Y)$. As above, we think of γ as a homotopy, $I\times\mathbb{R}^{n-1}\to Y$, which we decompose, $\gamma=\gamma^V+\gamma^W$ as in theorem 6.1.1, where γ^V and γ^W have compact support. The assumption $\partial[[\gamma]]=0$ then means that the zero map

$$\mathbb{R}^{n-1}\to V\cap W$$

is homotopic to γ_1^V by a homotopy $\lambda: I\times\mathbb{R}^{n-1}\to V\cap W$ with compact support. We can arrange that λ is stationary near $\{0,1\}\times\mathbb{R}^{n-1}$. Then $\gamma^V+\lambda$ and $\gamma^W-\lambda$ are mapping cycles from $I\times\mathbb{R}^{n-1}$ to V and W, respectively. Both vanish outside a compact subset of $]0,1[\times\mathbb{R}^{n-1}$ and so can be viewed as mapping cycles with compact support defined on all of $\mathbb{R}^n$. Hence they represent elements in $H_n(V)$ and $H_n(W)$ whose images in $H_n(Y)$ add up to $[[\gamma]]$.
(iii) We show that the composition

$$H_{n+1}(Y)\xrightarrow{\partial}H_n(V\cap W)\longrightarrow H_n(V)\oplus H_n(W)\ .$$

is zero. We can assume $n\geq 0$. Represent an element in $H_{n+1}(Y)$ by a mapping cycle $\gamma:\mathbb{R}^{n+1}\to Y$ with compact support contained in $]0,1[\times\mathbb{R}^n$. Decompose as usual, and obtain $\partial[[\gamma]]=[\gamma_1^Y]$. Now $\gamma_1^V=-\gamma_1^W$ viewed as a mapping cycle $\mathbb{R}^n\to V$ with compact support is homotopic to zero by the homotopy γ^V. Therefore $\partial[\gamma]$ maps to zero in $H_n(V)$. A similar calculation shows that it maps to zero in $H_n(W)$.
(iv) Finally let $\varphi:\mathbb{R}^n\to V\cap W$ be a mapping cycle with compact support and suppose that $[[\varphi]]\in H_n(V\cap W)$ is in the kernel of the homomorphism $H_n(V\cap W)\to H_n(V)\oplus H_n(W)$. Choose a homotopy $\gamma^V: I\times\mathbb{R}^n\to V$ from zero to φ, and choose another homotopy $\gamma^W: I\times\mathbb{R}^n\to W$ from zero to $-\varphi$, both with compact support and both stationary near $\{0,1\}\times\mathbb{R}^n$. Then $\gamma:=\gamma^V+\gamma^W$ has compact support contained in $]0,1[\times\mathbb{R}^n$ and so can be viewed as a mapping cycle with compact support defined on all of $\mathbb{R}^{n+1}$. As such it represents a class $[[\gamma]]\in H_{n+1}(Y)$. It is clear that $\partial[[\gamma]]=[[\varphi]]$. □

Remark 6.2.5. The Mayer-Vietoris sequence has a naturality property. The statement is complicated. Suppose that Y and Y' are topological spaces, $g: Y\to Y'$ is a continuous map, $Y=V\cup W$ where V and W are open subsets, $Y'=V'\cup W'$ where V'' and W' are open subsets, $g(V)\subset V'$ and $g(W)\subset W'$. Then the Mayer-Vietoris sequences for Y,V,W

and Y', V', W' can be arranged in a ladder-shaped diagram

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
H_{n+1}(Y) \arrow[r, "g_*"] \arrow[d, "\partial"] & H_{n+1}(Y') \arrow[d, "\partial"] \\
H_n(V \cap W) \arrow[r, "g_*"] \arrow[d] & H_n(V' \cap W') \arrow[d] \\
H_n(V) \oplus H_n(W) \arrow[r, "g_*"] \arrow[d] & H_n(V') \oplus H_n(W') \arrow[d] \\
H_n(Y) \arrow[r, "g_*"] \arrow[d, "\partial"] & H_n(Y') \arrow[d, "\partial"] \\
H_{n-1}(V \cap W) \arrow[r, "g_*"] \arrow[d] & H_{n-1}(V' \cap W') \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

This diagram is *commutative*; that is the naturality statement. The proof is not complicated (it is by inspection).

Often this can be usefully combined with the following observation: if, in the Mayer-Vietoris sequence for Y and V, W we interchange the roles (order) of V and W, then the homomorphisms ∂ and $H_n(V \cap W) \to H_n(V) \oplus H_n(W)$ change sign. To be more precise, we set up a diagram

$$\begin{tikzcd}
H_{n+1}(Y) \arrow[r, "="] \arrow[d, "\partial"] & H_{n+1}(Y) \arrow[d, "\partial"] \\
H_n(V \cap W) \arrow[r, "="] \arrow[d] & H_n(W \cap V) \arrow[d] \\
H_n(V) \oplus H_n(W) \arrow[r, "\cong"] & H_n(W) \oplus H_n(V)
\end{tikzcd}$$

where the columns are bits from the Mayer-Vietoris sequence of Y, V, W and Y, W, V, respectively. The diagram is *not* (always) commutative; instead each of the small squares in it commutes up to a factor (-1). The proof is by inspection.

CHAPTER 7

Homology of spheres and applications

7.1. Homology of spheres

PROPOSITION 7.1.1. *The homology groups of* S^1 *are* $H_0(S^1) \cong \mathbb{Z}$, $H_1(S^1) \cong \mathbb{Z}$ *and* $H_k(S^1) = 0$ *for all* $k \neq 0, 1$.

PROOF. Choose two distinct points p and q in S^1. Let $V \subset S^1$ be the complement of p and let $W \subset S^1$ be the complement of q. Then $V \cup W = S^1$. Clearly V is homotopy equivalent to a point, W is homotopy equivalent to a point and $V \cap W$ is homotopy equivalent to a discrete space with two points. Therefore $H_k(V) \cong H_k(W) \cong \mathbb{Z}$ for $k = 0$ and $H_k(V) \cong H_k(W) = 0$ for all $k \neq 0$. Similarly $H_k(V \cap W) \cong \mathbb{Z} \oplus \mathbb{Z}$ for $k = 0$ and $H_k(V \cap W) = 0$ for all $k \neq 0$. The exactness of the Mayer-Vietoris sequence associated with the open covering of S^1 by V and W implies immediately that $H_k(S^1) = 0$ for $k \neq 0, 1$. The part of the Mayer-Vietoris sequence which remains interesting after this observation is

$$0 \longrightarrow H_1(S^1) \xrightarrow{\partial} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H_0(S^1) \longrightarrow 0$$

Since S^1 is path-connected, the group $H_0(S^1)$ is isomorphic to $\mathbb{Z}$. The homomorphism from $\mathbb{Z} \oplus \mathbb{Z}$ to $H_0(S^1)$ is onto by exactness, so its kernel is isomorphic to $\mathbb{Z}$. Hence the image of the homomorphism $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is isomorphic to $\mathbb{Z}$, so its kernel is again isomorphic to $\mathbb{Z}$. Now exactness at $H_1(S^1)$ leads to the conclusion that $H_1(S^1) \cong \mathbb{Z}$. $\square$

THEOREM 7.1.2. *The homology groups of* S^n *(for* $n > 0$*) are*

$$H_k(S^n) \cong \begin{cases} \mathbb{Z} & \text{if } k = n \\ \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. We proceed by induction on n. The induction beginning is the case $n = 1$ which we have already dealt with separately in proposition 7.1.1. For the induction step, suppose that $n > 1$. We use the Mayer-Vietoris sequence for S^n and the open covering $\{V, W\}$ with $V = S^n \setminus \{p\}$ and $W = S^n \setminus \{q\}$ where $p, q \in S^n$ are the north and south pole, respectively. We will also use the homotopy invariance of homology. This gives us

$$H_k(V) \cong H_k(W) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

because V and W are homotopy equivalent to a point. Also we get

$$H_k(V \cap W) \cong \begin{cases} \mathbb{Z} & \text{if } k = n-1 \\ \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

by the induction hypothesis, since $V \cap W$ is homotopy equivalent to S^{n-1}. Furthermore it is clear what the inclusion maps $V \cap W \to V$ and $V \cap W \to W$ induce in homology:

an isomorphism in H_0 and (necessarily) the zero map in H_k for all $k \neq 0$. Thus the homomorphism

$$H_k(V \cap W) \longrightarrow H_k(V) \oplus H_k(W)$$

from the Mayer-Vietoris sequence takes the form

$$\mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$$

when $k = 0$, and

$$\mathbb{Z} \longrightarrow 0$$

when $k = n - 1$. In all other cases, its source and target are both zero. Therefore the exactness of the Mayer-Vietoris sequence implies that $H_0(S^n)$ and $H_n(S^n)$ are isomorphic to $\mathbb{Z}$, while $H_k(S^n) = 0$ for all other $k \in \mathbb{Z}$. □

THEOREM 7.1.3. *Let* $f\colon S^n \to S^n$ *be the antipodal map. The induced homomorphism* $f_*\colon H_n(S^n) \to H_n(S^n)$ *is multiplication by* $(-1)^{n+1}$.

PROOF. We proceed by induction again. For the induction beginning, we take $n = 1$. The antipodal map $f : S^1 \to S^1$ is homotopic to the identity, so that $f^* : H_1(S^1) \to H_1(S^1)$ has to be the identity, too. For the induction step, we use the setup and notation from the previous proof. Exactness of the Mayer-Vietoris sequence for S^n and the open covering $\{V, W\}$ shows that

$$\partial : H_n(S^n) \longrightarrow H_{n-1}(V \cap W)$$

is an isomorphism. The diagram

$$\begin{tikzcd} H_n(S^n) \arrow[r, "\partial"] \arrow[d, "f_*"'] & H_{n-1}(V \cap W) \arrow[d, "f_*"'] \\ H_n(S^n) \arrow[r, "\partial"] & H_{n-1}(W \cap V) \end{tikzcd}$$

is meaningful because f takes $V \cap W$ to $V \cap W = W \cap V$. But the diagram is not commutative (i.e., it is not true that $f_* \circ \partial$ equals $\partial \circ f_*$). The reason is that f interchanges V and W, and it does matter in the Mayer-Vietoris sequence which of the two comes first. Therefore we have instead

$$f_* \circ \partial = -\partial \circ f_*$$

in the above square. By the inductive hypothesis, the f_* in the left-hand column of the square is multiplication by $(-1)^n$, and therefore the f^* in the right-hand column of the square must be multiplication by $(-1)^{n+1}$. □

7.2. The usual applications

THEOREM 7.2.1. (Brouwer's fixed point theorem). *Let* $f : D^n \to D^n$ *be a continuous map, where* $n \geq 1$. *Then* f *has a fixed point, i.e., there exists* $y \in D^n$ *such that* $f(y) = y$.

PROOF. Suppose for a contradiction that f does not have a fixed point. For $x \in D^n$, let $g(x)$ be the point where the ray (half-line) from $f(x)$ to x intersects the boundary S^{n-1} of the disk D^n. Then g is a continuous map from D^n to S^{n-1}, and we have $g|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}$. Summarizing, we have

$$S^{n-1} \xrightarrow{j} D^n \xrightarrow{g} S^{n-1}$$

where j is the inclusion, $g \circ j = \mathrm{id}$. Therefore we get

$$H_{n-1}(S^{n-1}) \xrightarrow{j_*} H_{n-1}(D^n) \xrightarrow{g_*} H_{n-1}(S^{n-1})$$

where $g_* j_* = \mathrm{id}$. Thus the abelian group $H_{n-1}(S^{n-1})$ is isomorphic to a direct summand of $H_{n-1}(D^n)$. But from our calculations above, we know that this is not true. If $n > 1$ we have $H_{n-1}(D^n) = 0$ while $H_{n-1}(S^{n-1})$ is not trivial. If $n = 1$ we have $H_{n-1}(D^n) \cong \mathbb{Z}$ while $H_{n-1}(S^{n-1}) \cong \mathbb{Z} \oplus \mathbb{Z}$. □

Let $f\colon S^n \to S^n$ be any continuous map, $n > 0$. The induced homomorphism f_* from $H_n(S^n)$ to $H_n(S^n)$ is multiplication by some number $n_f \in \mathbb{Z}$, since $H_n(S^n)$ is isomorphic to $\mathbb{Z}$.

Definition 7.2.2. The number n_f is the *degree* of f.

Remark. The degree n_f of $f\colon S^n \to S^n$ is clearly an invariant of the homotopy class of f.

Remark. In the case $n = 1$, the definition of degree as given just above agrees with the definition of degree given in section 1. See exercises.

Example 7.2.3. According to theorem 7.1.3, the degree of the antipodal map $S^n \to S^n$ is $(-1)^{n+1}$.

Proposition 7.2.4. *Let $f\colon S^n \to S^n$ be a continuous map. If $f(x) \neq x$ for all $x \in S^n$, then f is homotopic to the antipodal map, and so has degree $(-1)^{n+1}$. If $f(x) \neq -x$ for all $x \in S^n$, then f is homotopic to the identity map, and so has degree 1.*

Proof. Let $g\colon S^n \to S^n$ be the antipodal map, $g(x) = -x$ for all x. Assuming that $f(x) \neq x$ for all x, we show that f is homotopic to g. We think of S^n as the unit sphere in $\mathbb{R}^{n+1}$, with the usual notion of distance. We can make a homotopy $(h_t : S^n \to S^n)_{t\in[0,1]}$ from f to g by "sliding" along the unique minimal geodesic arc from $f(x)$ to $g(x)$, for every $x \in S^n$. In other words, $h_t(x) \in S^n$ is situated $t \cdot 100$ percent of the way from $f(x)$ to $g(x)$ along the minimal geodesic arc from $f(x)$ to $g(x)$. (The important thing here is that $f(x)$ and $g(x)$ are not antipodes of each other, by our assumptions. Therefore that minimal geodesic arc is unique.)
Next, assume $f(x) \neq -x$ for all $x \in S^n$. Then, for every x, there is a unique minimal geodesic from x to $f(x)$, and we can use that to make a homotopy from the identity map to f. □

Corollary 7.2.5. (Hairy ball theorem). *Let ξ be a tangent vector field* (explanations follow) *on S^n. If $\xi(z) \neq 0$ for every $z \in S^n$, then n is odd.*

Comments. A tangent vector field on $S^n \subset \mathbb{R}^{n+1}$ can be defined as a continuous map ξ from S^n to the vector space $\mathbb{R}^{n+1}$ such that $\xi(x)$ is perpendicular to (the position vector of) x, for every $x \in S^n$. We say that vectors in $\mathbb{R}^{n+1}$ which are perpendicular to $x \in S^n$ are *tangent* to S^n at x because they are the velocity vectors of smooth curves in $S^n \subset \mathbb{R}^n$ as they pass through x.

Proof. Define $f : S^n \to S^n$ by $f(x) = \xi(x)/\|\xi(x)\|$. Then $f(x) \neq x$ and $f(x) \neq -x$ for all $x \in S^n$, since $f(x)$ is always perpendicular to x. Therefore f is homotopic to the antipodal map, and also homotopic to the identity. It follows that the antipodal map is homotopic to the identity. Therefore n is odd by theorem 7.1.3. □

Remark 7.2.6. Theorem 7.1.3 has an easy generalization which says that the degree of the map $f\colon S^n \to S^n$ given by

$$(x_1, x_2, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_k, -x_{k+1}, \ldots, -x_{n+1})$$

is $(-1)^{n+1-k}$. Here we assume $n \geq 1$ as usual. The proof can be given by induction on $n+1-k$. The induction step is now routine, but the induction beginning must cover all cases where $n = 1$. This leaves the three possibilities $k = 0, 1, 2$. One of these gives the identity map $S^1 \to S^1$, and another gives the antipodal map $S^1 \to S^1$ which is homotopic to the identity. The interesting case which remains is the map $f\colon S^1 \to S^1$ given by $f(x_1, x_2) = (x_1, -x_2)$. We need to show that it has degree -1, in the sense of definition 7.2.2. One way to do this is to use the following diagram

$$\begin{tikzcd} H_1(S^1) \arrow[r, "f_*"] \arrow[d, "\partial"] & H_1(S^1) \arrow[d, "\partial"] \\ H_0(V \cap W) \arrow[r, "f_*"] & H_0(W \cap V) \end{tikzcd}$$

where $V = S^1 \smallsetminus \{(0,1)\}$ and $W = S^1 \smallsetminus \{(0,-1)\}$. We know from the previous chapter that it commutes up to a factor (-1). In the lower row, we have the identity homomorphism

$$\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}.$$

The vertical arrows are injective (seen earlier in the proof of proposition 7.1.1). Therefore the upper horizontal arrow is multiplication by -1.

To state this result in a more satisfying manner, let us note that the orthogonal group $O(n+1)$ (the group of orthogonal $(n+1) \times (n+1)$-matrices with real entries) is a topological group which has two path components. The two path components are the preimages of $+1$ and -1 under the homomorphism

$$\det\colon O(n+1) \to \{-1, +1\}.$$

Let $f\colon S^n \to S^n$ be given by $f(z) = Az$ for some $A \in O(n+1)$. Because $\deg(f)$ depends only on the homotopy class of f, it follows that $\deg(f)$ depends only on the path component of A in $O(n+1)$, and hence only on $\det(A)$. What we have just shown means that $\deg(f)$ is *equal* to $\det(A)$.

Remark 7.2.7. In our definition of the degree of a map from S^n to S^n, where $n > 0$, we did not use a specific isomorphism from $H_n(S^n)$ to $\mathbb{Z}$ and we did not have to use one. It was enough to know that $H_n(S^n)$ is isomorphic to $\mathbb{Z}$. But it is possible to specify a preferred isomorphism from $H_n(S^n)$ to $\mathbb{Z}$ by saying that the continuous map $\mathrm{id}\colon S^n \to S^n$ viewed as a mapping cycle $S^n \to S^n$ and then as an element

$$[[\mathrm{id}]] \in \frac{[[S^n, S^n]]}{[[\star, S^n]]} = H_n(S^n)$$

shall correspond to $1 \in \mathbb{Z}$. There is something to prove, though: we must show that $[[\mathrm{id}]]$ is a generator of the abelian group $H_n(S^n) \cong \mathbb{Z}$. *Proof*: we observe that $H_n(S^n)$ is a ring R. (Represent elements by mapping cycles $S^n \to S^n$; addition of mapping cycles defines the addition in R and composition defines the multiplication. It takes a little work to show that composition is well defined.) Clearly $[[\mathrm{id}]]$ is the multiplicative unit of the ring R. If a ring R has underlying additive group isomorphic to $\mathbb{Z}$, then its unit element must be a generator of the underlying additive group.

It follows easily that the degree of a map $f\colon S^n \to S^n$ is equal to

$$[[f]] \in \frac{[[S^n, S^n]]}{[[\star, S^n]]} = H_n(S^n) = \mathbb{Z},$$

and here it is obviously important that we have selected an isomorphism $H_n(S^n) \to \mathbb{Z}$. $\square$

CHAPTER 8

Proving the homotopy decomposition theorem

8.1. Reductions

Here we reduce the proof of the homotopy decomposition theorem to the following lemmas.

LEMMA 8.1.1. *Let Z be a paracompact topological space, Y any topological space. Let $\beta\colon Z \times [0,1] \to Y$ be a mapping cycle. Write $\iota_0, \iota_1\colon Z \to Z \times [0,1]$ for the maps given by $\iota_0(z) = (z,0)$ and $\iota_1(z) = (z,1)$. If there exists a decomposition*

$$\beta \circ \iota_0 = \beta_0^V + \beta_0^W$$

where β_0^V and β_0^W are mapping cycles from Z to V and W, respectively, then there exists a decomposition $\beta \circ \iota_1 = \beta_1^V + \beta_1^W$.

LEMMA 8.1.2. *In the situation of lemma 8.1.1, every element of Z has an open neighborhood U such that the restriction $\beta_{U\times[0,1]}$ of β to $U \times [0,1]$ admits a decomposition*

$$\beta_{U\times[0,1]} = \beta^V_{U\times[0,1]} + \beta^W_{U\times[0,1]}$$

where $\beta^V_{U\times[0,1]}$ and $\beta^W_{U\times[0,1]}$ are mapping cycles from $U\times[0,1]$ to V and W, respectively.

SHOWING THAT LEMMA 8.1.2 IMPLIES LEMMA 8.1.1. In the situation of lemma 8.1.1, choose an open cover $(U_k)_{k\in\Lambda}$ such that the restriction $\beta_{[k]}$ of β to $U_k \times [0,1]$ admits a decomposition

$$\beta_{[k]} = \beta^V_{[k]} + \beta^W_{[k]} \ .$$

Such an open cover exists by lemma 8.1.2. Since Z is paracompact, there is no loss of generality in assuming that the open cover is locally finite. Moreover, there exists a partition of unity $(\varphi_k)_{k\in\Lambda}$ subordinate to the cover $(U_k)_{k\in\Lambda}$. Choose a total ordering of Λ. If Λ is finite, we can proceed as follows. We may assume that Λ is $\{1,2,3,\dots,m\}$ for some m, with the standard ordering. For $k \in \{0,1,\dots,m\}$ let

$$f_k\colon Z \to Z \times [0,1]$$

be the function $z \mapsto (z, \sum_{\ell=1}^{k} \varphi_\ell)$. Then $f_0 = \iota_0$ and $f_m = \iota_1$ in the notation of lemma 8.1.1. By induction on k we define a decomposition

$$\beta \circ f_k = (\beta \circ f_k)^V + (\beta \circ f_k)^W \ .$$

For $k = 0$ this decomposition (of $\beta \circ f_0 = \beta \circ \iota_0$) is already given to us. If we have constructed the decomposition for $\beta \circ f_{k-1}$, where $0 < k \le m$, we define it for $\beta \circ f_k$ in such a way that

$$(\beta \circ f_k)^V = (\beta \circ f_{k-1})^V + \beta^V_{[k]} \circ f_k - \beta^V_{[k]} \circ f_{k-1}$$

on $U_k \subset Z$ and $(\beta \circ f_k)^V = (\beta \circ f_{k-1})^V$ outside the support of φ_k. Similarly, define

$$(\beta \circ f_k)^W = (\beta \circ f_{k-1})^W + \beta^W_{[k]} \circ f_k - \beta^W_{[k]} \circ f_{k-1}$$

on U_k and $(\beta \circ f_k)^W = (\beta \circ f_{k-1})^W$ outside the support of φ_k. Then on U_k we have

$$(\beta \circ f_k)^V + (\beta \circ f_k)^W = \beta \circ f_{k-1} + \beta \circ f_k - \beta \circ f_{k-1} = \beta \circ f_k$$

and outside the support of φ_k we have

$$(\beta \circ f_k)^V + (\beta \circ f_k)^W = (\beta \circ f_{k-1})^V + (\beta \circ f_{k-1})^W = \beta \circ f_{k-1} = \beta \circ f_k \ .$$

Therefore $(\beta \circ f_k)^V + (\beta \circ f_k)^W = \beta \circ f_k$ as required. The case $k = m$ is the decomposition of $\beta \circ \iota_1 = \beta \circ f_m$ that we are after.

If Λ is not finite, we can proceed as follows. Choose $z \in Z$ and an open neighborhood Q of z in Z such that the set

$$J = \{ k \in \Lambda \mid Q \cap U_k \neq \emptyset \}$$

is finite. Now J is a finite set with a total ordering, and the φ_j where $j \in J$ constitute a partition of unity for Q, subordinate to the open cover $(U_k \cap Q)_{k \in J}$ of Q. Use this as above to find a decomposition of $\beta \circ \iota_1$, restricted to Q, into summands which are mapping cycles from Q to V and W, respectively. Do this for every z and open neighborhood Q. The decompositions obtained match on overlaps, and so define a decomposition of $\beta \circ \iota_1$ of the required sort. □

Showing that lemma 8.1.1 implies the homotopy decomposition theorem. Given X, Y and a mapping cycle $\gamma \colon X \times [0,1] \to Y$, we look for a decomposition $\gamma = \gamma^V + \gamma^W$ where γ^V and γ^W are mapping cycles from $X \times [0,1]$ to V and W, respectively. There is an additional condition to be satisfied. Namely, γ is zero on an open neighborhood U of $(X \times \{0\}) \cup (C \times [0,1])$ in $X \times [0,1]$, and we want γ^V, γ^W to be zero on some (perhaps smaller) open neighborhood U' of $(X \times \{0\}) \cup (C \times [0,1])$ in $X \times [0,1]$.

Put $Z = X \times [0,1]$. Since X was assumed to be paracompact, Z is also paracompact; it is a general topology fact that the product of a paracompact space with a compact Hausdorff space is paracompact. We have a map

$$h \colon Z \times [0,1] \to Z$$

defined by $h((x,s),t)) = (x, st)$ for $(x,t) \in X \times [0,1] = Z$ and $t \in [0,1]$. Now $\beta := \gamma \circ h$ is a mapping cycle from $Z \times [0,1]$ to Y. In the notation of lemma 8.1.1, we have

$$\beta \circ \iota_1 = \gamma, \qquad \beta \circ \iota_0 \equiv 0 \ .$$

There exists a decomposition $\beta_0 = \beta_0^V + \beta_0^W$ because we can take $\beta_0^V \equiv 0$ and $\beta_0^W \equiv 0$. Therefore, by lemma 8.1.1, there exists a decomposition $\beta \circ \iota_1 = \beta_1^V + \beta_1^W$, and we can write that in the form

$$\gamma = \beta_1^V + \beta_1^W \ .$$

This is a decomposition of the kind that we are looking for. Unfortunately there is no reason to expect that β_1^V, β_1^W are zero on $(X \times \{0\}) \cup (C \times [0,1])$, or on a neighborhood of that in $X \times [0,1]$.

But it is easy to construct a continuous map $\psi \colon X \times [0,1] \to X \times [0,1]$ such that $\psi(X \times [0,1])$ is contained in the open set U specified above, and such that ψ agrees with the identity on some open neighborhood U' of $(X \times \{0\}) \cup (C \times [0,1])$ in $X \times [0,1]$. Then obviously $U' \subset U$. Now let

$$\gamma^V = \beta_1^V - (\beta_1^V \circ \psi), \qquad \gamma^W = \beta_1^W - (\beta_1^W \circ \psi).$$

Then $\gamma^V + \gamma^W = (\beta_1^V + \beta_1^W) - (\beta_1^V + \beta_1^W) \circ \psi = \gamma - \gamma \circ \psi$. Furthermore $\gamma \circ \psi$ is zero because γ is zero on U and the image of ψ is contained in U. So $\gamma^V + \gamma^W = \gamma$. Also γ^V and γ^W are zero on U' by construction, since ψ agrees with the identity on U'. □

8.2. Local homotopy decomposition

Proof of lemma 8.1.2. Call an open subset P of $Z \times [0,1]$ *good* if the mapping cycle $\beta_{|P}$ from P to Y can be written as the sum of a mapping cycle from P to V and a mapping cycle from P to W. The goal is to show that every $z \in Z$ has an open neighborhood U such that $U \times [0,1]$ is good.
The proof is based on two observations.

- Every element of $Z \times [0,1]$ admits a good open neighborhood.
- If U is open in Z and A, B are open subsets of $[0,1]$ which are also intervals, and if $U \times A$ and $U \times B$ are both good, then $U \times (A \cup B)$ is good.

To prove the first observation, fix $(z,t) \in Z \times [0,1]$ and choose an open neighborhood Q of that in $Z \times [0,1]$ such that $\beta_{|Q}$ can be written as a formal linear combination, with coefficients in $\mathbb{Z}$, of continuous maps from Q to Y. Such a Q exists by the definition of *mapping cycle*. Making Q smaller if necessary, we can arrange that each of the (finitely many) continuous maps which appear in that formal linear combination is either a map from Q to V or a map from Q to W. It follows immediately that Q is good.
In proving the second observation, we can easily reduce to a situation where $A \cap B$ contains an element t_0, where $0 < t_0 < 1$, and $A \cup B$ is the union of $A \cap [0,t_0]$ and $B \cap [t_0,1]$. Choose a continuous map $\psi\colon B \to B \cap A$ such that $\psi(s) = s$ for all $s \in B \cap [0,t_0]$. Since $P := U \times A$ is good by assumption, we can write

$$\beta_{|P} = \beta^{V,P} + \beta^{W,P}$$

where the summands in the right-hand side are mapping cycles from P to V and from P to W, respectively. Similarly, letting $Q := U \times B$ we can write

$$\beta_{|Q} = \beta^{V,Q} + \beta^{W,Q}\,.$$

Let $\varphi\colon Q \to P \cap Q$ be given by $\varphi(z,t) = (z, \psi(t))$. Define $\beta^{V,P\cup Q}$, a mapping cycle from $P \cup Q$ to V, as follows:

$$\beta^{V,P\cup Q} = \begin{cases} \beta^{V,P} & \text{on} \quad P \cap (U \times [0,t_0[\,) \\ \beta^{V,Q} - (\beta^{V,Q} \circ \varphi) + (\beta^{V,P} \circ \varphi) & \text{on} \quad Q\,. \end{cases}$$

This is well defined because the two formulas agree on the intersection of Q and $U \times [0,t_0[$, where φ agrees with the identity. Similarly, define $\beta^{W,P\cup Q}$, a mapping cycle from $P \cup Q$ to W, as follows:

$$\beta^{W,P\cup Q} = \begin{cases} \beta^{W,P} & \text{on} \quad P \cap (U \times [0,t_0[\,) \\ \beta^{W,Q} - (\beta^{W,Q} \circ \varphi) + (\beta^{W,P} \circ \varphi) & \text{on} \quad Q\,. \end{cases}$$

An easy calculation shows that $\beta^{V,P\cup Q} + \beta^{W,P\cup Q} = \beta_{|P\cup Q}$. Therefore $P \cup Q = U \times (A \cup B)$ is good. The second observation is established.
Now fix $z_0 \in Z$. By the first of the observations, it is possible to choose for each $t \in [0,1]$ a good open neighborhood Q_t of (z_0,t) in $Z \times [0,1]$. By a little exercise, there exists an open neighborhood U of z_0 in Z and a small number $\delta = 1/n$ (where n is a positive integer) such that each of the open sets

$$U \times [0,2\delta[\,, \quad U \times]1\delta, 3\delta[\,, \quad U \times]2\delta, 4\delta[\,, \quad \ldots,$$

$$U \times]1-3\delta, 1-1\delta[\,, \quad U \times]1-2\delta, 1]$$

in $Z \times [0,1]$ is contained in Q_t for some $t \in [0,1]$. Therefore these open sets

$$U \times [0,2\delta[\,,\ U \times]1\delta, 3\delta[\,, \ldots$$

are also good. By the second of the two observations, applied $(n-2)$ times, their union, which is $U \times [0,1]$, is also good. □

8.3. Relationship with fiber bundles

The proof of the homotopy decomposition theorem as given above has many surprising similarities with proofs in section 3 related to fiber bundles (theorem 3.4, corollaries 3.7 and 3.8., and improvements in section 3.4). I cannot resist the temptation to indicate where these similarities come from.

Let E and B be topological spaces and let $p\colon E \to B$ be a fiber bundle. We need to be a little more precise by requiring that $p\colon E \to B$ be a fiber bundle *with fiber* F, for a fixed topological space F. This is supposed to mean that every fiber of p is homeomorphic to F in some way. (We learned in section 2 that every fiber bundle over a path connected space is a fiber bundle with fiber F, for some F.) With this situation we can associate two presheaves $\mathcal{T}$ and $\mathcal{H}_F$ on B.

- For an open set U in B, let $\mathcal{H}_F(U)$ be the group of homeomorphisms h from $U \times F$ to $U \times F$ respecting the projection to U.
- For an open set U in B let $\mathcal{T}(U)$ be the set of trivializations of the fiber bundle $E_{|U} \to U$, that is, the set of all homeomorphisms $p^{-1} \to U \times F$ respecting the projections to U.
- An inclusion of open sets $U_0 \hookrightarrow U_1$ in B induces maps

$$\mathcal{H}_F(U_1) \to \mathcal{H}_F(U_0), \qquad \mathcal{T}(U_1) \to sT(U_0)$$

  by restriction of homeomorphisms.

In fact it is clear that $\mathcal{T}$ and $\mathcal{H}_F$ are sheaves. Clearly $\mathcal{H}_F$ is a sheaf of groups, that is, each set $\mathcal{H}_F(U)$ comes with a group structure and the restriction maps $\mathcal{H}_F(U_1) \to \mathcal{H}_F(U_0)$ are group homomorphisms. By contrast $\mathcal{T}$ is not a sheaf of groups in any obvious way. But there is an *action* of the group $\mathcal{H}_F(U)$ on the set $\mathcal{T}(U)$ given by

$$(h, g) \mapsto h \circ g$$

(composition of homeomorphisms, where $h \in \mathcal{H}_F(U)$ and $g \in \mathcal{T}(U)$). This is compatible with restriction maps (reader, make this precise). Moreover:

(1) for any $g \in \mathcal{T}(U)$, the map $\mathcal{H}_F(U) \to \mathcal{T}(U)$ given by $h \mapsto h \circ g$ is a bijection;
(2) every $z \in B$ has an open neighborhood U such that $\mathcal{T}(U) \neq \emptyset$.

(Of course, despite (1), it can happen that $\mathcal{T}(U)$ is empty for some open subsets U of B, for example, $U = B$.) The proof of (1) is easy and by inspection; (2) holds by the definition of *fiber bundle*. There are words and expressions to describe this situation: we can say that $\mathcal{H}_F$ is a sheaf of groups on B and $\mathcal{T}$ is an $\mathcal{H}_F$*-torsor*.
This reasoning shows that a fiber bundle on B with fiber F determines an $\mathcal{H}_F$-torsor on B. It is also true (and useful, and not very hard to prove, though it will not be explained here) that the process can be reversed: every $\mathcal{H}_F$-torsor on B determines a fiber bundle with fiber F on B.

Now try to forget fiber bundles for a while. We return to the homotopy decomposition theorem. Assume that $Y = V \cup W$ as in the homotopy decomposition theorem. Let Z be any topological space and fix α, a mapping cycle from Z to Y. We introduce two presheaves $\mathcal{F}$ and $\mathcal{G}$ on Z.

- For an open set U in Z, let $\mathcal{G}(U)$ be the abelian group of mapping cycles from U to $V \cap W$.
- For open U in Z let $\mathcal{F}(U)$ be the set of mapping cycles β from U to V such that $\alpha_{|U} - \beta$ is a mapping cycle from U to W. To put it differently: an element β of $\mathcal{F}(U)$ is, or amounts to, a sum decomposition
$$\alpha_{|U} = \beta + (\alpha_{|U} - \beta)$$
where the two summands β and $\alpha_{|U} - \beta$ are mapping cycles from U to V and from U to W, respectively.
- An inclusion of open sets $U_0 \hookrightarrow U_1$ in Z induces maps
$$\mathcal{G}(U_1) \to \mathcal{G}(U_0), \qquad \mathcal{F}(U_1) \to sF(U_0)$$
by restriction of mapping cycles.

It is easy to see that $\mathcal{F}$ and $\mathcal{G}$ are *sheaves*, and $\mathcal{G}$ is even a sheaf of abelian groups on Z. By contrast $\mathcal{F}$ is not in an obvious way a sheaf of abelian groups. But there is an *action* of the group $\mathcal{G}(U)$ on the set $\mathcal{F}(U)$ given by
$$(\lambda, \beta) \mapsto \lambda + \beta\,.$$
(In this formula, $\lambda \in \mathcal{G}(U)$ and $\beta \in \mathcal{F}(U)$; then $\lambda + \beta$ can be viewed as a mapping cycle from U to V and it turns out to be an element of $\mathcal{F}(U)$.) Moreover:

(1) for any $\beta \in \mathcal{F}(U)$, the map $\mathcal{G}(U) \to \mathcal{F}(U)$ given by $\lambda \mapsto \lambda + \beta$ is a bijection;
(2) every $z \in Z$ has an open neighborhood U such that $\mathcal{F}(U) \neq \emptyset$.

(Of course it is quite possible, despite (1), that $\mathcal{F}(U)$ is empty for some open subsets U of Z, for example, $U = Z$.) The proof of (1) is easy and by inspection; the proof of (2) was given in a special case earlier, but it can be repeated. Choose a neighborhood U of z such that $\alpha_{|U}$ can be represented by a formal linear combination, with integer coefficients, of continuous maps from U to Y. Making U smaller if necessary, we can assume that each of the (finitely many) continuous maps which appear in that formal linear combination is either a map from U to V or a map from U to W. Then it is clear that $\alpha_{|U}$ can be written as a sum of two mapping cycles, one from U to V and the other from U to W. So $\mathcal{F}(U)$ is nonempty.

So we see that $\mathcal{G}$ is a sheaf of abelian groups on Z and $\mathcal{F}$ is a $\mathcal{G}$-torsor. Again we are interested in questions like this one: is $\mathcal{F}(Z)$ nonempty? This is equivalent to asking whether our fixed mapping cycle α from Z to Y can be written as a sum of two mapping cycles, one from Z to V and one from Z to W.

CHAPTER 9

Combinatorial description of some spaces

9.1. Vertex schemes and simplicial complexes

Definition 9.1.1. A *vertex scheme* consists of a set V and a subset $\mathcal{S}$ of the power set $\mathcal{P}(V)$, subject to the following conditions: every $T \in \mathcal{S}$ is finite and nonempty, every subset of V which has exactly one element belongs to $\mathcal{S}$, and if T' is a nonempty subset of some $T \in \mathcal{S}$, then $T' \in \mathcal{S}$.
The elements of V are called *vertices* (singular: *vertex*) of the vertex scheme. The elements of $\mathcal{S}$ are called *distinguished subsets* of V.

Example 9.1.2. The following are examples of vertex schemes:

(i) Let $V = \{1, 2, 3, \dots, 10\}$. Define $\mathcal{S} \subset \mathcal{P}(V)$ so that the elements of $\mathcal{S}$ are the following subsets of V: all the singletons, that is to say $\{1\}, \{2\}, \dots, \{10\}$, and $\{1, 2\}$, $\{2, 3\}$, ..., $\{9, 10\}$ as well as $\{10, 1\}$.
(ii) Let $V = \{1, 2, 3, 4\}$ and define $\mathcal{S} \subset \mathcal{P}(V)$ so that the elements of $\mathcal{S}$ are exactly the subsets of V which are nonempty and not equal to V.
(iii) Let V be any set and define $\mathcal{S}$ so that the elements of $\mathcal{S}$ are exactly the nonempty finite subsets of V.
(iv) Take a regular icosahedron. Let V be the set of its vertices (which has 12 elements). Define $\mathcal{S} \subset \mathcal{P}(V)$ in such a way that the elements of $\mathcal{S}$ are all singletons, all doubletons which are connected by an edge, and all tripletons which make up a triangular face of the icosahedron. (There are twenty such tripletons, which is supposed to explain the name *icosahedron*.)

The *simplicial complex* determined by a vertex scheme $(V, \mathcal{S})$ is a topological space $X = |V|_{\mathcal{S}}$. We describe it first as a set. An element of X is a function $f\colon V \to [0, 1]$ such that

$$\sum_{v \in V} f(v) = 1$$

and the set $\{v \in V \mid f(v) > 0\}$ is an element of $\mathcal{S}$.
It should be clear that X is the union of certain subsets $\Delta(T)$, where $T \in \mathcal{S}$. Namely, $\Delta(T)$ consists of all the functions $f\colon V \to [0, 1]$ for which $\sum_{v \in V} f(v) = 1$ and $f(v) = 0$ if $v \notin T$. The subsets $\Delta(T)$ of X are not always disjoint. Instead we have $\Delta(T) \cap \Delta(T') = \Delta(T \cap T')$ if $T \cap T'$ is nonempty; also, if $T \subset T'$ then $\Delta(T) \subset \Delta(T')$.
The subsets $\Delta(T)$ of X, for $T \in \mathcal{S}$, come equipped with a preferred topology. Namely, $\Delta(T)$ is (identified with) a subset of a finite dimensional real vector space, the vector space of all functions from T to $\mathbb{R}$, and as such gets a subspace topology. (For example, $\Delta(T)$ is a single point if T has one element; it is homeomorphic to an edge or closed interval if T has two elements; it looks like a compact triangle if T has three elements; etc. We say that $\Delta(T)$ is a *simplex* of dimension m if T has cardinality $m + 1$.) These topologies are compatible in the following sense: if $T \subset T'$, then the inclusion $\Delta(T) \to \Delta(T')$ makes a

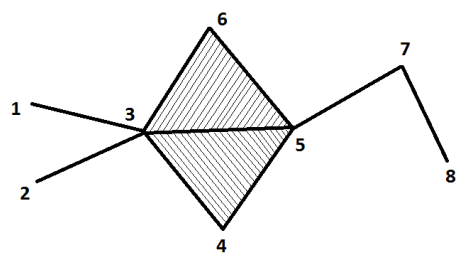
homeomorphism of $\Delta(T)$ with a subspace of $\Delta(T')$.
We decree that a subset W of X shall be *open* if and only if $W \cap \Delta(T)$ is open in $\Delta(T)$, for every T in $\mathcal{S}$. Equivalently, and perhaps more usefully: a map g from X to another topological space Y is continuous if and only if the restriction of g to $\Delta(T)$ is a continuous from $\Delta(T)$ to Y, for every $T \in \mathcal{S}$.

Example 9.1.3. The simplicial complex associated to the vertex scheme (i) in example 9.1.2 is homeomorphic to S^1. In (ii) and (iv) of example 9.1.2, the associated simplicial complex is homeomorphic to S^2.

Example 9.1.4. The simplicial complex associated to the vertex scheme $(V, \mathcal{S})$ where $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and

$$\mathcal{S} = \left\{ \begin{array}{l} \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{1, 3\}, \{2, 3\}, \{3, 4\}, \\ \{3, 5\}, \{3, 6\}, \{4, 5\}, \{5, 6\}, \{5, 7\}, \{7, 8\}, \{3, 4, 5\}, \{3, 5, 6\} \end{array} \right\}$$

looks like this:



Lemma 9.1.5. *The simplicial complex* $X = |V|_{\mathcal{S}}$ *associated with a vertex scheme* $(V, \mathcal{S})$ *is a Hausdorff space.*

Proof. Let f and g be distinct elements of X. Keep in mind that f and g are functions from V to $[0, 1]$. Choose $v_0 \in V$ such that $f(v_0) \neq g(v_0)$. Let $\varepsilon = |f(v_0) - g(v_0)|$. Let U_f be the set of all $h \in X$ such that $|h(v_0) - f(v_0)| < \varepsilon/2$. Let U_g be the set of all $h \in X$ such that $|h(v_0) - g(v_0)| < \varepsilon/2$. From the definition of the topology on X, the sets U_f and U_g are open. They are also disjoint, for if $h \in U_f \cap U_g$ then $|f(v_0) - g(v_0)| \leq |f(v_0) - h(v_0)| + |h(v_0) - g(v_0)| < \varepsilon$, contradiction. Therefore f and g have disjoint neighborhoods in X. □

Lemma 9.1.6. *Let* $(V, \mathcal{S})$ *be a vertex scheme and* $(W, \mathcal{T})$ *a vertex sub-scheme, that is,* $W \subset V$ *and* $\mathcal{T} \subset \mathcal{S} \cap \mathcal{P}(W)$. *Then the evident map* $\iota\colon |W|_{\mathcal{T}} \to |V|_{\mathcal{S}}$ *is a closed, continuous and injective map and therefore a homeomorphism onto its image.*

Proof. The map ι is obtained by viewing functions from W to $[0, 1]$ as functions from V to $[0, 1]$ by defining the values on elements of $V \smallsetminus W$ to be 0. A subset A of $|V|_{\mathcal{S}}$ is closed if and only if $A \cap \Delta(T)$ is closed for the standard topology on $\Delta(T)$, for every $T \in \mathcal{S}$. Therefore, if A is a closed subset of $|V|_{\mathcal{S}}$, then $\iota^{-1}(A)$ is a closed subset of $|W|_{\mathcal{T}}$; and if C is a closed subset of $|W|_{\mathcal{S}}$, then $\iota(C)$ is closed in $|V|_{\mathcal{S}}$. □

Remark 9.1.7. The notion of a simplicial complex is old. Related vocabulary comes in many dialects. I have taken the expression *vertex scheme* from Dold's book *Lectures on*

algebraic topology with only a small change (for me, $\emptyset \notin \mathcal{S}$). It is in my opinion a good choice of words, but the traditional expression for that appears to be *abstract simplicial complex*. Most authors agree that a *simplicial complex* (non-abstract) is a topological space with additional data. For me, a simplicial complex is a space of the form $|V|_{\mathcal{S}}$ for some vertex scheme $(V, \mathcal{S})$; other authors prefer to write, in so many formulations, that a simplicial complex is a topological space X together with a homeomorphism $|V|_{\mathcal{S}} \to X$, for some vertex scheme $(V, \mathcal{S})$.

9.2. Semi-simplicial sets and their geometric realizations

Semi-simplicial sets are closely related to vertex schemes. A semi-simplicial set has a *geometric realization*, which is a topological space; this is similar to the way in which a vertex scheme determines a simplicial complex.

Definition 9.2.1. A semi-simplicial set Y consists of a sequence of sets

$$(Y_0, Y_1, Y_2, Y_3, \ldots)$$

(each Y_k is a set) and, for each injective order-preserving map

$$f\colon \{0, 1, 2, \ldots, k\} \longrightarrow \{0, 1, 2, \ldots, \ell\}$$

where $k, \ell \geq 0$, a map $f^*\colon Y_\ell \to Y_k$. The maps f^* are called *face operators* and they are subject to conditions:

- if f is the identity map from $\{0, 1, 2, \ldots, k\}$ to $\{0, 1, 2, \ldots, k\}$ then f^* is the identity map from Y_k to Y_k.
- $(g \circ f)^* = f^* \circ g^*$ when $g \circ f$ is defined (so $f\colon \{0, 1, \ldots, k\} \to \{0, 1, \ldots, \ell\}$ and $g\colon \{0, 1, \ldots, \ell\} \to \{0, 1, \ldots, m\}$).

Elements of Y_k are often called k-*simplices* of Y. If $x \in Y_k$ has the form $f^*(y)$ for some $y \in Y_\ell$, then we may say that x is a *face* of y corresponding to face operator f^*.

Remark 9.2.2. The definition of a semi-simplicial set can be reformulated in category language as follows. There is a category $\mathcal{C}$ whose objects are the sets $[n] = \{0, 1, \ldots, n\}$, where n can be any non-negative integer. A morphism in $\mathcal{C}$ from $[m]$ to $[n]$ is an order-preserving injective map from the set $[m]$ to the set $[n]$. Composition of morphisms is, by definition, composition of such order-preserving injective maps.
A semi-simplicial set is a contravariant functor Y from $\mathcal{C}$ to the category of sets. We like to write Y_n when we ought to write $Y([n])$. We like to write $f^*\colon Y_n \to Y_m$ when we ought to write $Y(f)\colon Y([n]) \to Y([m])$, for a morphism $f\colon [m] \to [n]$ in $\mathcal{C}$.
Nota bene: if you wish to define (invent) a semi-simplicial set Y, you need to invent sets $Y_0, Y_1, Y_2, \ldots$ (one set Y_n for each integer $n \geq 0$) *and* you need to invent maps $f^*\colon Y_n \to Y_m$, one for each order-preserving injective map $f\colon [m] \to [n]$. *Then* you need to convince yourself that $(g \circ f)^* = f^* \circ g^*$ whenever $f\colon [k] \to [\ell]$ and $g\colon [\ell] \to [m]$ are order-preserving injective maps.

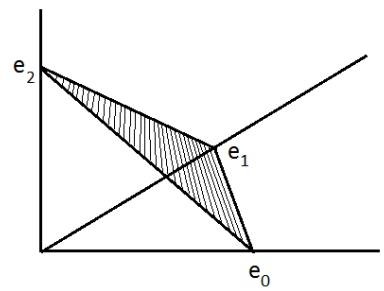
Example 9.2.3. Let $(V, \mathcal{S})$ be a vertex scheme as in the preceding (sub)section. Choose a total ordering of V. From these data we can make a semi-simplicial set Y as follows.

- Y_n is the set of all order-preserving injective maps β from $\{0, 1, \ldots, n\}$ to V such that $\mathrm{im}(\beta) \in \mathcal{S}$. Note that for each $T \in \mathcal{S}$ of cardinality $n + 1$, there is exactly one such β.
- For an order-preserving injective $f\colon \{0, 1, \ldots, m\} \to \{0, 1, \ldots, n\}$ and $\beta \in Y_n$, define $f^*(\beta) = \beta \circ f \in Y_m$.

In order to warm up for geometric realization, we introduce a (covariant) functor from the category $\mathcal{C}$ in remark 9.2.2 to the category of topological spaces. On objects, the functor is given by

$$\{0, 1, 2, \ldots, m\} \mapsto \Delta^m$$

where Δ^m is the space of functions u from $\{0, 1, \ldots, m\}$ to $\mathbb{R}$ which satisfy the condition $\sum_{j=0}^{m} u(j) = 1$. (As usual we view this as a subspace of the finite-dimensional real vector space of all functions from $\{0, 1, \ldots, n\}$ to $\mathbb{R}$. It is often convenient to think of $u \in \Delta^n$ as a vector, $(u_0, u_1, \ldots, u_m)$, where all coordinates are ≥ 0 and their sum is 1.) Here is a picture of Δ^2 as a subspace of $\mathbb{R}^3$ (with basis vectors e_0, e_1, e_2):



For a morphism f, meaning an order-preserving injective map

$$f\colon \{0, 1, 2, \ldots, m\} \longrightarrow \{0, 1, 2, \ldots, n\},$$

we want to see an induced map

$$f_*\colon \Delta^m \to \Delta^n.$$

This is easy: for $u = (u_0, u_1, \ldots, u_m) \in \Delta^m$ we define

$$f_*(u) = v = (v_0, v_1, \ldots, v_n) \in \Delta^n$$

where $v_j = u_i$ if $j = f(i)$ and $v_j = 0$ if $j \notin \mathrm{im}(f)$.
(Keep the following conventions in mind. For a covariant functor G from a category $\mathcal{A}$ to a category $\mathcal{B}$, and a morphism $f\colon x \to y$ in $\mathcal{A}$, we often write $f_*\colon G(x) \to G(y)$ instead of $G(f)\colon G(x) \to G(y)$. For a contravariant functor G from a category $\mathcal{A}$ to a category $\mathcal{B}$, and a morphism $f\colon x \to y$ in $\mathcal{A}$, we often write $f^*\colon G(y) \to G(x)$ instead of $G(f)\colon G(y) \to G(x)$.)

The geometric realization $|Y|$ of a semi-simplicial set Y is a topological space defined as follows. Our goal is to have, for each $n \geq 0$ and $y \in Y_n$, a preferred continuous map

$$c_y\colon \Delta^n \to |Y|$$

(the *characteristic map* associated with the simplex $y \in Y_n$). These maps should match in the sense that whenever we have an injective order-preserving

$$f\colon \{0, 1, \ldots, m\} \to \{0, 1, \ldots, n\}$$

and $y \in Y_n$, so that $f^*y \in Y_m$, then the diagram

$$\begin{tikzcd} \Delta^n \arrow[r, "c_y"] & |Y| \\ \Delta^m \arrow[u, "f_*"] \arrow[r, "c_{f^*y}"] \arrow[ur, phantom, "="] & |Y| \end{tikzcd}$$

is commutative. There is a "most efficient" way to achieve this. As a set, let $|Y|$ be the set of all symbols $\bar{c}_y(u)$ where $y \in Y_n$ for some $n \geq 0$ and $u \in \Delta^n$, modulo the relations[1]

$$\bar{c}_y(f_*(u)) \sim \bar{c}_{f^*y}(u)$$

(notation and assumptions as in that diagram). This ensures that we have maps c_y from Δ^n to $|Y|$, for each $y \in Y_n$, given in the best tautological manner by

$$c_y(u) := \text{ equivalence class of } \bar{c}_y(u) \ .$$

Also, those little squares which we wanted to be commutative are now commutative because we enforced it. Finally, we say that a subset U of $|Y|$ shall be *open* (definition coming) if and only if $c_y^{-1}(U)$ is open in Δ^n for each characteristic map $c_y \colon \Delta^n \to |Y|$.

A slightly different way (shorter but possibly less intelligible) to say the same thing is as follows:

$$|Y| := \left(\coprod_{n \geq 0} Y_n \times \Delta^n \right) \Big/ \sim$$

where $\sim$ is a certain equivalence relation on $\coprod_n Y_n \times \Delta^n$. It is the smallest equivalence relation which has $(y, f_*(u))$ equivalent to (f^*y, u) whenever $f \colon \{0, 1, \ldots, m\} \to \{0, 1, \ldots, n\}$ is injective order-preserving and $y \in Y_n$, $u \in \Delta^m$. Note that, where it says $Y_n \times \Delta^n$, the set Y_n is regarded as a topological space with the discrete topology, so that $Y_n \times \Delta^n$ has meaning; we could also have written $\coprod_{y \in Y_n} \Delta^n$ instead of $Y_n \times \Delta^n$.
This new formula for $|Y|$ emphasizes the fact that $|Y|$ is a *quotient space* of a topological disjoint union of many standard simplices Δ^n (one simplex for every pair (n, y) where $y \in Y_n$). Go ye forth and look up *quotient space* or *identification topology* in your favorite book on point set topology.— To match the second description of $|Y|$ with the first one, let the element of $|Y|$ represented by $(y, u) \in Y_n \times \Delta^n$ in the second description correspond to the element which we called $c_y(u)$ in the first description of $|Y|$.

Example 9.2.4. Fix an integer $n \geq 0$. We might like to invent a semi-simplicial set

$$Y = \underline{\Delta}^n$$

such that $|Y|$ is homeomorphic to Δ^n. The easiest way to achieve that is as follows. Define Y_k to be the set of all order-preserving injective maps from $\{0, 1, \ldots, k\}$ to $\{0, 1, \ldots, n\}$. So Y_k has $\binom{n+1}{k+1}$ elements (which implies $Y_k = \emptyset$ if $k > n$). For an injective order-preserving map

$$g \colon \{0, 1, \ldots, k\} \to \{0, 1, \ldots, \ell\},$$

define the face operator $g^* \colon Y_\ell \to Y_k$ by $g^*(f) = f \circ g$. This makes sense because $f \in Y_\ell$ is an order-preserving injective map from $\{0, 1, \ldots, \ell\}$ to $\{0, 1, \ldots, n\}$. There is a unique

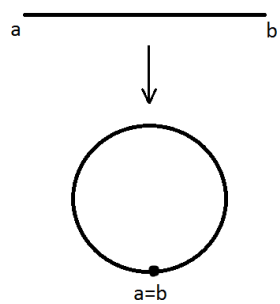
[1] *Modulo the relations* is short for the following process: form the smallest equivalence relation on the set of all those symbols $\bar{c}_y(u)$ which contains the stated relation. Then pass to the set of equivalence classes for that equivalence relation. That set of equivalence classes is $|Y|$.

element $y \in Y_n$, corresponding to the identity map of $\{0, 1, \ldots, n\}$. It is an exercise to verify that the characteristic map $c_y \colon \Delta^n \to |Y|$ is a homeomorphism.

EXAMPLE 9.2.5. Up to relabeling there is a unique semi-simplicial set Y such that Y_0 has exactly one element, Y_1 has exactly one element, and $Y_n = \emptyset$ for $n > 1$. Then $|Y|$ is homeomorphic to S^1. More precisely, let $z \in Y_1$ be the unique element; then the characteristic map

$$c_z \colon \Delta^1 \longrightarrow |Y|$$

is an identification map. (Translation: it is surjective and a subset of the target is open in the target if and only if its preimage is open in the source.) The only identification taking place is $c_z(a) = c_z(b)$, where a and b are the two boundary points of Δ^1.



9.3. Technical remarks concerning the geometric realization

Let Y be a semi-simplicial set. We reformulate the definition of the geometric realization $|Y|$ once again.

From the semi-simplicial set Y, we make a category $\mathcal{C}_Y$ as follows. An object is a pair (n, z) where n is a non-negative integer and $z \in Y_n$. A morphism from (m, y) to (n, z) is, by definition, an order-preserving injective map $g \colon \{0, 1, 2, \ldots, m\}$ to $\{0, 1, 2, \ldots, n\}$ which has the property $g^*(z) = y$ (where $g^* \colon Y_n \to Y_m$ is the face operator determined by g).

We define a covariant functor F_Y from $\mathcal{C}_Y$ to the category of topological spaces as follows. The definition of F_Y on objects is simply

$$F_Y(n, z) = \Delta^n$$

where Δ^n is the standard n-simplex. (Recall that this is the space of all functions u from $\{0, 1, \ldots, n\}$ to $[0, 1]$ which satisfy $\sum_j u(j) = 1$, viewed as a subspace of the real vector space of all functions from $\{), 1, \ldots, n\}$ to $\mathbb{R}$.) If we have a morphism from (m, y) to (n, z) given by an an order-preserving injective map $g \colon \{0, 1, 2, \ldots, m\}$ to $\{0, 1, 2, \ldots, n\}$, then we define

$$F_Y(f) = g_* \colon \Delta^m \to \Delta^n,$$

that is to say, $F_Y(f)(u_1, \ldots, u_m) = (v_1, \ldots, v_n)$ where $v_i = u_j$ if $i = g(j)$ and $v_i = 0$ if i is not of the form $g(j)$. Note that I have written u_i instead of $u(i)$ etc. ; strictly speaking $u(i)$ is correct because we said that u is a function from $\{0, 1, \ldots, m\}$ to $[0, 1]$.

Now the definition of $|Y|$ can be recast as follows:

$$|Y| = \left(\coprod_{(n,z)} F_Y(n,z) \right) \Big/ \sim$$

where $\sim$ is the equivalence relation generated by

$$F_Y(m,y) \ \ni \ (u_1,\dots,u_m) \sim F_Y(g)(u_1,\dots,u_m) \in F_Y(n,z)$$

whenever g is a morphism from (m,y) to (n,z); in other words g is an order-preserving injective map from $\{0,1,2,\dots,m\}$ to $\{0,1,2,\dots,n\}$ which has $g^*(z) = y$. It may look as if the formula defines $|Y|$ only as a set, but we want to view it as a formula defining a topology on $|Y|$ as well, namely, the *quotient topology*. Therefore, a subset of $|Y|$ is considered to be *open* (definition) if and only if its preimage in $\coprod_{(n,z)} F_Y(n,z)$ is open.

Warning: do not read these $2\frac{1}{2}$ *lines unless you are somewhat familiar with category theory.* You will notice that $|Y|$ has been defined to be the direct limit (also called colimit) of the functor F_Y.

Example 9.3.1. Let $(V,\mathcal{S})$ be a vertex scheme, choose a total ordering on V, and let Y be the associated semi-simplicial set, as in example 9.2.3. We are going to show that the geometric realization $|Y|$ is homeomorphic to the simplicial complex $|V|_{\mathcal{S}}$.
An element of Y_n is an order-preserving injective map from $\{0,1,\dots,n\}$ to V. This is determined by its image T, a distinguished subset of V (where *distinguished* means that $T \in \mathcal{S}$). So we can pretend that Y_n is simply the set of all distinguished subsets of V that have exactly $n+1$ elements. Furthermore, if $T' \in Y_m$ and $T \in Y_n$, then there exists at most one morphism from T' to T in the category $\mathcal{C}_Y$. It exists if and only if $T' \subset T$. Therefore we may say that $\mathcal{C}_Y$ is the category whose objects are the distinguished subsets $T, T', \dots$ of V, with exactly one morphism from T' to T if $T' \subset T$, and no morphism from T' to T otherwise. In this description, the functor F_Y is given on objects by

$$F_Y(T) = \Delta(T)$$

where $\Delta(T)$ replaces Δ^n (assuming that T has exactly $n+1$ elements) and means: the space of functions u from T to $[0,1]$ that satisfy $\sum_{j\in T} u(j) = 1$. For $T' \subset T$ we have exactly one morphism from T' to T, and the induced map $F_Y(T') = \Delta(T') \to \Delta(T) = F_Y(T)$ is given by $u \mapsto v$ where $v(t) = u(t)$ if $t \in T'$ and $v(t) = 0$ if $t \in T \smallsetminus T'$. Therefore

$$|Y| = \Big(\coprod_{T\in\mathcal{S}} \Delta(T) \Big) \Big/ \sim$$

where the equivalence relation is generated by $u \in \Delta(T') \sim v \in \Delta(T)$ if $T' \subset T$ and $v(t) = u(t)$ for $t \in T'$, $v(t) = 0$ for $t \in T \smallsetminus T'$.
There is a map of sets

$$\coprod_{T\in\mathcal{S}} \Delta(T) \longrightarrow |V|_{\mathcal{S}}$$

which is equal to the inclusion $\Delta(T) \to |V|_{\mathcal{S}}$ on each $\Delta(T)$. That map clearly determines a *bijective* map

$$|Y| = \Big(\coprod_{T\in\mathcal{S}} \Delta(T) \Big) \Big/ \sim \qquad \longrightarrow \qquad |V|_{\mathcal{S}} \ .$$

By our definition of the topology on $|V|_{\mathcal{S}}$, a subset of $|V|_{\mathcal{S}}$ is open if and only if its preimage in $\coprod_{T\in\mathcal{S}} \Delta(S)$ is open; and by our definition of the topology in $|Y|$, that happens if and only if its preimage in $|Y|$ is open. So that bijective map from $|Y|$ to $|V|_{\mathcal{S}}$ is a homeomorphism.

LEMMA 9.3.2. *Let Y be any semi-simplicial set. For every element a of $|Y|$ there exist unique $m \geq 0$ and $(z, w) \in Y_m \times \Delta^m$ such that $a = c_z(w)$ and w is in the "interior" of Δ^m, that is, the coordinates $w_0, w_1, \ldots, w_m$ are all strictly positive.*
Furthermore, if $a = c_x(u)$ for some $(x, u) \in Y_k \times \Delta^k$, then there is a unique order-preserving injective $f\colon \{0, 1, \ldots, m\} \to \{0, 1, 2, \ldots, k\}$ such that $f^*(x) = z$ and $f_*(w) = u$, for the above-mentioned $(z, w) \in Y_m \times \Delta^m$ with $w_0, w_1, \ldots, w_m > 0$.

PROOF. Let us call such a pair (z, w) with $a = c_z(w)$ a *reduced presentation* of a; the condition is that all coordinates of w must be positive. More generally we say that (x, u) is a *presentation* of a if $(x, u) \in Y_k \times \Delta^k$ for some $k \geq 0$ and $a = c_x(u)$. First we show that a admits a reduced presentation and then we show uniqueness.
We know that $a = c_x(u)$ for some $(x, u) \in Y_k \times \Delta^k$. Some of the coordinates $u_0, \ldots, u_k$ can be zero (not all, since their sum is 1). Suppose that $m + 1$ of them are nonzero. Let $f\colon \{0, 1, \ldots, m\} \to \{0, 1, \ldots, k\}$ be the unique order-preserving map such that $u_{f(j)} \neq 0$ for $j = 0, 1, 2, \ldots, m$. Then $a = c_z(w)$ where $z = f^*(x)$ and $w \in \Delta^m$ with coordinates $w_j = u_{f(j)}$. (Note that $f_*(w) = u$.) So (z, w) is a reduced presentation of a.
We have also shown that any presentation (x, u) of a (whether reduced or not) determines a reduced presentation. Namely, there exist unique m, f and $w \in \Delta^m$ such that $v = f_*(w)$ for some $w \in \Delta^m$ with all $w_i > 0$; then $(f^*(x), w)$ is a reduced presentation of a.
It remains to show that if a has two presentations, say $(x, u) \in Y_k \times \Delta^k$ and $(y, v) \in Y_\ell \times \Delta^\ell$, then they determine the *same* reduced representation of a. If indeed $a = c_x(u) = c_y(v)$ then $\bar{c}_x(u)$ and $\bar{c}_y(v)$ are equivalent, and so (recalling how that equivalence relation was defined) we find that there is no loss of generality in assuming that $x = g^*(y)$ and $v = g_*(u)$ for some order-preserving injective $g\colon \{0, 1, \ldots, k\} \to \{0, 1, \ldots, \ell\}$. Now determine the unique m and order-preserving injective $f\colon \{0, 1, \ldots, m\} \to \{0, 1, \ldots, k\}$ such that $u = f_*(w)$ where $w \in \Delta^m$ and all $w_i > 0$. Then we also have $v = g_*(u) = g_*(f_*(w)) = (g \circ f)_*(w)$ and it follows that we get the same reduced presentation, $(f^*(x), w) = ((g \circ f)^*(y), w)$, in both cases. □

COROLLARY 9.3.3. *The space $|Y|$ is a Hausdorff space.*

PROOF. For $a \in Y$ with reduced presentation $(z, w) \in Y_m \times \Delta^m$ and $\varepsilon > 0$, define $N(a, \varepsilon) \subset |Y|$ as follows. It consists of all $b \in |Y|$ with reduced presentation $(x, u) \in Y_k \times \Delta^k$ such that there exists an order-preserving injective $f\colon \{0, 1, \ldots, m\} \to \{0, 1, \ldots, k\}$ for which $f^*(x) = z$ and $f_*(w)$ is ε-close to u, that is, the maximum of the numbers $|w_{f(j)} - u_j|$ is $< \varepsilon$. From the definitions, $N(a, \varepsilon)$ is open in $|Y|$; so it is a neighborhood of a.
Let $a' \in |Y|$ be another element, with reduced presentation $(y, v) \in Y_n \times \Delta^n$. We assume $a \neq a'$ and proceed to show that $N(a', \varepsilon) \cap N(a, \varepsilon) = \emptyset$ if ε is small enough. More precisely, we take ε to be less than half the minimum of the coordinates of v and w; and if it should happen that $m = n$ and $y = z$, then we know $v, w \in \Delta^m$ but $v \neq w$, and we take ε to be less than half the maximum of the $|v_j - w_j|$ as well. Now suppose for a contradiction that $b \in N(a, \varepsilon) \cap N(a', \varepsilon)$ and that b has reduced presentation $(x, u) \in Y_k \times \Delta^k$. Then there exist order-preserving injective $f\colon \{0, 1, \ldots, m\} \to \{0, 1, \ldots, k\}$ and $g\colon \{0, 1, \ldots, n\} \to \{0, 1, \ldots, k\}$ such that $f^*(x) = z$, $g^*(x) = y$ and $f_*(w), g_*(v)$ are both ε-close to u in Δ^k. Then $f_*(w)$ is 2ε-close to $g_*(v)$ in Δ^k, and now we can deduce that $m = n$ and $f = g$. (Otherwise there is some $j \in \{0, 1, \ldots, k\}$ which is in the image of g but not in the image of f, or vice versa, and then the j-th coordinate of $g_*(w)$ differs by more than 2ε from the j-th coordinate of $f_*(v)$.) Therefore $z = f^*(x) = g^*(x) = y$ and so a has reduced presentation (z, w) while a' has reduced presentation (z, v), with

$v, w \in \Delta^m$ and the same $z \in Y_m$. It follows that v and w are already 2ε-close in Δ^m. This contradicts our choice of ε. □

REMARK 9.3.4. In the proof above, and in a similar proof in the previous section, arguments involving distances make an appearance, suggesting that we have a metrizable situation. To explain what is going on let me return to the situation of a vertex scheme $(V, \mathcal{S})$ with simplicial complex $|V|_{\mathcal{S}}$, which is easier to understand. A metric on the set $|V|_{\mathcal{S}}$ can be introduced for example by $d(f,g) = (\sum_v (f(v)-g(v))^2)^{1/2}$ or $d(f,g) = \sum_v |f(v)-g(v)|$. Here we insist/remember that elements of $|V|_{\mathcal{S}}$ are functions $f, g, \ldots : V \to [0,1]$ subject to some conditions. The sums in the formulas for $d(f,g)$ are finite, even though V might not be a finite set. It is not hard to show that the two formulas for $d(f,g)$, although different as metrics, determine the same topology. However the topology on $|V|_{\mathcal{S}}$ that we have previously decreed (let me call it the *weak* topology) is not in all cases the same as that metric topology. Every subset of $|V|_{\mathcal{S}}$ which is open in the metric topology is also open in the weak topology. But the weak topology can have more open sets. (We reasoned that the weak topology is Hausdorff because it has all the open sets that the metric topology has, and perhaps a few more, and the metric topology is certainly Hausdorff.) In the case where V is finite, weak topology and metric topology on $|V|_{\mathcal{S}}$ coincide. (Exercise.)

9.4. A shorter but less conceptual definition of semi-simplicial set

Every injective order-preserving map from $[k] = \{0, 1, \ldots, k\}$ to $[\ell] = \{0, 1, \ldots, \ell\}$ is a composition of $\ell - k$ injective order preserving maps

$$[m-1] \longrightarrow [m]$$

where $k < m \leq \ell$. It is easy to list the injective order-preserving maps from $[m-1]$ to $[m]$; there is one such map f_i for every $i \in [m]$, characterized by the property that the image of f_i is

$$[m] \smallsetminus \{i\} .$$

(This f_i really depends on two parameters, m and i. Perhaps we ought to write $f_{m,i}$, but it is often practical to suppress the m subscript.) We have the important relations

$$f_i f_j = f_j f_{i-1} \quad \text{if } j < i \tag{9.4.1}$$

(You are allowed to read this from left to right or from right to left! It is therefore a formal consequence that $f_i f_j = f_{j+1} f_i$ when $j \geq i$.) These *generators and relations* suffice to describe the category $\mathcal{C}$ (lecture notes week 11) whose objects are the sets $[k] = \{0, 1, \ldots, k\}$ for $k \geq 0$ and whose morphisms are the order-preserving injective maps between those sets. In other words, the structure of $\mathcal{C}$ as a category is pinned down if we say that it has objects $[k]$ for $k \geq 0$ and that, for every $k > 0$ and $i \in \{0, 1, \ldots, k\}$, there are certain morphisms $f_i \colon [k-1] \to [k]$ which, under composition when it is applicable, satisfy the relations (9.4.1). Prove it!

Consequently a semi-simplicial set Y, which is a contravariant functor from $\mathcal{C}$ to spaces, can also be described as a sequence of sets $Y_0, Y_1, Y_2, \ldots$ and maps

$$d_i \colon Y_k \to Y_{k-1}$$

which are subject to the relations

$$d_j d_i = d_{i-1} d_j \quad \text{if } j < i \tag{9.4.2}$$

Here $d_i \colon Y_k \to Y_{k-1}$ denotes the map induced by $f_i \colon [k-1] \to [k]$, whenever $0 \leq i \leq k$. Because of contravariance, we have had to reverse the order of composition in translating relations (9.4.1) to obtain relations (9.4.2).

CHAPTER 10

CW-spaces

10.1. CW-Spaces: definition and examples

CW-spaces are generalizations of simplicial complexes and geometric realizations of semi-simplicial sets (see Lecture notes WS13-14). To be more precise: a simplicial complex is a topological space $|V|_S$ which has been obtained from a vertex scheme (V, S) , and a semi-simplicial set X has a geometric realization $|X|$ which is a topological space. Both $|V|_S$ and $|X|$ have the additional structure that they need in order to qualify as CW-spaces.
In describing a CW-space, we do not begin with combinatorial data in order to make a space out of them. We begin with a space and we put additional structure on it by specifying an increasing sequence of subspaces. The definition is a great achievement due to J.H.C. Whitehead (probably 1949).

Definition 10.1.1. A *CW-space* is a space X together with an increasing sequence of subspaces

$$\emptyset = X^{-1} \subset X^0 \subset X^1 \subset X^2 \subset X^3 \subset \dots$$

subject to the following conditions.

(1) $X = \bigcup_{n \geq -1} X^n$ and a subset A of X is closed if and only if $A \cap X^n$ is closed in X^n for all n.
(2) For every $n \geq 0$ there *exists* a pushout square of spaces (see remark 10.1.2)

$$\begin{tikzcd} \coprod_{\lambda \in \Lambda_n} S^{n-1} \arrow[r, "\text{incl.}"] \arrow[d] & \coprod_{\lambda \in \Lambda_n} D^n \arrow[d] \\ X^{n-1} \arrow[r, "\text{incl.}"] & X^n \end{tikzcd}$$

where Λ_n is a set (and D^n, S^{n-1} are unit disk and unit sphere in $\mathbb{R}^n$, respectively).

Let us unravel this and derive some of the easier consequences.

- Condition (2) implies that X^{n-1} is a closed subspace of X^n.
- Using that, we can deduce from condition (1) that X^n is a closed subspace of X, for each n.
- X is a normal space (disjoint closed sets have disjoint open neighborhoods) and therefore also Hausdorff. Sketch proof: let A_1 and A_2 be disjoint closed subsets of X. Inductively, construct disjoint open neighborhoods $U_{1,n}$ and $U_{2,n}$ in X^n of $A_1 \cap X^n$ and $A_2 \cap X^n$, respectively. Do this in such a way that $U_{1,n-1} = U_{1,n} \cap X^{n-1}$ and $U_{2,n-1} = U_{2,n} \cap X^{n-1}$. Then by condition (1), the sets $U_1 := \bigcup_n U_{1,n}$ and $U_2 := \bigcup_n U_{1,n}$ are open in X and they are disjoint neighborhoods of A_1 and A_2, respectively.

- A subset Y of X is closed in X if and only if its intersection with every compact subset C of X is closed in C. (This property has a name: *compactly generated*.) Proof: one direction is trivial. Suppose that $Y \cap C$ is closed in C for every compact subset C of X. It suffices to show that $Y \cap X^n$ is closed in X^n, for every n. We proceed by induction on n. For the induction step, assume that $Y \cap X^{n-1}$ is closed in X^{n-1}. Choose a pushout square as in condition (2). The intersection of Y with the image of each copy of D^n under the right-hand vertical arrow is closed in that image, by assumption. Therefore the preimage of $Y \cap X^n$ is closed in $\Lambda_n \times D^n$. Therefore $Y \cap X^n$ is closed in X^n by the definition of *pushout square*.
- Condition (2) implies that $X^n \smallsetminus X^{n-1}$, which is open in X^n, is homeomorphic (with the subspace topology) to $\Lambda_n \times (D^n \smallsetminus S^{n-1})$, or equivalently to $\Lambda_n \times \mathbb{R}^n$. In other words $X^n \smallsetminus X^{n-1}$ is homeomorphic to a disjoint union of copies of $\mathbb{R}^n$. These copies of $\mathbb{R}^n$ are well-defined subspaces of X because they are also the connected components of $X^n \smallsetminus X^{n-1}$. They are called the n-*cells* of X. Thus the n-cells of X are homeomorphic to $\mathbb{R}^n$. No specific homeomorphism with $\mathbb{R}^n$ is provided. The vertical arrows in the square of (2) are not *given* as part of the structure of a CW-space, they only *exist*.
- Let S be a subset of X such that the intersection of S with every cell of X is a finite set. Then S is a closed subset of X. Sketch proof: It is enough to show that $S \cap X^n$ is closed in X^n for all n. We proceed by induction on n; so assume for the induction step that $S \cap X^{n-1}$ is closed in X^{n-1}. Now $S \cap X^n$ is the union of $S \cap X^{n-1}$, which is closed in X^{n-1} and therefore closed in X^n, and a subset T of $X^n \smallsetminus X^{n-1}$ which has finite intersection with every n-cell. By condition (2), the set T is closed in X^n.
- Let S be a subset of X such that the intersection of S with every cell of X is a finite set. Then S is discrete with the subspace topology. Proof: Every subset of S is closed in X (by the same reasoning that we applied to S) and therefore closed in S.
- Let C be a compact subspace of X. Then C is contained in a union of finitely many cells of X. Proof: Suppose not. Then there is an infinite subset S of C such that S has at most one point in common with each cell. We know already that S is closed in X and discrete. Therefore S is closed in C and discrete. Therefore S is compact, discrete and infinite, contradiction.
- The closure in X of every cell of X is contained in a finite union of cells. Proof: condition (2) implies that the closure of every n-cell is compact in X^n, being equal to the image of a continuous map from D^n to the Hausdorff space X^n. Therefore it is compact in X and so (by the previous results) it is contained in a finite union of cells.
- Every compact subspace of X (and in particular the closure of any cell in X) is contained in a compact subspace of X which is a finite union of cells. Proof: by the previous it suffices to show that any n-cell E of X is contained in a compact subspace of X which is a finite union of cells. The closure $\bar{E}$ of E in X is compact and therefore contained in a finite union of cells. These cells might be called $E = E_0$, $E_1, E_2, \ldots, E_k$ (where the indexing has nothing to do with their dimension). But we know that $\bar{E} \smallsetminus E$ is contained in X^{n-1} by condition (2). Therefore cells $E_1, E_2, \ldots, E_k$ have dimension $< n$. By inductive assumption (yes, we are doing an induction on n) each E_i where $i = 1, 2, \ldots, k$ is contained

in a compact subspace C_k of X which is a finite union of cells of X. Take the union K of $C_1 \cup C_2 \cup \dots C_k$ and $\bar{E}$, which is the same as the union of $C_1 \cup C_2 \cup \dots C_k$ and E. Therefore K is compact and it is a finite union of cells of X.

According to Whitehead himself, the letters C and W in CW-space are for *weak topology*, expressed in condition (1), and *closure finiteness*, as in: *the closure of every cell is contained in a finite union of cells.* But perhaps he meant a selection of initials from his full name John Henry Constantine Whitehead. (Against that theory, I believe his preferred first name was Henry, not Constantine.)

In a CW-space X, the subspace X^n is called the n*-skeleton* of X. If $Z \subset X$ is an n-cell, that is to say, a connected component of $X^n \smallsetminus X^{n-1}$, then by condition (2) above we know that there *exists* a continuous map

$$\varphi\colon D^n \to X$$

which restricts to a homeomorphism from $D^n \smallsetminus S^{n-1}$ to Z. Such a φ is called a *characteristic map* for the cell.

Remark 10.1.2. A commutative square of spaces and maps

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "u"] \\ C \arrow[r, "v"] & D \end{tikzcd}$$

is a *pushout square* if the resulting map

$$\big(B \sqcup C\big)\big/\sim \longrightarrow D$$

determined by u on B and v on C is a homeomorphism. Here "$\sim$" denotes the equivalence relation on the disjoint union $B \sqcup C$ generated by $f(x) \sim g(x)$ for all $x \in A$. (Intuitively, $f(x) \in B \subset B \sqcup C$ is glued to $g(x) \in C \subset B \sqcup C$.) In such a square, the space D and the maps u and v are in some sense completely determined by A, B, C and f, g, because D is $\big(B \sqcup C\big)\big/\sim$ up to renaming of elements, and u, v are the standard maps from B and C to that. — Note that in this situation a subset E of D is open in D if and only if $u^{-1}(E)$ is open in B and $v^{-1}(E)$ is open in C.
Also note that if $f\colon A \to B$ happens to be injective, then $v\colon C \to D$ is injective and $B \smallsetminus f(A)$ is homeomorphic to $D \smallsetminus v(C)$.

Example 10.1.3. Let $(V, \mathcal{S})$ be a vertex scheme. (So V is a set and $\mathcal{S}$ is a collection of nonempty finite subsets of V, and if T, S are nonempty finite subset of V such that $T \subset S$ and $S \in \mathcal{S}$, then $T \in sS$.) Recall that $|V|_{\mathcal{S}}$ is the set of functions $f\colon V \to [0,1]$ with the property that $V \smallsetminus f^{-1}(0)$ is an element of $\mathcal{S}$ and $\sum_{v \in V} f(v) = 1$. We defined a topology on that (perhaps not the one you think; see lecture notes WS13). Let $X = |V|_{\mathcal{S}}$ with that topology and let X^n consist of all the $f \in X$ such that $V \smallsetminus f^{-1}(0)$ has at most $n+1$ elements. Then X with these subspaces X^n is a CW-space. There is not much to prove here; it is almost true by the definition of $|V|_{\mathcal{S}}$. This CW-space has one n-cell for every element of $\mathcal{S}$ which has cardinality $n+1$ (as a subset of V).

Example 10.1.4. Let Y be a semi-simplicial set. Let $Y^{(n)}$ be the semi-simplicial subset of Y generated by the elements $y \in Y_k$ where $k \le n$. Then the geometric realization

$|Y|$ is a CW-space with the subspace $|Y^{(n)}|$ as its n-skeleton. Again there is not much to prove here. This CW-space has one n-cell for every $z \in Y_n$.

EXAMPLE 10.1.5. The sphere S^k has a structure of CW-space X where X^n is a single point for $n < k$ and $X^n = S^k$ when $n \geq k$. This CW-space has exactly two cells, one of dimension 0 and one of dimension k. (This example is also a special case of example 10.1.4.)

EXAMPLE 10.1.6. From the sequence of inclusions $\mathbb{R}^0 \subset \mathbb{R}^1 \subset \mathbb{R}^2 \subset \cdots \subset \mathbb{R}^k$ and the corresponding sequence of inclusions

$$\emptyset = S^{-1} \subset S^0 \subset S^1 \subset S^2 \subset \cdots \subset S^{k-1} \subset S^k$$

we obtain another CW-structure on $X = S^k$ where $X^n = S^n$ if $n \leq k$ and $X^n = S^k$ if $n \geq k$. (This example is not a special case of example 10.1.4 if $k > 1$.)

EXAMPLE 10.1.7. The CW-structure on $X = S^k$ in the previous example is invariant under the antipodal involution on S^k; that is to say, the antipodal map $X \to X$ takes each skeleton X^n to itself. Therefore or (preferably) by inspection, $Y = \mathbb{R}P^k$ has a CW-structure where Y^n is $\mathbb{R}P^n$ for $n \leq k$ and $Y^n = \mathbb{R}P^k$ if $n \geq k$.

EXAMPLE 10.1.8. A more difficult and more interesting example of a CW-space is the Grassmannian $G_{p,q}$ of p-dimensional linear subspaces in $\mathbb{R}^{p+q}$ with the CW-structure due to Schubert. (I believe Schubert found this in the 19th century, long before CW-spaces were invented.) The Grassmannian is probably well known to you from courses on differential topology or differential geometry as a fine example of a smooth manifold. Here we are not so interested in the manifold aspect, but we need to know that $G_{p,q}$ is a topological space. Write $n = p + q$. A p-dimensional linear subspace V of $\mathbb{R}^n$ determines a linear map $\mathbb{R}^n \to \mathbb{R}^n$, orthogonal projection to V. It has the following properties: self-adjoint, idempotent, rank p. In this way, $G_{p,q}$ can be identified with the set of $n \times n$-matrices which are symmetric, idempotent and of rank p. So $G_{p,q}$ is "contained" in the finite dimensional real vector space of real $n \times n$-matrices, which has a standard topology ... and we can give it the subspace topology.
Let $E(k)$ be the linear span of the first k standard basis vectors in $\mathbb{R}^n$. So we have an increasing sequence of real vector spaces

$$0 = E(0) \subset E(1) \subset E(2) \subset \cdots \subset E(n-1) \subset E(n) = \mathbb{R}^n.$$

Now let $V \in G_{p,q}$, that is to say, V is a p-dimensional linear subspace of $\mathbb{R}^n = E(n)$. Let $f_V(k) = \dim(V \cap E(k))$ for $k = 0, 1, 2, \ldots, n$. So V determines a function f_V from $\{0, 1, 2, \ldots, n\}$ to $\{0, 1, \ldots, p\}$. The function is a nondecreasing and surjective and satisfies $f_V(0) = 0$ and $f_V(n) = p$. Schubert's idea was to say: we put two elements V, W of $G_{p,q}$ in the same equivalence class if $f_V = f_W$. Let us see whether these equivalence classes are cells and if so, what their dimensions are. So fix a nondecreasing surjective f from $\{0, 1, \ldots, n\}$ to $\{0, 1, \ldots, p\}$ which satisfies $f(0) = 0$, $f(n) = p$, and let us be interested in the set of $V \in G_{p,q}$ having $f_V = f$. Let

$$f_! \colon \{1, \ldots, p\} \to \{1, \ldots, n\}$$

be the *injective* monotone function such that $f_!(j)$ is the minimal element among the i having $f(i) = j$. Form the set A_f of real $n \times p$-matrices

$$(M_{ij})$$

where $M_{ij} = 0$ if $i > f_!(j)$, $M_{ij} = 1$ if $i = f_!(j)$, and $M_{ij} = 0$ if $i = f_!(k)$ for some $k < j$. The columns are linearly independent. So we can make a map from A_f to $G_{p,q}$ by taking

(M_{ij}) to its column span. Etc. etc. ; this gives a homeomorphism from A_f to the set of $V \in G_{p,q}$ having $f_V = f$. Now clearly A_f is an affine subspace of $\mathbb{R}^{p\times q}$ (translate of a linear subspace) and its dimension is

$$\sum_{k=1}^{p} (f_!(k) - 1) - (k-1) = \sum_{k=1}^{p} f_!(k) - k\ .$$

Therefore we are allowed to say that the set of $V \in G_{p,q}$ having $f_V = f$ is a cell. It will be left as an exercise to show that Schubert's partition of $G_{p,q}$ into cells is in fact a structure of CW-space (where the n-skeleton, obviously, has to be the union of all cells whose dimension is at most n). There are $\binom{n}{p}$ cells in the structure; the maximum of their dimensions is

$$n + (n-1) + \cdots + (n-p+1) - (1+2+\cdots+p) = p(n-p) = pq$$

and there is exactly one cell which has the maximal dimension. It corresponds to the $f\colon \{0,1,2,\dots,n\} \to \{0,1,2,\dots,p\}$ which has $f(x) = x-(n-p)$ for $x > n-p$ and $f(x) = 0$ otherwise.

10.2. CW-subspaces and CW quotient spaces

Proposition 10.2.1. *Let X be a CW-space and $A \subset X$ a closed subspace such that A is a union of cells of X. Then A becomes a CW-space in its own right if we define $A^n := X^n \cap A$.*

In this situation we call A a *CW-subspace* of X.

Sketch proof. There is not much to prove here. Let $Z \subset X$ be an n-cell which is contained in A. Let $\varphi_Z\colon D^n \to X$ be a characteristic map for Z, so that φ_Z restricts to a homeomorphism from $D^n \smallsetminus S^{n-1}$ to Z. The image of φ_Z is contained in A because it is the closure $\bar{Z}$ of Z in X, and $\bar{Z} \subset A$ because $Z \subset A$ and A is closed in X. Therefore we can write $\varphi_Z\colon D^n \to Z$ without lying very hard. Now choose characteristic maps for all the n-cells of X, giving a pushout square

$$\begin{tikzcd}
\coprod_{\lambda\in\Lambda_n} S^{n-1} \arrow[r, "\text{incl.}"] \arrow[d] & \coprod_{\lambda\in\Lambda_n} D^n \arrow[d] \\
X^{n-1} \arrow[r, "\text{incl.}"] & X^n
\end{tikzcd}$$

as in definition 10.1.1. Here Λ_n is in a (chosen) bijection with the set of n-cells of X. Let $\Lambda'_n \subset \Lambda_n$ be the subset corresponding to the n-cells which are contained in A. Then by what we have just seen there is a commutative square

$$\begin{tikzcd}
\coprod_{\lambda\in\Lambda'_n} S^{n-1} \arrow[r, "\text{incl.}"] \arrow[d] & \coprod_{\lambda\in\Lambda'_n} D^n \arrow[d] \\
X^{n-1} \cap A \arrow[r, "\text{incl.}"] & X^n \cap A
\end{tikzcd}$$

which is obtained from the previous square by appropriate restrictions. It is easy to show that this is again a pushout square. This verifies condition (2) in definition 10.1.1 for the space A. $\square$

PROPOSITION 10.2.2. *Under assumptions as in proposition 10.2.1, the quotient space* X/A *is also a CW-space with the definition*

$$(X/A)^n := X^n/A^n = X^n/(X^n \cap A).$$

Remark. It is wise to define the quotient space X/A as the pushout of $X \leftarrow A \rightarrow \star$ where, as usual, $\star$ denotes a singleton space and the left-hand arrow is the inclusion. This removes an ambiguity which would otherwise arise if A is empty. Namely, if A is empty, then X/A is homeomorphic to $X \sqcup \star$. (Consequently it is *not* quite correct to say that X/A is the quotient space of X by the equivalence relation which is generated by $x \sim y$ if $x, y \in A$. That statement is only correct when A is nonempty.) It follows that X/A is always a based space, i.e., it has a distinguished element or singleton subspace which we can again denote by $\star$ without lying too hard.

PROOF OF PROPOSITION 10.2.2. In the notation of the proof of proposition 10.2.1: a choice of characteristic maps for the n-cells of X gives us a pushout square

$$\begin{tikzcd} \coprod_{\lambda \in \Lambda_n} S^{n-1} \arrow[r, "\text{incl.}"] \arrow[d] & \coprod_{\lambda \in \Lambda_n} D^n \arrow[d] \\ X^{n-1} \arrow[r, "\text{incl.}"] & X^n \end{tikzcd}$$

and if $n > 0$ this determines a pushout square

$$\begin{tikzcd} \coprod_{\lambda \in \Lambda_n \smallsetminus \Lambda'_n} S^{n-1} \arrow[r, "\text{incl.}"] \arrow[d] & \coprod_{\lambda \in \Lambda_n \smallsetminus \Lambda'_n} D^n \arrow[d] \\ X^{n-1}/A^{n-1} \arrow[r, "\text{incl.}"] & X^n/A^n \end{tikzcd}$$

Here the vertical maps are obtained by using the chosen characteristic maps for the n-cells of X and composing with the quotient map $X^n \to X^n/A^n$, or $X^{n-1} \to X^{n-1}/A^{n-1}$ where appropriate. The case $n = 0$ is different: we have $(X/A)^0 = X^0/A^0 \cong \Lambda_0/\Lambda'_0$ which is *not* identifiable with $\Lambda_0 \smallsetminus \Lambda'_0$ because it has one extra element. That extra element accounts for the base point of X/A, which is a 0-cell in X/A. $\square$

EXAMPLE 10.2.3. In the notation of example 10.1.7, the quotient space $\mathbb{R}P^k/\mathbb{R}P^n$ where $0 < n < k$ is a CW-space which has one 0-cell (base point), then one cell exactly in each of the dimensions $n+1, n+2, \ldots, k$, and no cells in other dimensions. These based spaces are called *stunted projective spaces*.

CHAPTER 11

Cellular maps and cellular homotopies

11.1. The homotopy extension property

LEMMA 11.1.1. *Let X be a CW-space and let A be a CW-subspace of X. Let Y be any space, $f\colon X \to Y$ a continuous map and $(h_t\colon A \to Y)_{t\in[0,1]}$ a homotopy such that $h_0 = f_{|A}$. Then there exists a homotopy*

$$(\bar{h}_t\colon X \to Y)_{t\in[0,1]}$$

such that $\bar{h}_t{}_{|A} = h_t$ for all $t \in [0,1]$ and $\bar{h}_0 = f$.

Remark. In the language of homotopy theory, this can be stated by saying that the inclusion $A \to X$ has the HEP, homotopy extension property. Equivalently, the inclusion $A \to X$ is a *cofibration*.

PROOF. We construct homotopies

$$(\bar{h}_{t,n}\colon X^n \to Y)_{t\in[0,1]}$$

by induction on n. These will be compatible, in the sense that $\bar{h}_{t,n-1}$ is the restriction of $\bar{h}_{t,n}$ to $X^{n-1} \times [0,1]$. Then we can define $\bar{h}_t$ so that it agree with $\bar{h}_{t,n}$ on $X^n \times [0,1]$. Because of condition (1) in the definition of CW-space, there is no continuity problem. Therefore, for the induction step, assume that the homotopy

$$(\bar{h}_{t,n-1}\colon X^{n-1} \to Y)_{t\in[0,1]}$$

has already been constructed, and that it agrees with the prescribed $(h_t)_{t\in[0,1]}$ on $A^{n-1} \times [0,1]$, and also that $\bar{h}_{0,n-1}$ agrees with f on X^{n-1}. We wish to construct

$$(\bar{h}_{t,n-1}\colon X^n \to Y)_{t\in[0,1]}$$

which, to be honest, is a map $X^n \times [0,1] \to Y$. This map is already defined for us on $X^{n-1} \times [0,1]$ and on $A^n \times [0,1]$. What this means is that it is not defined on the n-cells of X which are not contained in A. Choose characteristic maps for these to get a pushout square

$$\begin{tikzcd}
\coprod_{\Lambda_n \smallsetminus \Lambda'_n} S^{n-1} \arrow[r] \arrow[d] & \coprod_{\Lambda_n \smallsetminus \Lambda'_n} D^n \arrow[d, "\varphi"] \\
X^{n-1} \cup A^n \arrow[r] & X^n
\end{tikzcd}$$

whre Λ_n is an indexing set for the n-cells of X, and $\Lambda'_n \subset \Lambda_n$ corresponds to the n-cells which are in A. By the good properties of pushouts, it is now enough to define a homotopy

$$(g_t\colon \coprod D^n \to Y)_{t\in[0,1]}$$

which agrees with $\bar{h}_{t,n-1} \circ \varphi$ on $\coprod S^{n-1}$ and, for $t = 0$, with $f \circ \varphi$ on $\coprod D^n$. The coproducts are indexed by $\Lambda_n \smallsetminus \Lambda'_n$. By the good properties of coproducts, it is then also enough to define for each $\lambda \in \Lambda_n \smallsetminus \Lambda_{n-1}$ a homotopy

$$(g_{t,\lambda} \colon D^n \to Y)_{t \in [0,1]}$$

which agrees with $\bar{h}_{t,n-1} \circ \varphi$ on that copy of S^{n-1} and, for $t = 0$, with $f \circ \varphi$ on that copy of D^n (where *that copy* refers to the copy corresponding to λ). Of course, the homotopy $(g_{t,\lambda})_{t \in [0,1]}$ is really a map

$$D^n \times [0,1] \to Y$$

to be constructed which is already defined for us on $(D^n \times \{0\}) \cup (S^{n-1} \times [0,1])$. Therefore it suffices to show: *every* continuous map

$$u \colon (D^n \times \{0\}) \cup (S^{n-1} \times [0,1]) \longrightarrow Y$$

admits an extension to a continuous map $v \colon D^n \times [0,1] \to Y$. A solution to that is $v = u \circ r$ where

$$r \colon D^n \times [0,1] \longrightarrow (D^n \times \{0\}) \cup (S^{n-1} \times [0,1])$$

is a map which agrees with the identity on $(D^n \times \{0\}) \cup (S^{n-1} \times [0,1])$. Such a map r can be obtained as follows. View $D^n \times [0,1]$ as a subspace of $\mathbb{R}^n \times \mathbb{R}$ in the most obvious way. Let z be the point $(0,0,0,\ldots,0,2)$ in $\mathbb{R}^n \times \mathbb{R}$. Define r in such a way that $r(x)$ is the unique point where the line through x and z intersects $(D^n \times \{0\}) \cup (S^{n-1} \times [0,1])$. $\square$

11.2. Cellular maps

Definition 11.2.1. Let $f \colon X \to Y$ be a continuous map, where X and Y are CW-spaces. The map f is called *cellular* if $f(X^n) \subset Y^n$ for all $n \geq 0$.

Example 11.2.2. View S^1 as the unit circle in $\mathbb{C}$. For $n \in \mathbb{Z}$, the map $f \colon S^1 \to S^1$ defined by $f(z) = z^n$ is a cellular map if we use the CW-structure on S^1 which has 0-skeleton equal to $\{1\}$ and 1-skeleton equal to all of S^1. If instead we use the CW-structure on S^1 with 0-skeleton S^0 and 1-skeleton equal to all of S^1, then f is also a cellular map.

Example 11.2.3. The antipodal map $g \colon S^n \to S^n$ is not a cellular map if we use a CW-structure on S^n with exactly one 0-cell and exactly one n-cell and no other cells.

11.3. Approximation of maps by cellular maps

Lemma 11.3.1. *Let U be an open subset of $\mathbb{R}^n$ and $f \colon U \to \mathbb{R}^{n+k}$ a continuous map such that $f^{-1}(0)$ is compact, where $k > 0$. Then for any $\varepsilon > 0$ there exists a map $g \colon U \to \mathbb{R}^{n+k}$ such that $\|g - f\| \leq \varepsilon$, the support of $g - f$ is compact and $g^{-1}(0) = \emptyset$.*

Proof. There are two well-known methods for this. One is to use Sard's theorem. Choose open sets V_1 and V_2 in $\mathbb{R}^n$ such that $V_1 \cup V_2 = U$, where V_1 has compact closure in U and contains $f^{-1}(0)$. Choose a smooth function $\varphi \colon U \to [0,1]$ with compact support such that $\operatorname{supp}(1-\varphi) \cap f^{-1}(0) = \emptyset$. Without loss of generality, ε is less than the minimum of $\|f\|$ on the compact set $\operatorname{supp}(\varphi) \cap \operatorname{supp}(1-\varphi)$. It is easy to construct a smooth map g_1 from U to $\mathbb{R}^{n+k}$ such that $\|f(x) - g_1(x)\| < \varepsilon/2$ for all $x \in U$. As a special case of Sard's theorem, the image of g_1 is a set of Lebesgue measure zero in $\mathbb{R}^{n+k}$. Hence there exists $y \in \mathbb{R}^{n+k}$, not in the image of g_1, such that $\|y\| < \varepsilon/2$. Let $g_2 = g_1 - y$, so that 0 is not in the image of g_2. By construction, $\|f(x) - g_2(x)\| < \varepsilon$ for all $x \in U$. Let $g = \varphi \cdot f + (1-\varphi) \cdot g_2$. This g has all the properties that we require.
The other method would be to use piecewise linear approximation. This is more elementary but also much more tedious. ... *Under construction.* $\square$

Corollary 11.3.2. *Let U be an open subset of $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}^{n+k}$ a continuous map such that $f^{-1}(0)$ is compact, $k > 0$. Then there exist a map $g\colon U \to \mathbb{R}^{n+k}$ such that $g^{-1}(0) = \emptyset$ and a homotopy $(h_t\colon U \to \mathbb{R}^{n+k})_{t\in[0,1]}$ such that $h_0 = f$, $h_1 = g$ and $(h_t)_{t\in[0,1]}$ is stationary*[1] *outside a compact subset K of U.*

Proof. Take g as in lemma 11.3.1. Put $h_t(x) := (1-t)f(x) + tg(x)$. □

Lemma 11.3.3. *Let $f\colon D^n \to X$ be a continuous map, where X is a CW-space. Suppose that $f(S^{n-1}) \subset X^{n-1}$. Then there exists a homotopy*

$$(h_t\colon D^n \to X)_{t\in[0,1]}$$

which is stationary on S^{n-1} and such that $h_0 = f$ while $h_1(D^n) \subset X^n$.

Proof. The image of f is compact, therefore contained in a compact CW-subspace Y of X (which must have finitely many cells only, as it is compact). We choose Y as small as possible. Suppose that the maximal dimension of the cells in Y is $n+k$, where $k > 0$. The $(n+k)$-cells in Y all have nonempty intersection with the image of f, otherwise the choice of Y was not minimal. Choose one of them, say $E \subset Y$, and let $U = f^{-1}(E) \subset D^n \smallsetminus S^{n-1}$, an open set. The restriction of f to U can be viewed as a map from U to $E \cong \mathbb{R}^{n+k}$. This is (after some more reparameterization) the situation of corollary 11.3.2. Therefore we can make a homotopy $(\alpha_t)_{t\in[0,1]}$ from f to a map $f_1\colon D^n \to X$ as in that corollary. (The homotopy is stationary outside a compact subset K of U, that is to say, it associates a *constant* path $t \mapsto \alpha_t(z)$ in X to every element z of $D^n \smallsetminus K$.) The advantage of f_1 compared with f is that it avoids the point p in $E \subset Y$ which corresponds to the origin of $\mathbb{R}^{n+k}$ in our parametrization of E. But the image of f_1 is still contained in Y. Now it is easy to make a homotopy

$$(\beta_t\colon Y \smallsetminus \{p\} \to Y)_{t\in[0,1]}$$

where β_0 is the inclusion and β_1 lands in the CW-subspace $Y \smallsetminus E$, and $(\beta_t)_{t\in[0,1]}$ is stationary on $Y \smallsetminus E$. Composing this homotopy with f_1, where we view f_1 as a map from D^n to $Y \smallsetminus \{p\}$ we get a homotopy

$$(\beta_t \circ f_1)_{t\in[0,1]}$$

from f_1 to a map $f_2 = \beta_1 \circ f_1$ which avoids the cell E entirely. The combined homotopy from f to f_2 is stationary on S^{n-1} by construction. We have made progress in the sense that the image of f_2 is contained in $Y \smallsetminus E$, a compact CW-subspace of X with fewer $(n+k)$-dimensional cells than Y. Carry on like this, treating f_2 as we treated f before. □

Corollary 11.3.4. *Every map $f\colon X \to Y$ between CW-spaces X and Y is homotopic to a cellular map.*

Proof. Let $a(n) = 1 - 2^{-n-1}$ for $n = -1, 0, 1, 2, 3, \ldots$. We write $f = f_{-1}$ and we construct maps $f_n\colon X \to Y$ such that f_n is cellular on X^n, and for each $n \geq 0$ a homotopy

$$(h_t\colon X \to Y)_{t\in[a(n-1),a(n)]}$$

which is stationary on X^{n-1} and such that $h_{a(n-1)} = f_{n-1}$ and $h_{a(n)} = f_n$.
Suppose that f_{n-1} and h_t for $0 \leq t \leq a(n-1)$ have already been constructed. By

[1] A homotopy $(\gamma_t\colon A \to B)_{t\in[0,1]}$ is *stationary* on a subspace C of A if the path $t \mapsto \gamma_t(x)$ is constant for every $x \in C$.

condition (2) in the definition of a CW-space and by lemma 11.3.3, we can define a homotopy

$$(g_t\colon X^n \to Y)_{t\in[a(n-1),a(n)]}$$

which is stationary on X^{n-1} and such that $g_{a(n)}(X^n) \subset Y^n$, and $g_{a(n-1)}$ agrees with f_{n-1} on X^{n-1}. By the homotopy extension property, lemma 11.1.1, that homotopy can be extended to a homotopy $(h_t\colon X \to Y)_{t\in[a(n-1),a(n)]}$, where $h_{a(n-1)} = f_{n-1}$. This completes the induction step. Now observe that the maps h_t so far constructed define a homotopy

$$(h_t\colon X \to Y)_{t\in[0,1]}$$

from $f = f_{-1}$ to another map $h_1 = f_\infty$, if we define h_1 so that it agrees with h_t on X^n for all $t \in [a(n), 1[$. The map f_∞ is cellular. □

11.4. Products of CW-spaces

This is quite an educational topic. Why are we interested in it here? Because we want say something about cellular approximation of homotopies. In connection with that we need to know that for a CW-space Y, the product $Y \times [0,1]$ is also a CW-space in a preferred way.

Lemma 11.4.1. (Kuratowski)*Let Y by any space and K a compact*[2] *space. Then the projection $p\colon Y\times K \to Y$ is a closed map, i.e., for any closed subset A of $Y\times K$ the image $p(A)$ is closed in Y.*

Proof. Choose closed $A \subset Y \times K$. Choose $z \in Y \smallsetminus p(A)$. Then $\{z\} \times K$ has empty intersection with the closed set A in $Y \times K$. So by definition of the topology on $Y \times K$, there exist open sets $U_\lambda \subset Y$ and $V_\lambda \subset K$ (depending on an index $\lambda \in \Lambda$) such that

$$\{z\} \times K \;\subset\; \bigcup_{\lambda\in\Lambda} \big(U_\lambda \times V_\lambda\big) \;\subset\; (Y \times K) \smallsetminus A\,.$$

By the compactness of K, we can assume that Λ is a finite set. We can also assume $z \in U_\lambda$ for all $\lambda \in \Lambda$. Then $\bigcap_\lambda U_\lambda$ is an open neighborhood of z which has empty intersection with $p(A)$. □

Proposition 11.4.2. (J.H.C. Whitehead) *Let $g\colon Y \to Z$ be a continuous map of spaces which is a quotient map*[3]*. Let K be a locally compact space. Then the map $Y\times K \to Z\times K$ defined by $(y,k) \mapsto (g(y),k)$ is also a quotient map.*

Proof. ... Later ... the proof will probably use lemma 11.4.1. □

Corollary 11.4.3. *Let X be a CW-space and let Y be a locally compact CW-space. Then the product $X \times Y$, with the product topology, becomes a CW-space if we define*

$$(X \times Y)^n := \bigcup_{p+q=n} X^p \times Y^q\ .$$

Proof. Let Λ be the set of cells of X and Θ the set of cells of Y. Choose characteristic maps

$$\varphi_\lambda\colon D^{n(\lambda)} \to X\,, \qquad \psi_\theta\colon D^{n(\theta)} \to Y$$

for the cells of X and Y. Then we have (in sloppy notation) maps

$$\varphi_\lambda \times \psi_\theta\colon D^{n(\lambda)} \times D^{n(\theta)} \longrightarrow X \times Y$$

[2]Not necessarily Hausdorff.

[3]Means: a subset W of Z is open if and only if $g^{-1}(W)$ is open in Y.

for each pair (λ, θ). We need to show mainly that the resulting map

$$\coprod_{(\lambda,\theta)\in\Lambda\times\Theta} D^{n(\lambda)} \times D^{n(\theta)} \longrightarrow X \times Y$$

is a quotient map. (Everything else that we might want to know follows easily from that. Note in particular that $D^{n(\lambda)} \times D^{n(\theta)}$ is homeomorphic to $D^{n(\lambda)+n(\theta)}$, so we can use the maps $\varphi_\lambda \times \psi_\theta$ as characteristic maps for cells in $X \times Y$.) To show this we write that map as a composition of two:

$$\coprod_{(\lambda,\theta)\in\Lambda\times\Theta} D^{n(\lambda)} \times D^{n(\theta)} \longrightarrow \coprod_{\lambda\in\Lambda} D^{n(\lambda)} \times Y$$

and

$$\coprod_{\lambda\in\Lambda} D^{n(\lambda)} \times Y \longrightarrow X \times Y\,.$$

It is easy to see that the first of these maps is a quotient map, because for each fixed λ the map from $\coprod_\theta D^{n(\lambda)} \times D^{n(\theta)}$ to $D^{n(\lambda)} \times Y$ is a quotient map. (Here we don't need Whitehead's proposition because it is a standard case of a surjective map from one compact Hausdorff space to another.) The second of these maps is a quotient map by Whitehead's proposition 11.4.2. □

11.5. Cellular approximation of homotopies

The goal is to prove:

THEOREM 11.5.1. *Let X and Y be CW-spaces and let $f, g\colon X \to Y$ be cellular maps. Suppose that f is homotopic to g. Then there exists a* cellular *homotopy from f to g, that is to say, a cellular map $H\colon X \times [0,1] \longrightarrow Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$.*

Here we are using the standard CW-structure on $[0,1]$ with two 0-cells $\{0\}$ and $\{1\}$ and one 1-cell, and we are using the product CW-structure on $X \times [0,1]$. This is the reason why we had to discuss products of CW-spaces in the previous (sub)section.

The proof is a special case of a slight refinement of corollary 11.3.4. The refinement is formulated in the following remark.

REMARK 11.5.2. Let $f\colon X \to Y$ be a map between CW-spaces and let $A \subset X$ be a CW-subspace such that $f_{|A}$ is already cellular. Then there exists a homotopy h from f to a map $g\colon X \to Y$ such that g is cellular, and the homotopy is stationary on A. The homotopy can be constructed exactly as in the proof of corollary 11.3.4; in step number n, worry only about the n-cells of X which are not in A.

PROOF OF THEOREM 11.5.1. It is a direct application of remark 11.5.2: but for X, A, f in the remark substitute $X \times [0,1]$, $X \times \{0,1\}$, H as in the statement of the theorem, respectively. □

CHAPTER 12

Homology of CW-spaces

12.1. Chain complexes

Definition 12.1.1. A chain complex, graded over $\mathbb{Z}$, is a family of abelian groups $(C_n)_{n\in\mathbb{Z}}$ together with homomorphisms $d_n\colon C_n \to C_{n-1}$ satisfying the condition $d_{n-1}\circ d_n = 0$ for all $n \in \mathbb{Z}$. (The homomorphisms d_n are sometimes called *boundary operators*, sometimes *differentials*.)

$$\cdots \xleftarrow{d_{n-1}} C_{n-1} \xleftarrow{d_n} C_n \xleftarrow{d_{n+1}} C_{n+1} \xleftarrow{d_{n+2}} \cdots$$

Example 12.1.2. Examples of chain complexes were seen in the last sections of last year's lecture notes, in connection with the homology of simplicial complexes and (geometric realizations of) semi-simplicial sets. We will see such examples again in connection with CW-spaces and their homology, soon. Here I want to give an indication of how we can associate a chain complex to a CW-space in an elementary way without knowing a great deal about homology. (A certain willingness to cheat is assumed.) So let X be a CW-space and let Λ_n be the set of n-cells of X (probably I mean: an indexing set for the n-cells of X). We want to build a chain complex

$$\cdots \xleftarrow{d_{n-1}} C(X)_{n-1} \xleftarrow{d_n} C(X)_n \xleftarrow{d_{n+1}} C(X)_{n+1} \xleftarrow{d_{n+2}} \cdots$$

called the *cellular chain complex* of X, and for that purpose we define provisionally

$$C(X)_n := \bigoplus_{\lambda\in\Lambda_n} \mathbb{Z}$$

(a direct sum of copies of $\mathbb{Z}$, one for each n-cell of X). That is the definition for $n \geq 0$, and for $n < 0$ we take $C(X)_n := 0$.

Therefore, although d_n has not been defined so far, we know already that it comes as a matrix with entries $a_{\sigma,\tau} \in \mathbb{Z}$, one entry for each $\sigma \in \Lambda_{n-1}$ and $\tau \in \Lambda_n$. (Each column of the matrix, corresponding to a fixed $\tau \in \Lambda_n$, can only have finitely many nonzero entries.)

To describe $a_{\sigma,\tau}$ we choose characteristic maps $D^n \to X^n$ and $D^{n-1} \to X^{n-1}$ for the n-cell corresponding to τ and the $(n-1)$-cell corresponding to σ. Restrict the first of these to get

$$S^{n-1} \to X^{n-1},$$

the *attaching map* for the cell corresponding to τ. The other one should be composed with the quotient map from X^{n-1} to $X^{n-1}/X^{n-1}_{\neg\sigma}$ where $X^{n-1}_{\neg\sigma}$ is the CW-subspace of X^{n-1} obtained by deleting the cell corresponding to σ from X^{n-1}. Because we have chosen a characteristic map for the cell of σ, that quotient space is now identified with $D^{n-1}/S^{n-2} \cong S^{n-1}$ and so that quotient map takes the form

$$X^{n-1} \to S^{n-1}.$$

I call it the *collapse map* for the cell corresponding to σ. It is clear what to do next: we compose the attaching map for the cell corresponding to τ with the collapse map for the cell corresponding to σ and we obtain a map $S^{n-1} \to S^{n-1}$. That map has a degree which is by definition

$$a_{\sigma,\tau} \in \mathbb{Z} \ .$$

A number of questions can be raised:

- Is $a_{\sigma,\tau}$ well defined? (It turns out that it is well defined up to sign only, and we need to do something about the sign problem later.)
- Is it really true that each column of the matrix $d_n = (a_{\sigma,\tau})$ has only finitely many nonzero entries? (Good exercise for you.)
- Is it really true that $d_{n-1} \circ d_n = 0$ for all n? (If we choose characteristic maps for all cells of X, once and for all, then d_n and d_{n-1} are defined and it turns out that $d_{n-1} \circ d_n$ is indeed 0, but I am not aware of a very short argument for that.)
- Is there an elementary definition of the degree of a map from S^n to S^n ? (Good question. John Milnor wrote a little book *Topology from the differentiable viewpoint* where he defines the degree of such a map using approximation by a smooth map and then Sard's theorem, and the concept of *regular value*. That's not soooo elementary but it is probably more elementary than using homology to define the degree.)

Definition 12.1.3. Let $C = (C_n, d_n)_{n\in\mathbb{Z}}$ and $D = (D_n, d'_n)$ be chain complexes. A *chain map* $f\colon C \to D$ is a family of homomorphisms $f_n\colon C_n \to D_n$ satisfying $d'_n \circ f_n = f_{n-1} \circ d_n$ for all n.

Example 12.1.4. A cellular map $f\colon X \to Y$ between CW-spaces determines a chain map $C(f)\colon C(X) \to C(Y)$ between their cellular chain complexes. ...

If $f\colon C \to D$ is a chain map and $g\colon D \to E$ is a chain map, then $g \circ f$ can be defined by means of $(g \circ f)_n = g_n \circ f_n$ and it is then a chain map from C to E. (Therefore chain complexes and chain maps from a category. The category is an *additive category*. In other words the set of chain maps from $C \to D$ is always an abelian group and composition is bilinear — more correctly, *bi-additive*.)

Definition 12.1.5. Let $C = (C_n, d_n)_{n\in\mathbb{Z}}$ and $D = (D_n, d'_n)_{n\in\mathbb{Z}}$ be chain complexes. A *chain homotopy* from a chain map $f\colon C \to D$ to a chain map $g\colon C \to D$ is a family of homomorphisms $h_n\colon C_n \to D_{n+1}$ satisfying

$$d'_{n+1} \circ h_n + h_{n-1} \circ d_n = g_n - f_n$$

for all n. If such a chain homotopy exists, then f and g are said to be *chain homotopic*.

It is fairly clear from the definition that chain homotopy is an equivalence relation on the abelian group of chain maps from C to D, and in fact a congruence relation, so that the set of equivalence classes is again an abelian group. This can be denoted by $[C, D]$ where necessary.

Example 12.1.6. A cellular homotopy h between cellular maps $f, g\colon X \to Y$ (between CW-spaces) determines a chain homotopy $C(h)$ connecting the chain maps $C(f)$ and $C(g)$ from $C(X)$ to $C(Y)$. ...

It is again fairly easy to show that the relation of chain homotopy is compatible with composition. That is, if $e\colon B \to C$ and $f, g\colon C \to D$ are chain maps and f, g are chain

homotopic, then $f \circ e$ is chain homotopic to $g \circ e$. Also if $f, g\colon C \to D$ are chain maps which are chain homotopic, and $k\colon D \to E$ is another chain map, then $k \circ g$ is chain homotopic to $k \circ f$. Therefore we have a well defined composition

$$[D, E] \times [C, D] \longrightarrow [C, E]$$

which takes a pair represented by chain maps $u\colon E \to D$ and $v\colon C \to D$ to $u \circ v$. (Therefore chain complexes and chain maps up to chain homotopy form a category. It is still an additive category.)

Definition 12.1.7. The direct sum of two chain complexes C and D is ... (exactly what you think it is).

Example 12.1.8. Let X and Y be CW-spaces. Then the cellular chain complex $C(X \sqcup Y)$ is isomorphic to the direct sum $C(X) \oplus C(Y)$.

Definition 12.1.9. The tensor product $C \otimes D$ of two chain complexes C and D is defined as follows:

$$(C \otimes D)_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

and the differential $(C \otimes D)_n \to (C \otimes D)_{n-1}$ is determined by

$$x \otimes y \ \mapsto \ \big(d(x) \otimes y\big) + (-1)^p \big(x \otimes d(y)\big)$$

for $x \in C_p$ and $y \in D_q$, assuming $p + q = n$. (A "generic" d has been used for the differentials in C and D.)

Remark 12.1.10. Write d'' for the differential in $C \otimes D$. With notation as above we have

$$\begin{aligned} & d''(d''(x \otimes y)) = d''\big(d(x) \otimes y\big) + (-1)^p \big(x \otimes d(y)\big) \\ = \ & d(d(x)) \otimes y + (-1)^{p-1} d(x) \otimes d'(y) + (-1)^p (d(x) \otimes d(y)) + x \otimes d(d(y)) \\ = \ & 0. \end{aligned}$$

Obviously the sign $(-1)^p$ is important to ensure that $d''d'' = 0$. There is a rule of thumb for this: if, in a product-like expression you move a term of degree u past a term of degree v, then you should probably introduce a sign $(-1)^{pq}$. For example $d''(x \otimes y) = d(x) \otimes y + (-1)^p x \otimes d(y)$ because it feels like moving the d, which has degree -1, past the x which was assumed to have degree p. Another application of this useful rule: $C \otimes D$ is isomorphic to $D \otimes C$ by the isomorphism taking $x \otimes y$ to $(-1)^{pq} y \otimes x$, where $x \in C_p$ and $y \in D_q$.

Example 12.1.11. Let X and Y be CW-spaces. Assume for simplicity that Y is locally compact (equivalently, every point in Y has a neighborhood which meets only finitely many cells). Then we know that $X \times Y$ is a again a CW-space where $(X \times Y)^n := \bigcup_{p+q=n} X^p \times Y^q$. For the cellular chain complexes we might reasonably expect to get

$$C(X \times Y) \cong C(X) \otimes C(Y) \ .$$

This is strictly true with our provisional definition of $C(X)$ etc. if we choose characteristic maps $\varphi_\lambda\colon D^p \to X$ and $\varphi_\sigma\colon D^q \to Y$ for all cells of X and Y and use these to choose characteristic maps for the cells of $X \times Y$:

$$D^p \times D^q \longrightarrow X \times Y \, ; (w, z) \mapsto (\varphi_\lambda(w), \varphi_\sigma(z)) \ .$$

It would probably require a proof, but we can easily see that

$$C_n(X \times Y) \cong \bigoplus_{p+q=n} C_p(X) \otimes C_q(Y)$$

because the left-hand side is a free abelian group with one generator for each cell of $X \times Y$, while the right-hand side is the free abelian group with one generator for each pair consisting of a cell in x and a cell in Y.

EXAMPLE 12.1.12. Let C be the chain complex which has $C_0 = \mathbb{Z} \oplus \mathbb{Z}$ and $C_1 = \mathbb{Z}$, all other chain groups equal to 0, and differential $d\colon C_1 \to C_0$ given by $d(c) = -a \oplus b$ where a, b, c are the preferred generators. Think of this as the cellular chain complex of $[0, 1]$. Let D and E be some other chain complexes. A chain map α from $C \otimes D$ to E is exactly the same thing as a triple consisting of two chain maps $f, g\colon D \to E$ and a homotopy h from f to g. Namely, given α define

$$f(x) = \alpha(a \otimes x), \quad g(x) = \alpha(b \otimes x), \quad h(x) = \alpha(c \otimes x)\ .$$

Then

$$\begin{aligned} d_E(h(x)) = &\ d_E(\alpha(c \otimes x)) = \alpha(d_{C\otimes D}(c \otimes x)) = \alpha(d(c) \otimes x - c \otimes d(x)) \\ = &\ \alpha(-a \otimes x + b \otimes x - c \otimes d(x)) \\ = &\ -f(x) + g(x) - h(d_D(x)) \end{aligned}$$

and therefore $d_E \circ h + h \circ d_D = -f + g$ as claimed.
This means that *chain homotopy* is a concept analogous to *homotopy* in the setting of spaces, because a homotopy between maps from X to Y is the same thing as a map from $[0, 1] \times X$ to Y. (And the product $\times$ of spaces corresponds to the tensor product $\otimes$ of chain complexes, and the unit interval $[0, 1]$ corresponds to the chain complex that we have called C.)

DEFINITION 12.1.13. Let C be a chain complex with differential d. The homology group $H_n(C)$ is the (group-theoretic) quotient

$$\frac{\ker[d\colon C_n \to C_{n-1}]}{\operatorname{im}[d\colon C_{n+1} \to C_n]}\ .$$

(Elements of $\ker[d\colon C_n \to C_{n-1}]$ are also called n-*cycles*, and elements of $\operatorname{im}[d\colon C_{n+1} \to C_n]$ are called n-dimensional boundaries. The equation $dd = 0$ ensures that the subgroup of n-dimensional boundaries in C_n is contained in the subgroup of n-dimensional cycles; the quotient n-*cycles modulo* n-*boundaries* is the n-th homology group of C.)

PROPOSITION 12.1.14. *The homology group H_n is a functor (from the category of chain complexes and chain maps to the category of abelian groups). More precisely, a chain map $f\colon C \to D$ determines a homomorphism of abelian groups $f_*\colon H_n(C) \to H_n(D)$ and the conditions for a functor are satisfied.*

The definition of f_* is: $f_*[x] := [f(x)]$ where x is an n-cycle in C_n , representing an element $[x]$ of $H_n(C)$. The main point is to show that this is well defined: if $[x] = [y]$ then $y = x + d(z)$ for some $z \in C_{n+1}$, and so

$$f(y) = f(x + d(z)) = f(x) + f(d(z)) = f(x) + d(f(z))$$

which tells us that $[f(y)] = [f(x)] \in H_n(D)$.

PROPOSITION 12.1.15. *If $f, g\colon C \to D$ are homotopic chain maps, then*

$$f_* = g_*\colon H_n(C) \to H_n(D).$$

PROOF. Let $[x] \in H_n(C)$ and let h be a chain homotopy from f to g. Then $f_*[x] = [f(x)]$ whereas $g_*[x] = [g(x)]$. But $g(x) = f(x) + h(d(x)) + d(h(x)) = f(x) + d(h(x))$, where we have used $d(x) = 0$. So $[g(x)] = [f(x)]$. $\square$

Here are some definitions related to the word *exact*. A diagram of abelian groups and homomorphisms

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is called *exact* if $\ker(f) = \mathrm{im}(g)$. This implies $g \circ f = 0$, but it is a stronger condition. We also say that a longer string of morphisms such as

$$\cdots \xleftarrow{e_{n-1}} C_{n-1} \xleftarrow{e_n} C_n \xleftarrow{e_{n+1}} \cdots$$

is *exact* if $\ker(e_{n-1}) = \mathrm{im}(e_n)$ for all n. An exact diagram of the form

$$0 \to A \to B \to C \to 0$$

is also called *short exact*. This means the the homomorphism $A \to B$ in the diagram is injective (because its kernel is zero, because the image of the previous arrow is zero) and the homomorphism $B \to C$ in the diagram is surjective (because its image is everything, because the kernel of the next arrow is everything) and the kernel of $B \to C$ is the image of $A \to B$. In this situation of a short exact sequence, it is not far from the truth to say that A is a subgroup of B and C is the quotient group B/A. (Remember that these groups are abelian.)

We can also speak of a short exact sequence of chain complexes:

$$0 \to A \to B \to C \to 0\ .$$

The correct interpretation of this is that A, B, C are chain complexes and that we have a chain map $A \to B$ and a chain map $B \to C$ such that, for every $n \in \mathbb{Z}$, the given homomorphisms $A_n \to B_n$ and $B_n \to C_n$ make up a short exact sequence

$$0 \to A_n \to B_n \to C_n \to 0\ .$$

Lemma 12.1.16. *A short exact sequence of chain complexes*

$$0 \longrightarrow A \xrightarrow{j} B \xrightarrow{p} C \longrightarrow 0$$

determines homomorphisms $\partial\colon H_n(C) \to H_{n-1}(A)$ *for all* $n \in \mathbb{Z}$ *by the formula*

$$\partial([x]) := [d_B(y)]$$

for $x \in C_n$ *with* $d_C(x) = 0$*, where* $y \in C_n$ *satisfies* $p(y) = x$.

Proof. This lemma is also meant as a definition, but we still need to verify that the definition makes sense and is unambiguous. We may pretend that A is a subcomplex of B and that $C = B/A$, but it is still useful to have the name p for the projection $B \to B/A$. First of all, $[x] \in H_n(C)$ is represented by $x \in C_n$ with $d_C(x) = 0$. What is y ? It is an element of B_n which is mapped to x by p. We know that y exists because p is surjective. But we do not know that $d_B(y) = 0$ and this is exactly where the idea of this definition comes from. We do know that $d_B(y) \in B_{n-1}$ and that $p(d_B(y)) = d_C(p(y)) = d_C(x) = 0$. It follows by the supposed exactness that $d_B(y)$ is in the subgroup $A_{n-1} \subset B_{n-1}$. Also it is clear that $d_A d_B(y) = d_B d_B(y) = 0$ since A is a subcomplex of B. Therefore $d_B(y)$ represents an element $[d_B(y)]$ of $H_{n-1}(A)$.

Is this well defined? Instead of y, we could have selected another element $z \in B_n$ such that $p(z) = x$. Then $p(z - y) = 0$, so $z - y \in A_n$ by exactness. Therefore $[d_B(z)] - [d_B(y)] = [d_A(z - y)]$. And $[d_A(z - y)]$ is zero in $H_{n-1}(A)$ by the definition of $H_{n-1}(A)$. $\square$

THEOREM 12.1.17. (The long exact sequence of homology groups of a short exact sequence of chain complexes.) *In the situation of lemma 12.1.16, the sequence of homomorphisms*

$$\cdots \xrightarrow{p_*} H_{n+1}(C) \xrightarrow{\partial} H_n(A) \xrightarrow{j_*} H_n(B) \xrightarrow{p_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{j_*} \cdots$$

is exact.

PROOF. This is awfully routine. As before we can pretend that $A \subset B$ and $C = B/A$. There are six sub-statements to prove.

- Showing $p_* j_* = 0$: because $p_* j_* = (pj)_* = 0$.
- Showing $\partial p_* = 0$: for $y \in C_n$ with $d_C(y) = 0$, we have $p_*([y]) = [p(y)]$. So $\partial(p_*([y])) = \partial([p(y)]) = [d_B(y)] = 0$ by definition of ∂.
- Showing $j_* \partial = 0$: for $[x] \in H_n(C)$ and $y \in B_n$ with $p(y) = x$ we have $\partial[x] = [d_B(y)] \in H_{n-1}(A)$. So $j_* \partial[x] = [d_B(y)] \in H_{n-1}(B)$ which is zero since $d_B(y)$ is obviously in the image of $d_B \colon B_n \to B_{n-1}$.
- im $\supset$ ker at $H_n(B)$: if $[y] \in H_n(B)$ and $p_*[y] = [p(y)] = 0 \in H_n(C)$, then $\exists x \in C_{n+1}$ satisfying $d_C(x) = p(y)$. Then $\exists z \in B_{n+1}$ satisfying $p(z) = x$. So $[y] = [y']$ where $y' = y - d_B(z)$, but now $p(y') = 0$. So $y' \in A_n$ by exactness. Now $[y'] \in H_n(A)$ satisfies $j_*[y'] = [y]$.
- im $\supset$ ker at $H_n(C)$: if $[x] \in H_n(C)$ and $\partial([x]) = 0$ and $x = p(y)$ for some $y \in B_n$, then $[d_B(y)] = 0 \in H_{n-1}(A)$. So $\exists w \in A_n$ satisfying $d_A(w) = d_B(y)$, and so $d_B(y - w) = 0$, and so $[y - w] \in H_n(B)$ is defined. Then $p_*[y - w] = [p(y) - p(w)] = [p(y)] = [x]$.
- im $\supset$ ker at $H_n(A)$: if $[w] \in H_n(A)$ and $[w] = 0 \in H_n(B)$, then $\exists v \in B_{n+1}$ satisfying $d_B(v) = w$. Then $d_C(p(v)) = p(w) = 0$, so $[p(v)] \in H_{n+1}(C)$ is defined, and $\partial[p(v)] = [d_B(v)] = [w]$. $\square$

The following lemma is often useful in connection or conjunction with theorem 12.1.17.

LEMMA 12.1.18. (The Five lemma) *Suppose given a commutative diagram of abelian groups*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & B \arrow[r, "u"] \arrow[d, "g"] & C \arrow[r, "v"] \arrow[d, "h"] & D \arrow[r] \arrow[d, "j"] & E \arrow[d, "k"] \\
A' \arrow[r] & B' \arrow[r, "u'"] & C' \arrow[r, "v'"] & D' \arrow[r] & E'
\end{tikzcd}$$

with exact rows. If f, g, j *and* k *are isomorphisms, then* h *is also an isomorphism.*

PROOF. It is a good idea to reduce as quickly as possible to the situation where A, A', D and D' are zero (so that the rows are *short* exact). To achieve this we replace the above diagram by

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \operatorname{coker}(A \to B) \arrow[r] \arrow[d, "g_1"] & C \arrow[r] \arrow[d, "h"] & \ker(D \to E) \arrow[r] \arrow[d, "j_1"] & 0 \arrow[d] \\
0 \arrow[r] & \operatorname{coker}(A' \to B') \arrow[r] & C' \arrow[r] & \ker(D' \to E') \arrow[r] & 0
\end{tikzcd}$$

where $\operatorname{coker}(A \to B)$ means $B/\operatorname{im}(A \to B)$. The homomorphism j_1 is obtained from j by restriction and g_1 is obtained from g by passing to quotients. In this new diagram, g_1 and j_1 are still isomorphisms, if f, g, j, k were isomorphisms in the old one. So we have achieved the reduction.

We return to the notation *and to the diagram* of the lemma. We may now assume $A = A' = D = D' = 0$ and as before we assume that g and j are isomorphisms. Given $y \in C'$

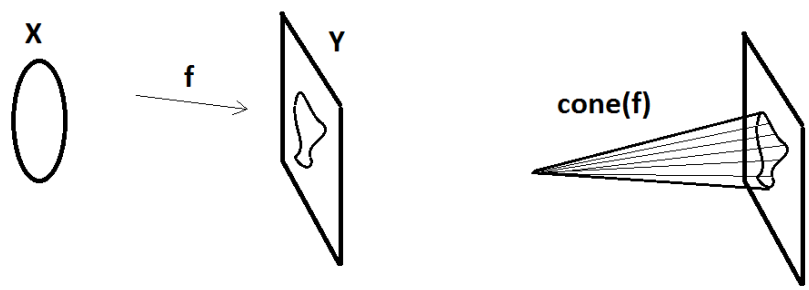
there is $x \in C$ such that $j(v(x)) = v'(y)$. Then $v'(y - h(x)) = 0$, so $y - h(x) = u'(z) = h(u((g^{-1}(z)))$ for some $z \in B$. So $y - h(x)$ is in the image of h, and so y is in the image of h. So h is surjective. For $x \in C$ with $h(x) = 0$ we know $j(v(x)) = 0$, so $v(x) = 0$ and so $x = u(z)$ for some $z \in B$ and we must have $g(z) = 0$ and so $z = 0$ and so $x = 0$. Therefore h is *injective*. (The method of proof is called diagram chasing and we should probably not be proud of it. There ought to be a better way.) □

12.2. Mapping cones

Definition 12.2.1. Let $f\colon X \to Y$ be a map of spaces. The *mapping cone* of f is the quotient space

$$\frac{Y \sqcup [0,1] \times X \sqcup \{1\}}{\sim}$$

where "$\sim$" is the smallest equivalence relation such that $(0, x) \sim f(x) \in Y$ for all $x \in X$ and $(1, x) \sim 1 \in \{1\}$ for all $x \in X$. Notation: $\mathrm{cone}(f)$.



The mapping cone has a distinguished base point 1; this is sometimes important.[1]

Suppose that X is a closed subset of Y and $f\colon X \to Y$ is the inclusion map. Then there is a comparison map

$$p\colon \mathrm{cone}(f) \longrightarrow Y/X$$

where Y/X is understood to be $\{\infty\} \sqcup Y$ modulo the smallest equivalence relation which has $y \sim \infty$ for all $y \in X$. (Note that Y/X also has a distinguished base point ∞ by construction.[2]) The formula for the comparison map is: equivalence class of (t, x) maps to the base point ∞ for all $(t, x) \in [0,1] \times X$; equivalence class of $y \in Y$ maps to equivalence class of y.

[1] Question for the gentle reader: what does $\mathrm{cone}(f)$ look like when X is empty ?

[2] What does Y/X look like when X is empty ?

PROPOSITION 12.2.2. *If the inclusion* $f\colon X \to Y$ *is a cofibration* (has the homotopy extension property)*, then the comparison map* $p\colon \mathrm{cone}(f) \to Y/X$ *is a homotopy equivalence.*

PROOF. Let $j\colon Y \to \mathrm{cone}(f)$ be the obvious inclusion. The composition

$$jf\colon X \to \mathrm{cone}(f)$$

has a nullhomotopy $(h_t\colon X \to \mathrm{cone}(f))_{t\in[0,1]}$ given by

$$h_t(x) = \text{equivalence class of } (t,x) \text{ in } \mathrm{cone}(f),$$

so that $h_0 = jf$ and h_1 is constant (with value 1). Since f has the homotopy extension property, there exists a homotopy $(H_t\colon Y \to \mathrm{cone}(f))_{t\in[0,1]}$ such that $H_0 = j$ and $H_t f = h_t$ for all $t \in [0,1]$. Then H_1 is a map from Y to $\mathrm{cone}(f)$ which maps all of X to the base point 1. So H_1 can be viewed as a map q from Y/X to $\mathrm{cone}(f)$. We will show that $pq \sim \mathrm{id}_{Y/X}$ and $qp \sim \mathrm{id}_{\mathrm{cone}(f)}$. *First claim*: pq is homotopic to $\mathrm{id}_{Y/X}$ by the homotopy $(pH_{1-t})_{t\in[0,1]}$. Strictly speaking pH_{1-t} is a map from Y to Y/X, but it maps all of X to the base point. *Second claim*: qp is homotopic to $\mathrm{id}_{\mathrm{cone}(f)}$ by the homotopy which agrees with $(H_{1-t})_{t\in[0,1]}$ on $Y \subset \mathrm{cone}(f)$ and which agrees with $((s,x) \mapsto (1-t+ts,x))_{t\in[0,1]}$ on points of the form (s,x) in $\mathrm{cone}(f)$, where $x \in X$ and $s \in [0,1]$. □

Let's note that all the maps (and homotopies) in this proof were base-point preserving. So it can be said that $p\colon \mathrm{cone}(f) \to Y/X$ is a pointed homotopy equivalence, in the situation of the proposition.

12.3. Homology of the mapping cone

DEFINITION 12.3.1. The *reduced homology* of a space X with base point $\star$ is

$$\tilde{H}_n(X) := H_n(X)/H_n(\star)$$

by which is meant the cokernel of the inclusion-induced (injective) map from $H_n(\star)$ to $H_n(X)$.

Clearly $H_n(X) = \tilde{H}_n(X)$ for $n \neq 0$, since $H_n(\star)$ is nonzero only for $n = 0$. The tilde notation is therefore mostly welcome when we are tired of making exceptions for $n = 0$. (It is also customary to define the reduced n-th homology of a nonempty space X with no specified base point as the kernel of the homomorphism $H_n(X) \to H_n(\star)$ induced by the unique map $X \to \star$. This is clearly isomorphic to the above definition of reduced homology when X *has* a chosen base point.)

PROPOSITION 12.3.2. *For a map* $f\colon X \to Y$*, there is a long exact sequence of homology groups*

$$\cdots \longrightarrow H_n(X) \xrightarrow{f_*} H_n(Y) \xrightarrow{j_*} \tilde{H}_n(\mathrm{cone}(f)) \longrightarrow H_{n-1}(X) \xrightarrow{f_*} H_{n-1}(Y) \xrightarrow{j_*} \cdots$$

PROOF. This is essentially the Mayer-Vietoris sequence of the open covering of $\mathrm{cone}(f)$ by open subsets $V = \mathrm{cone}(f) \smallsetminus \star$ and $W = \mathrm{cone}(f) \smallsetminus Y$, where $\star$ is the base point (also known as 1). So let us look at this MV sequence:

$$\cdots \to H_n(V \cap W) \to H_n(V) \oplus H_n(W) \to H_n(\mathrm{cone}(f)) \to H_{n-1}(V \cap W) \to \cdots$$

It should be clear that W is contractible; the picture of $\mathrm{cone}(f)$ above illustrates that well. Also, it is not hard to see that the inclusion $Y \to V$ is a homotopy equivalence; the picture of $\mathrm{cone}(f)$ above illustrates that well, too! Last not least, $V \cap W$ is the same as

X times open interval, so homotopy equivalent to X. Taking all that into account, we can write the MV sequence in the form

$$\cdots \to H_n(X) \to H_n(Y) \oplus H_n(\star) \to H_n(\mathrm{cone}(f)) \to H_{n-1}(X) \to \cdots$$

Now we observe that exactness is not affected if we put a tilde over each $H_n(\star)$ and over each $H_n(\mathrm{cone}(f))$. Indeed, it means that we are taking out two copies of $\mathbb{Z}$ in adjacent locations of the long exact sequence (only where $n = 0$) and the homomorphism relating them maps one of these copies of $\mathbb{Z}$ isomorphically to the other. Then we have a long exact sequence

$$\cdots \to H_n(X) \to H_n(Y) \oplus \tilde{H}_n(\star) \to \tilde{H}_n(\mathrm{cone}(f)) \to H_{n-1}(X) \to \cdots$$

And now we conclude by observing that $\tilde{H}_n(\star)$ is always zero. So it can be deleted without loss. □

Corollary 12.3.3. *Let X be a closed subspace of Y such that the inclusion $X \to Y$ is a cofibration. Then there is a long exact sequence of homology groups*

$$\cdots \longrightarrow H_n(X) \xrightarrow{f_*} H_n(Y) \xrightarrow{p_*} \tilde{H}_n(Y/X) \longrightarrow H_{n-1}(X) \xrightarrow{f_*} H_{n-1}(Y) \xrightarrow{p_*} \cdots$$

□

Example 12.3.4. *This example is also a remark on an issue of normalization.* Take $Y = D^m$ and $X = S^{m-1}$ in corollary 12.3.3. Suppose that $m > 1$ to begin with. Since $H_n(D^m) = 0$ for $n \neq 0$, the map

$$\tilde{H}_m(D^m/S^{m-1}) \longrightarrow H_{m-1}(S^{m-1})$$

from the long exact sequence is an isomorphism. Both of these groups are identified with $\mathbb{Z}$ in a preferred way.

- For $H_{m-1}(S^{m-1})$ this was explained in remark 7.2.7.
- For D^m/S^{m-1} we have the preferred homeomorphism from S^m to $\mathbb{R}^m \cup \{\infty\}$ of remark 5.3.4 and a map from $\mathbb{R}^m \cup \{\infty\}$ to D^m/S^{m-1} given by $z \mapsto z$ for $\|z\| \leq 1$ and $z \mapsto \star$ for $\|z\| \geq 1$. The composite map $u\colon S^m \to D^m/S^{m-1}$ is a homotopy equivalence (easy). We specify an isomorphism

  $$\tilde{H}_m(D^m/S^{m-1}) \longrightarrow \mathbb{Z}$$

  by saying that the class of $[[u]]$ must go to $1 \in \mathbb{Z}$.

Therefore the above-mentioned isomorphism $\tilde{H}_m(D^m/S^{m-1}) \longrightarrow H_{m-1}(S^{m-1})$ becomes an isomorphism $\mathbb{Z} \to \mathbb{Z}$. I believe that it is the identity; I have made a special effort to ensure that it is the identity. (For example in the construction of the long exact sequence of proposition 12.3.2 there was a choice: which of the two open sets $\mathrm{cone}(f) \smallsetminus \{\star\}$ and $\mathrm{cone}(f) \smallsetminus Y$ is going to take the role of V and which the role of W? If roles had been assigned differently, that would have caused some unhelpful sign changes.)
In the case $m = 1$, the long exact sequence reduces to a short exact sequence

$$0 \to \tilde{H}_1(D^1/S^0) \to H_0(S^0) \to H_0(D^0) \longrightarrow 0\,.$$

There are preferred isomorphisms $H_0(S^0) \cong \mathrm{map}(S^0, \mathbb{Z})$ and $H_0(D^1) \cong \mathbb{Z}$ from example 5.3.2, and also $\tilde{H}_1(D^1/S^0) \cong \mathbb{Z}$ as above for $\tilde{H}_m(D^m/S^{m-1})$. Therefore that short exact sequence simplifies to

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{map}(S^0, \mathbb{Z}) \xrightarrow{f \mapsto \sum f(x)} \mathbb{Z} \longrightarrow 0\,.$$

I believe that $1 \in \mathbb{Z}$ on the left is taken to the element $e \in \mathrm{map}(S^0, \mathbb{Z})$ which has $e(1) = 1 \in \mathbb{Z}$ and $e(-1) = -1 \in \mathbb{Z}$.

12.4. The cellular chain complex of a CW-space

COROLLARY 12.4.1. *Let Y be a CW-space and let $X \subset Y$ be a CW-subspace of Y. Then there is a long exact sequence*

$$\cdots \longrightarrow H_n(X) \xrightarrow{f_*} H_n(Y) \xrightarrow{p_*} \tilde{H}_n(Y/X) \longrightarrow H_{n-1}(X) \xrightarrow{f_*} H_{n-1}(Y) \xrightarrow{p_*} \cdots$$

PROOF. The inclusion $X \to Y$ is a cofibration by lemma 11.1.1. □

Let m be a fixed non-negative integer and let Q be a CW-space with a distinguished 0-cell $\star$ (base point). We want to assume that all cells of Q have dimension m, with the possible exception of the distinguished 0-cell. (We allow $m = 0$.)

LEMMA 12.4.2. *Then $\tilde{H}_m(Q)$ is a direct sum of infinite cyclic groups, one summand for each m-cell, excluding the base point cell if $m = 0$. Moreover $\tilde{H}_n(Q) = 0$ for $n \neq m$.*

PROOF. The case $m = 0$ is easy, so we assume $m > 0$. Let Λ be an indexing set for the m-cells of Z. For each m-cell $E_\lambda \subset Q$ let K_λ be the closure of E_λ. By the axioms for a CW-space, $K_\lambda = E_\lambda \cup \star$. Therefore K_λ is homeomorphic to a sphere S^m and has a distinguished base point. (But we did not *choose* a homeomorphism of K_λ with S^m.) Now let $Y = \coprod_{\lambda \in \Lambda} K_\lambda$ and $X = \coprod_{\lambda \in \Lambda} \star$. Then we can identify Q with Y/X. This leads to a long exact sequence in homology

$$\cdots \longrightarrow H_n(X) \longrightarrow H_n(Y) \longrightarrow \tilde{H}_n(Q) \longrightarrow H_{n-1}(X) \longrightarrow H_{n-1}(Y) \longrightarrow \cdots$$

The maps $H_n(X) \to H_n(Y)$ are injective because the inclusion $X \to Y$ admits a left inverse $Y \to X$. Therefore the long exact sequence breaks up into short exact sequences

$$0 \to H_n(X) \to H_n(Y) \to \tilde{H}_n(Q) \to 0.$$

In other words, $H_n(Q)$ is isomorphic to $H_n(Y)$ if $n > 0$, and zero if $n = 0$. Also $H_n(Y) = \bigoplus_{\lambda \in \Lambda} H_n(K_\lambda)$. Because K_λ is homeomorphic to S^m, the group $H_n(K_\lambda)$ is zero if $n > 0$, $n \neq m$ and infinite cyclic if $n = m$. □

Now in order to describe the homology of a CW-space X, we are going to proceed inductively by trying to understand the homology of the skeleton X^n for each n. There is a long exact sequence in homology relating the homology groups of X^{n-1}, X^n and X^n/X^{n-1}. Lemma 12.4.2 tells us what the homology of X^n/X^{n-1} is.

DEFINITION 12.4.3. The *cellular chain complex* $C(X)$ of a CW-space X has $C(X)_m = \tilde{H}_m(X^m/X^{m-1})$ and differential $d\colon C(X)_m \to C(X)_{m-1}$ equal to the composition

$$\tilde{H}_m(X^m/X^{m-1}) \xrightarrow{12.4.1} H_{m-1}(X^{m-1}) \xrightarrow{\text{projection}_*} \tilde{H}_{m-1}(X^{m-1}/X^{m-2}).$$

For $m = 0$, it is often more illuminating to write $C(X)_0 = H_0(X^0)$. This is justified because the composition $H_0(X^0) \to H_0(X^0/X^{-1}) \to \tilde{H}_0(X^0/X^{-1})$ is an isomorphism. From this point of view, $d\colon C(X)_1 \to C(X)_0$ is the homomorphism $\tilde{H}_1(X^1/X^0) \to H_0(X^0)$ of 12.4.1.

Remark: We should verify that $dd = 0$. According to the definition $d\colon C(X)_m \to C(X)_{m-1}$ is a composition of two homomorphisms; let's write it as $p_{m-1}\delta_m$. Therefore $dd = p_{m-2}\delta_{m-1}p_{m-1}\delta_m$. This is zero because $\delta_{m-1}p_{m-1}$ is the composition of two consecutive homomorphisms in the long exact sequence of corollary 12.4.1.

By lemma 12.4.2, the abelian group $C(X)_m$ is a direct sum of infinite cyclic groups, one summand for each m-cell. If we choose characteristic maps $\varphi_\lambda\colon D^m \to X$ for the m-cells, then we can identify X^m/X^{m-1} with a wedge $\bigvee_\lambda S^m$ of m-spheres (using a standard homeomorphism from D^m/S^{m-1} to S^m) and so $C(X)_m$ gets identified with $\bigoplus_\lambda \mathbb{Z}$. If we also choose characteristic maps for the $(m-1)$-cells, then the differential $d\colon C(X)_m \to C(X)_{m-1}$ is a homomorphism between two free abelian groups with preferred bases, so d has to be expressible as a matrix $(a_{\sigma,\tau})$ with entries in $\mathbb{Z}$, indexed by pairs (σ,τ) where σ is a label for an $(m-1)$-cell and τ is a label for an m-cell. The integer $a_{\sigma,\tau}$ is sometimes called an *incidence number*. We will return to it in proposition 12.4.9 below. (A preview was given in example 3.2.)

THEOREM 12.4.4. *For a CW-space X and integer $m \geq 0$ there is a natural isomorphism*

$$H_m(X) \to H_m(C(X)).$$

Here $H_m(X)$ is the m-th homology group of the space X (which was difficult to define) and $H_m(C(X))$ is the m-th homology group of the chain complex $C(X)$ (which was very easy to define). Therefore, in some sense, the theorem gives a rather good way to calculate the homology of X. Determining the chain groups $C(X)_m$ is typically not hard (you need to know how many m-cells X has), but determining $d\colon C(X)_m \to C(X)_{m-1}$ can be a little harder.

The word *natural* in theorem 12.4.4 obviously has to be there, but what does it mean? It has meaning only for *cellular* maps $f\colon X \to Y$ between CW-spaces. Such a cellular map induces base-point preserving maps $X^m/X^{m-1} \to Y^m/Y^{m-1}$ for every $m \geq 0$, therefore homomorphisms $f_*\colon C(X)_m \to C(Y)_m$ for every $m \geq 0$. These homomorphisms constitute a chain map, i.e., the diagrams

$$\begin{tikzcd} C(X)_m \arrow[r, "d"] \arrow[d, "f_*"] & C(X)_{m-1} \arrow[d, "f_*"] \\ Y(X)_m \arrow[r, "d"] & Y(X)_{m-1} \end{tikzcd}$$

commute. (The reason for that can be traced all the way back to naturality in proposition 12.3.2.)

The proof of theorem 12.4.4 is a combination of several lemmas. The first of these is basic, not specific to CW-spaces.

LEMMA 12.4.5. *Let K and X be spaces, K compact Hausdorff. For any mapping cycle α from K to X, there exists a compact subspace $X' \subset X$ such that α factors through X'.*

PROOF. Choose a finite open cover $(U_i)_{i=1,2,\ldots,k}$ of K such that α restricted to any U_i can be written as a formal linear combination, with integer coefficients, of (finitely many) continuous maps: $\sum_j a_{ij} f_{ij}$ where $a_{ij} \in \mathbb{Z}$ and the $f_{ij}\colon U_i \to X$ are continuous maps. Choose another finite open cover $(V_i)_{i=1,2,\ldots,k}$ of K such that the closure $\bar{V}_i$ of V_i in K is contained in U_i. (This is possible because K is compact Hausdorff.) Let $X' \subset X$ be the union of the finitely many compact sets $f_{ij}(\bar{V}_i)$. □

We now work with a fixed CW-space X as in theorem 12.4.4.

LEMMA 12.4.6. *For every $z \in H_k(X)$ there exists $m \geq 0$ such that z is in the image of the homomorphism $H_k(X^m) \to H_k(X)$ induced by the inclusion $X^m \to X$. If two elements*

of $H_k(X^m)$ *have the same image in* $H_k(X)$*, then there is* $n \geq m$ *such that they already have the same image in* $H_k(X^n)$*.*

Proof. Apply lemma 12.4.5 with $K = S^k$ to obtain the first statement, and with $K = S^k \times [0,1]$ for the second statement. Also, keep in mind that any compact subset X' of X must be contained in some skeleton X^m. □

Lemma 12.4.7. $H_n(X^m) = 0$ *for* $n > m$*.*

Proof. By induction on m. The cases $m = -1$ and/or $m = 0$ are obvious. For the induction step we have the long exact sequence

$$\cdots \longrightarrow H_n(X^{m-1}) \longrightarrow H_n(X^m) \longrightarrow \tilde{H}_n(X^m/X^{m-1}) \longrightarrow H_{n-1}(X^{m-1}) \longrightarrow \cdots$$

which is a special case of corollary 12.4.1. In addition we have the computation of lemma 12.4.2. □

Lemma 12.4.8. *The inclusion* $X^{m-1} \to X^m$ *induces a homomorphism from* $H_k(X^{m-1})$ *to* $H_k(X^m)$ *which is an isomorphism if* $k < m-1$*. There is an exact sequence*

$$0 \longrightarrow H_m(X^m) \xrightarrow{p_m} C(X)_m \xrightarrow{\delta_m} H_{m-1}(X^{m-1}) \longrightarrow H_{m-1}(X^m) \longrightarrow 0\ .$$

Proof. Use lemma 12.4.7, and use the same long exact sequence as in the proof of that lemma. □

Proof of theorem 12.4.4. We use the notation of lemma 12.4.8. By lemma 12.4.6 and lemma 12.4.8 we know that the inclusion $X^{m+1} \to X$ induces an isomorphism

$$H_m(X^{m+1}) \cong H_m(X)\ .$$

Then we compute $H_m(X^{m+1})$ using the exact sequence(s) of lemma 12.4.8:

$$H_m(X^{m+1}) \cong \frac{H_m(X^m)}{\mathrm{im}(\delta_{m+1})} \cong \frac{\mathrm{im}(p_m)}{\mathrm{im}(p_m\delta_{m+1})} = \frac{\ker(\delta_m)}{\mathrm{im}(p_m\delta_{m+1})} = \frac{\ker(p_{m-1}\delta_m)}{\mathrm{im}(p_m\delta_{m+1})}\ . \quad \square$$

To conclude, we need to look at the homomorphisms $d\colon C(X)_m \to C(X)_{m-1}$. Choose a characteristic map $\varphi_\tau\colon D^m \to X^m$ for an m-cell $E_\tau \subset X$ and a characteristic map φ_σ for an $(m-1)$-cell $E_\sigma \subset X$. Then we have the following commutative diagram

$$\begin{tikzcd} S^{m-1} \arrow[r] \arrow[d, "\psi_\tau"] & D^m \arrow[d, "\varphi_\tau"] \\ X^{m-1} \arrow[r] & X^m \end{tikzcd}$$

where the horizontal arrows are inclusion maps. (So ψ_τ is obtained from φ_τ by restriction.) Apply corollary 12.3.3 to the rows of this diagram and use naturality to obtain top

and middle row, both exact, of a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_m(S^{m-1}) \arrow[r] \arrow[d] & H_m(D^m) \arrow[r] \arrow[d] & \tilde{H}_m(D^m/S^{m-1}) \arrow[r, "e_m"] \arrow[d, "(\varphi_\tau/\psi_\tau)_*"] & H_{m-1}(S^{m-1}) \arrow[r] \arrow[d, "(\psi_\tau)_*"] & \cdots \\
\cdots \arrow[r] & H_m(X^{m-1}) \arrow[r] & H_m(X^m) \arrow[r] & \tilde{H}_m(X^m/X^{m-1}) \arrow[r, "\delta_m"] \arrow[dr, "d"'] & H_{m-1}(X^{m-1}) \arrow[r] \arrow[d, "p_{m-1}"] & \cdots \\
 & & & & \tilde{H}_{m-1}(X^{m-1}/X^{m-2}) \arrow[d, "(c_\sigma)_*"] & \\
 & & & & \tilde{H}_{m-1}(D^{m-1}/S^{m-2}) &
\end{tikzcd}$$

Here c_σ is the collapse map $X^{m-1}/X^{m-2} \to X^{m-1}/X^{m-1}_{\neg\sigma}$ followed by the identification of $X^{m-1}/X^{m-1}_{\neg\sigma}$ with D^{m-1}/S^{m-2} (which uses φ_σ). The entry $a_{\sigma,\tau} \in \mathbb{Z}$ of the "matrix" $d\colon C(X)_m \to C(X)_{m-1}$ is the homomorphism $(c_\sigma)_* \circ d \circ (\varphi_\tau/\psi_\tau)_*$, which we can view as a homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$ using the preferred isomorphisms of example 12.3.4. By the commutativity of the diagram, it is also $(c_\sigma)_* \circ p_{m-1} \circ (\psi_\tau)_* \circ e_m$. Since we have decided in example 12.3.4 that e_m is the identity map $\mathbb{Z} \to \mathbb{Z}$ when $m > 1$, we see that $(c_\sigma)_* \circ p_{m-1} \circ (\psi_\tau)_* \circ e_m$ as a map from $\mathbb{Z}$ to $\mathbb{Z}$ is multiplication with the degree of

$$S^{m-1} \xrightarrow{\psi_\tau} X^{m-1} \longrightarrow X^{m-1}/X^{m-1}_{\neg\sigma} \xrightarrow[\cong]{\text{inv. of quot. of } \varphi_\sigma} D^{m-1}/S^{m-2} \cong S^{m-1}$$

when $m > 1$. For $m = 1$ we get the same result using example 12.3.4, on the understanding that the degree of a map $S^0 \to S^0$ is 1 if it is the identity map, -1 if it is bijective but not the identity map, and 0 in all other cases. We formulate this in a proposition.

Proposition 12.4.9. *A choice of characteristic maps φ_λ for all cells E_λ of X determines isomorphisms*

$$C(X)_m \cong \bigoplus_{m\text{-cells } E_\lambda} \mathbb{Z}$$

so that $d\colon C(X)_m \to C(X)_{m-1}$ becomes a matrix with integer entries $a_{\sigma,\tau}$, one entry for each $(m-1)$-cell σ and m-cell τ. The number $a_{\sigma,\tau}$ is the degree of the map

$$S^{m-1} \xrightarrow{\text{res. of } \varphi_\lambda} X^{m-1} \longrightarrow X^{m-1}/X^{m-1}_{\neg\sigma} \xrightarrow[\cong]{\text{inv. of quot. of } \varphi_\sigma} D^{m-1}/S^{m-2} \cong S^{m-1}$$

where $X^{m-1}_{\neg\sigma}$ is $X^{m-1} \smallsetminus E_\sigma$. In the case $m = 1$, the degree of a map $g\colon S^0 \to S^0$ is defined to be 1 if g is the identity, -1 if g is bijective but $g \neq \mathrm{id}$, and 0 in all other cases.

Remark 12.4.10. On the ONF (outward normal first) convention for orienting the boundary of a smooth oriented manifold with boundary ... *under construction.*

Example 12.4.11. About the cellular chain complex of $|Y|$, where Y is a semi-simplicial set ... *under construction.*

CHAPTER 13

Suspension and the Mayer-Vietoris sequence in cohomology

13.1. Suspension

Definition 13.1.1. The *suspension* ΣY of a space Y is the pushout of

$$[0,1] \times Y \xleftarrow{\supset} \{0,1\} \times Y \xrightarrow{\text{proj.}} \{0,1\}.$$

Equivalently, ΣY is the mapping cone of the unique map $Y \to \{0\}$. Explicit description: Take the disjoint union of $[0,1] \times Y$ and $\{0,1\}$ and make identifications $(0,y) \sim 0$ as well as $(1,y) \sim 1$ for all $y \in Y$. (When Y is nonempty, ΣY is a quotient space of $[0,1] \times Y$ in an obvious way.)
Suspension is a functor: a map $f\colon X \to Y$ determines a map $\Sigma f\colon \Sigma X \to \Sigma Y$ given (mostly) by $(t,x) \mapsto (t, f(x))$ for $x \in X$ and $t \in [0,1]$.

Lemma 13.1.2. *Let X be a paracompact space, A a closed subspace. Then X/A is also paracompact.*

Proof. We can assume that X is nonempty; then there is the standard quotient map $q\colon X \to X/A$. Let $(U_\lambda)_{\lambda \in \Lambda}$ be an open covering of X/A. We need to construct a locally finite refinement of $(U_\lambda)_\lambda$. Choose λ_0 in Λ such that U_{λ_0} contains the base point of X/A, which is the class of all elements in A. Since X is normal, there exists an open neighborhood W of A in X such that $\bar{W} \subset q^{-1}(U_{\lambda_0})$, where $\bar{W}$ denotes the closure of W in X. Choose a locally finite open covering $(V_\kappa)_\kappa$ of X which refines the open covering $(q^{-1}(U_\lambda))_\lambda$ of X. Now the open sets $V_\kappa \smallsetminus \bar{W}$ together with U_{λ_0} form a locally finite open covering of X/A. □

Corollary 13.1.3. *If Y is paracompact, then ΣY is paracompact.*

Proof. ΣY can be obtained from $[0,1] \times Y$, which is paracompact, by dividing out first $\{0\} \times Y$ and then $\{1\} \times Y$. □

As we have seen, a map $f\colon X \to Y$ determines a map $\Sigma f\colon \Sigma X \to \Sigma Y$ by $\Sigma f(t,x) = (t, f(x))$. This procedure also respects homotopies. Therefore suspension of maps determines a map

$$[X,Y] \longrightarrow [\Sigma X, \Sigma Y]$$

where the square brackets indicate sets of homotopy classes. One might think that a map from $[[X,Y]]$ to $[[\Sigma X, \Sigma Y]]$ can be constructed in exactly the same way. But there are a few problems with that due to the fact that mapping cycles must be described germ-wise rather than pointwise. (It is not clear what the germ of Σf at $0 \in \Sigma X$ should look like when f is a mapping cycle from X to Y, for example.) Therefore we take some precautions. Firstly, we choose a continuous map $\psi\colon [0,1] \to [0,1]$ such that $\psi(t) = 0$ for all t in a neighborhood of 0 and $\psi(t) = 1$ for all t in a neighborhod of 1. A map $f\colon X \to Y$

determines a map $\Sigma_\psi f\colon \Sigma X \to \Sigma Y$ by $(x,t) \mapsto (f(x), \psi(t))$. Note that $\Sigma_\psi f$ is constant in a neighborhood of $0 \in \Sigma X$, and constant in a neighborhood of $1 \in \Sigma X$. Also, rather obviously, $\Sigma_\psi f$ is homotopic to Σf.
Secondly, before applying Σ_ψ to a *mapping cycle* $f\colon X \to Y$, let us demand that the composition of f with the unique continuous map $Y \to \star$ be the zero mapping cycle $X \to \star$. A mapping cycle with this property will be called *traceless*. In such a case $\Sigma_\psi f$ has meaning as a mapping cycle from ΣX to ΣY. It agrees with the zero mapping cycle[1] on a neighborhood of $\{0,1\} \subset \Sigma X$. Moreover $\Sigma_\psi f$ is a again traceless.

Proposition 13.1.4. *For spaces X and Y, where Y comes with a base point y_0, suspension of traceless mapping cycles defines a homomorphism*

$$\frac{[[X,Y]]}{[[X,\star]]} \longrightarrow \frac{[[\Sigma X, \Sigma Y]]}{[[\Sigma X, \star]]}.$$

Here ΣY has base point $(1, y_0)$ and the (injective) homomorphism $[[X,\star]] \to [[X,Y]]$ is defined by composing mapping cycles $X \to \star$ with the map $\star \to Y$ that has image $\{y_0\}$.

Proof. We almost proved it before stating the proposition. But for clarification let's recall that a mapping cycle from X to $\star$ is the same as a continuous map from X to $\mathbb{Z}$ and that two mapping cycles from X to $\star$ which are homotopic are necessarily equal. (See proposition 5.2.1.) If $f\colon X \to Y$ is any mapping cycle, we can make it traceless by subtracting qf, where $q\colon Y \to Y$ is given by $y \mapsto y_0$. In this way

$$\frac{[[X,Y]]}{[[X,\star]]}$$

can be understood as the abelian group of homotopy classes of traceless mapping cycles $f\colon X \to Y$. Then $[[f]] \mapsto [[\Sigma_\psi f]]$ is defined (as explained above), and it is well defined, and $\Sigma_\psi f$ is again traceless. □

Theorem 13.1.5. *Let X and Y be spaces, both nonempty, X paracompact, Y equipped with a base point. The homomorphism*

$$\frac{[[X,Y]]}{[[X,\star]] + [[\star,Y]]} \longrightarrow \frac{[[\Sigma X, \Sigma Y]]}{[[\Sigma X,\star]] + [[\star,\Sigma Y]]}$$

determined by suspension of traceless mapping cycles, as in proposition 13.1.4, is an isomorphism.

Comment. The notation suggests that $[[\star,Y]]$ is a subgroup of the abelian group $[[X,Y]]$, for example. More precisely there is a homomorphism from $[[\star,Y]]$ to $[[X,Y]]$ given by composing mapping cycles $\star \to Y$ with the unique continuous map from X to $\star$. It is injective because we can choose a continuous map $e\colon \star \to X$ to construct a homomorphism $[[X,Y]] \to [[\star,Y]]$ in a similar manner, by composition with e. That homomorphism is a left inverse for the other one.

Remark 13.1.6. Suppose that $X = S^n$. Then the theorem specializes to the statement

$$\tilde{H}_n(Y) \cong \tilde{H}_{n+1}(\Sigma Y)$$

[1] Zero mapping cycle means: zero element of the abelian group of mapping cycles ... no close relationship with $0 \in \Sigma Y$.

for nonempty Y. Here we define $\tilde{H}_n(Y)$ as the cokernel of the homomorphism from $H_n(\star)$ to $H_n(Y)$ induced by the map $\star \mapsto y_0$. Similarly, in the case $Y = S^n$ the theorem states that

$$\tilde{H}^n(X) \cong \tilde{H}^{n+1}(\Sigma X)$$

for nonempty X. In this case we have to use a definition of $H^n(X)$ as the cokernel of the homomorphism $H^n(\star) \to H^n(X)$ determined by the map $X \to \star$.

Proof of theorem 13.1.5. We use the homotopy decomposition theorem 6.1.1 to construct a homomorphism in the other direction. It is also convenient to make a choice of $x_0 \in X$. The abelian group

$$\frac{[[\Sigma X, \Sigma Y]]}{[[\Sigma X, \star]] + [[\star, \Sigma Y]]}$$

can be thought of in the following way. It is the group of homotopy classes of traceless mapping cycles $g\colon \Sigma X \to \Sigma Y$ such that ge is zero[2], where e is the injective map $[0,1] \to \Sigma X$ defined by $t \mapsto (t, x_0)$. (If $g\colon \Sigma X \to \Sigma Y$ is a traceless mapping that does not satisfy $ge = 0$, then replace g by $g - gu$ where $u\colon \Sigma X \to \Sigma X$ is defined by $u(t,x) = (t, x_0)$. It is easy to see that $[[gu]]$ is in the subgroup $[[\star, \Sigma Y]]$ of $[[\Sigma X, \Sigma Y]]$.)
We may also assume without loss of generality that g restricted to an open neighborhood of $0 \in \Sigma X$ is the zero mapping cycle. (If ge is zero, but g is not zero on any neighborhood of $0 \in \Sigma X$, then replace g by its composition with a map $\Sigma X \to \Sigma X$ of the form $(t,x) \mapsto (\psi(t), x)$, in the notation of the preliminaries to proposition 13.1.4.)
Once we have a mapping cycle $g\colon \Sigma X \to \Sigma Y$ satisfying all these good conditions, we obtain another mapping cycle

$$\gamma\colon [0,1] \times X \longrightarrow \Sigma Y$$

by composing with the quotient map $[0,1] \times X \to \Sigma X$. Then γ is zero (zero element in an abelian group of mapping cycles) on an open neighborhood of $\{0\} \times X$ and on $\{1\} \times X$. Now apply the homotopy decomposition theorem with $V = \Sigma Y \smallsetminus \{1\}$ and $W = \Sigma Y \smallsetminus \{0\}$, two open subsets of ΣY whose union is ΣY. What we get is

$$\gamma = \gamma^V + \gamma^W$$

where $\gamma^V\colon [0,1] \times X \to V$ and $\gamma^W\colon [0,1] \times X \to W$ are mapping cycles, both zero on an open neighborhood of $\{0\} \times X$. Restricting to $X \times \{1\} \subset X \times [0,1]$ we have

$$\gamma_1 = \gamma_1^V + \gamma_1^W$$

which we view as an equation relating mapping cycles from $X \cong \{1\} \times X$ to ΣY, V and W. But $\gamma_1 = 0$ by construction. It follows that γ_1^V is a mapping cycle from X to $V \cap W$, being equal to $-\gamma_1^W$. Also $V \cap W$ is homotopy equivalent to Y, by means of the projection $V \cap W =]0,1[\times Y \longrightarrow Y$. Therefore $[[\gamma_1^V]]$ can be regarded as an element of $[[X, Y]]$. Two things remain to be verified.

(1) The element $[[\gamma_1^V]] \in [[X, Y]]$ depends only on $[[g]]$ and $x_0 \in X$, on the understanding that γ_1^V is constructed from a representative g in the manner described above. Furthermore, replacing the choice $x_0 \in X$ by another element of X has no effect if we calculate modulo the subgroup $[[\star, Y]]$ of $[[X, Y]]$.

[2]Again this "zero" is the zero element of an abelian group (of mapping cycles), not to be confused with a certain element of ΣY.

(2) The formula $[[g]] \mapsto [[\gamma_1^V]]$ gives a homomorphism which is inverse to the homomorphism

$$\frac{[[X,Y]]}{[[X,\star]]+[[\star,Y]]} \longrightarrow \frac{[[\Sigma X,\Sigma Y]]}{[[\Sigma X,\star]]+[[\star,\Sigma Y]]}$$

given by $[[f]] \mapsto [[\Sigma_\psi f]]$.

Proof of (1). By linearity properties of the construction, it is enough to show that $[[\gamma_1^V]]$ is zero if $[[g]] = 0$. Let us first assume that the mapping cycle g itself is strictly zero. Keep x_0 fixed. Then γ^V is a mapping cycle from $[0,1] \times X$ to $V \cap W$ and as such it is a homotopy from zero to γ_1^V. Next, suppose that g is merely nullhomotopic. Choose a nullhomotopy

$$\bar{g}\colon \Sigma X \times [0,1] \to \Sigma Y\ .$$

Now we do to $\bar{g}$ what we did previously to g. Beware though: there is a small difference between $\Sigma X \times [0,1]$ and $\Sigma(X \times [0,1])$. Keep x_0 fixed. The mapping cycle $\bar{g}$ is automatically traceless. Without loss of generality, $\bar{g}$ is zero on $\Sigma\{x_0\} \times [0,1]$ and on an open neighborhood of $\{0\} \times [0,1]$ in $\Sigma X \times [0,1]$. From $\bar{g}$ we get a mapping cycle

$$\bar{\gamma}\colon [0,1] \times (X \times [0,1]) \to \Sigma Y$$

as before. The homotopy decomposition theorem can be applied and then $\bar{\gamma}_1^V$ from $X \times [0,1]$ to $V \cap W$ is a homotopy relating γ_1^V to another mapping cycle which we already know represents zero in $[[X,Y]]$, by part (1). Finally, replacing x_0 by another element of X has the effect of replacing γ by $\gamma - \alpha q$ where α is a mapping cycle from $[0,1]$ to ΣY and $q\colon \Sigma X \to [0,1]$ is the projection. Then γ_1^V gets replaced by γ_1^V minus a constant mapping cycle from X to $V \cap W$. (Here *constant* means that it is obtained by composing the map $X \to \star$ with a mapping cycle from $\star$ to $V \cap W$.)

Proof of (2). First let us show that if γ_1^V has been constructed from g as above and $g = \Sigma_\psi f$, where $f\colon X \to Y$ is a traceless mapping cycle, then $[[\gamma_1^V]] = [[f]]$. We also assume that f restricted to $\{x_0\}$ is zero. The mapping cycle

$$\gamma\colon X \times [0,1] \longrightarrow \Sigma Y$$

is the composition of $g = \Sigma_\psi f$ with the quotient map $[0,1] \times X \to \Sigma X$. Now we can make our own choice of γ^V and γ^W such that $\gamma = \gamma^V + \gamma^W$. Let $\theta(t) = \min(\psi(t), 1/2)$ for $t \in [0,1]$. Let γ^V be the composition of γ with the map $(t,x) \mapsto (\theta(t),x)$ from $[0,1] \times X$ to $[0,1] \times X$. Put $\gamma^W = \gamma - \gamma^V$. The conditions are satisfied and clearly γ_1^V as a mapping cycle from X to $V \cap W =]0,1[\times Y$ is f followed by the map $y \mapsto (y, 1/2)$. Therefore $[[\gamma_1^V]] = [[f]]$. — It remains to show that our formula $[[g]] \mapsto [[\gamma_1^V]]$ defines an *injective* homomorphism. So suppose that $[[\gamma_1^V]]$ is the zero element of

$$[[X,Y]]/([[\star,Y]] + [[X,\star]]).$$

Then it is already zero in $[[X,Y]]/[[\star,Y]]$ because it is traceless. This means that γ_1^V is homotopic to a constant mapping cycle from X to $V \cap W \simeq Y$, meaning one that is obtained by composing a mapping cycle $\star \to V \cap W$ with the unique map from X to $\star$. In this situation it is easy to modify γ^V in such a way that γ^V is actually constant on an open neighborhood of $X \times \{1\}$ in $X \times [0,1]$. Then it follows that γ^V and γ^W are mapping cycles from $X \times [0,1]$ to V and W respectively which can be written as compositions of the quotient map

$$[0,1] \times X \to \Sigma X$$

with mapping cycles g^V and g^W from ΣX to V and W, respectively. In other words we get $g = g^V + g^W$. The mapping cycles g^V and g^W are still traceless. Now it is enough to show that

$$[[g^V]] = 0 \in [[X, V]]/[[X, \star]]$$
$$[[g^W]] = 0 \in [[X, W]]/[[X, \star]].$$

But that is obvious. Indeed we have

$$[[X, V]]/[[X, \star]] = 0 = [[X, W]]/[[X, \star]]$$

because V and W are contractible. Therefore $[[g]] = 0$ in $[[\Sigma X, \Sigma Y]]/[[\star, \Sigma Y]]$, as was to be shown. □

13.2. Mayer-Vietoris sequence in cohomology

THEOREM 13.2.1. *Let X be a space, V and W open subsets of X such that $V \cup W = X$, and suppose that X, V, W are paracompact. Then there is a natural long exact sequence*

$$\begin{tikzcd}
\cdots & H^{n+1}(X) & & & \\
& H^n(V \cap W) \arrow[u, "\delta"] & H^n(V) \oplus H^n(W) \arrow[l, "e_V^* \oplus -e_W^*"'] & H^n(X) \arrow[l, "(j_V^*, j_W^*)"'] & \\
& & & H^{n-1}(V \cap W) \arrow[u, "\delta"] & \cdots \arrow[l]
\end{tikzcd}$$

where $e_V \colon V \cap W \to V$, $e_W \colon V \cap W \to W$, $j_V \colon V \to X$ and $j_W \colon W \to X$ are the inclusions.

We start by defining the as-yet-undefined homomorphism δ. Let X^e be the following substitute for X. As a set,

$$\begin{aligned} X^e &= \big\{(t, x) \in [0, 1] \times X \;\big|\; t = 0 \text{ if } x \notin W, \quad t = 1 \text{ if } x \notin V\big\} \\ &= \big(\{0\} \times V\big) \cup \big([0, 1] \times (V \cap W)\big) \cup \big(\{1\} \times W\big). \end{aligned}$$

But the topology is defined in such a way that the (obvious) surjection from the topological disjoint union $V \sqcup \big([0, 1] \times (V \cap W)\big) \sqcup W$ to X^e is an identification map; i.e., a subset of X^e is open if and only if its intersection with $[0, 1] \times (V \cap W)$, with $\{0\} \times V$ and with $\{1\} \times W$ is open. The projection map $q \colon X^e \to X$ given by $q(t, x) = x$ is a homotopy equivalence. To see this, choose a partition of unity (ψ_V, ψ_W) subordinate to the covering of X by V and W; so $\psi_V \colon X \to [0, 1]$ has support in V and $\psi_W \colon X \to [0, 1]$ has support in W and $\psi_V + \psi_W \equiv 1$. Define $s \colon X \to X^e$ by $s(z) = (\psi_W(z), z)$. Clearly $qs = \mathrm{id}_X$ and sq is homotopic to the identity on X^e.
There is a continuous map $p \colon X^e \to \Sigma(V \cap W)$ given by $(t, x) \mapsto (t, x)$ if $t \in [0, 1]$ and $x \in V \cap W$, and $(0, x) \mapsto 0$ for $x \in V$, $(1, x) \mapsto 1$ for $x \in W$. (It is continuous because we defined the topology on X^e as we did.) We define

$$\delta \colon H^n(V \cap W) \longrightarrow H^{n+1}(X)$$

as the composition

$$H^n(V \cap W) \xrightarrow{\Sigma} H^{n+1}(\Sigma(V \cap W)) \xrightarrow{p^*} H^{n+1}(X^e) \overset{q^*}{\cong} H^{n+1}(X).$$

Proof of theorem 13.2.1. Recall that a mapping cycle $f\colon A \to B$ *traceless* if the composition of f with the constant map $B \to \star$ is zero. Example: elements of $H^n(A)$ can be represented by traceless mapping cycles from A to $B = S^n$. Another example: we have seen that a traceless mapping cycle f from A to B can be suspended without great difficulty to give a traceless mapping cycle $\Sigma A \to \Sigma B$.

Here is an important principle which we shall use several times in the proof. Let A, B, C be spaces, let $f\colon A \to B$ be a map and let $g\colon B \to C$ be a map. If gf is homotopic to a constant map, then f can be extended to a map from $\mathrm{cone}(f)$ to C. Variant: let $f\colon A \to B$ be a map and let $g\colon B \to C$ be a traceless mapping cycle. If gf is homotopic to the zero mapping cycle, then gf can be extended to a traceless mapping cycle from $\mathrm{cone}(f)$ to C.

Showing $\ker \supset \mathrm{im}$ *at* $H^n(V) \oplus H^n(W)$. This is clear.

Showing $\ker \subset \mathrm{im}$ *at* $H^n(V) \oplus H^n(W)$. Suppose given classes in $H^n(V)$ and $H^n(W)$ represented by traceless mapping cycles $f\colon V \to S^n$ and $g\colon W \to S^n$. If $[[f]] \oplus [[g]]$ maps to zero under $e_V^* \oplus -e_W^*$, then there exists a mapping cycle

$$h\colon (V \cap W) \times [0,1] \to S^n$$

(a homotopy) such that $h_0 = f$ and $h_1 = g$. Without loss of generality the homotopy is stationary near $t = 0$ and $t = 1$. Then the union of f, g and h defines a traceless mapping cycle from X^e to S^n. The class of that in $H^n(X^e) \cong H^n(X)$ is the answer to our prayers.

Showing $\ker \supset \mathrm{im}$ *at* $H^n(X)$. We think of $H^n(X)$ as $H^n(X^e)$. For a class $[[f]]$ in $H^{n-1}(V \cap W)$, where $f\colon V \cap W \to S^{n-1}$ is traceless, the image of that class under $j_V^*\delta$ is $[[\Sigma_\psi f \circ p_{|V}]]$, where $\Sigma_\psi f \circ p_{|V}$ is a constant mapping cycle since p is constant on V. Since we can assume $n > 0$, it follows that $j_V^*\delta([[f]]) = 0$.

Showing $\ker \subset \mathrm{im}$ *at* $H^n(X)$. The case $n = 0$ is interesting but we leave it as an exercise. (Remember that $H^0(X)$ has been identified with the set of continuous maps from X to $\mathbb{Z}$.) Now we assume $n > 0$. Let $g\colon X^e \to S^n$ be a traceless mapping cycle such that $[[g]] \in H^n(X^e) \cong H^n(X)$ is taken to zero by j_V^* and j_W^*. Then g extends to a traceless mapping cycle G from

$$\mathrm{cone}(V) \cup X^e \cup \mathrm{cone}(W)$$

to S^n. Here $\mathrm{cone}(V) := \mathrm{cone}(\mathrm{id}_V)$ and $\mathrm{cone}(W) := \mathrm{cone}(\mathrm{id}_W)$. (*There should be a picture here ... under construction.*) Since the inclusion $V \sqcup W \to X^e$ is a cofibration, the projection

$$\mathrm{cone}(V) \cup X^e \cup \mathrm{cone}(W) \quad \longrightarrow \quad (X^e/V)/W = (X^e/W)/V = \Sigma(V \cap W)$$

is a homotopy equivalence. Therefore we can write

$$[[G]] \in H^n(\mathrm{cone}(V) \cup X^e \cup \mathrm{cone}(W)) \cong H^n(\Sigma(V \cap W)).$$

Since $n > 0$ we have $H^n(\Sigma(V \cap W)) = \tilde{H}^n(\Sigma(V \cap W))$ and by the suspension theorem, lecture notes week 5, that is isomorphic to $\tilde{H}^{n-1}(V \cap W)$, which we can interpret as a quotient of $H^{n-1}(V \cap W)$. So $[[G]]$ determines a class in $H^{n-1}(V \cap W)$ up to some ambiguity (if $n - 1 = 0$), and that class is taken to $[[g]]$ by the homomorphism δ.

Showing $\ker \supset \mathrm{im}$ *at* $H^n(V \cap W)$. It suffices to show that the composition δj_V^* is zero. By naturality, we can assume that $W = X$. Then $V \cap W$ is V. It is easy to show that $p\colon X^e \to \Sigma(V \cap W) = \Sigma V$ is homotopic to a constant map. Therefore δ is the zero homomorphism in this very special Mayer-Vietoris sequence.

Showing $\ker \subset \mathrm{im}$ *at* $H^n(V \cap W)$. We are no longer assuming $W = X$. It will be necessary to understand the mapping cone of $p\colon X^e \to \Sigma(V \cap W)$. That mapping cone contains the

mapping cone of the map $p^\sharp\colon V \sqcup W \to \{0,1\}$ which takes all of V to 0 and all of W to 1. (Remember that $\{0,1\} \subset \Sigma(V \cap W)$.) It is an exercise to show that the inclusion

$$\mathrm{cone}(p^\sharp) \longrightarrow \mathrm{cone}(p)$$

is a homotopy equivalence. Moreover $\mathrm{cone}(p^\sharp)$ is $\Sigma V \vee \Sigma W$, the quotient space of the topological disjoint union $\Sigma V \sqcup \Sigma W$ obtained by identifying $1 \in \Sigma V$ with $1 \in \Sigma W$. The composition

$$\Sigma(V \cap W) \xrightarrow{\subset} \mathrm{cone}(p) \simeq \mathrm{cone}(p^\sharp) = \Sigma V \vee \Sigma W \xrightarrow{\text{collapse } \Sigma W} \Sigma V$$

is homotopic to the inclusion $\Sigma(V \cap W) \to \Sigma V$ followed by the map $(t,x) \mapsto (1-t,x)$ from ΣV to itself and the composition

$$\Sigma(V \cap W) \xrightarrow{\subset} \mathrm{cone}(p) \simeq \mathrm{cone}(p^\sharp) = \Sigma V \vee \Sigma W \xrightarrow{\text{collapse } \Sigma V} \Sigma W$$

is homotopic to the inclusion $\Sigma(V \cap W) \to \Sigma V$. Now suppose that a class in $H^n(V \cap W)$ is represented by a traceless mapping cycle g from $V \cap W$ to S^n, and $\delta([[g]]) = 0 \in H^{n+1}(X)$. Then $p^*[[\Sigma g]]$ is zero in $H^{n+1}(X^e)$, where $p\colon X^e \to \Sigma(V \cap W)$ is the usual map and Σg, or more precisely $\Sigma_\psi g\colon \Sigma(V \cap W) \to \Sigma(S^n) = S^{n+1}$, is the suspension of g. This means that $\Sigma_\psi g \circ p$ is nullhomotopic, and so g can be extended to a traceless mapping cycle $G\colon \mathrm{cone}(p) \to S^{n+1}$. Then

$$\begin{aligned} [[G]] \in H^{n+1}(\mathrm{cone}(p)) &\xrightarrow{\cong} H^{n+1}(\mathrm{cone}(p^\sharp)) \\ &\xrightarrow{\cong} H^{n+1}(\Sigma V) \oplus H^{n+1}(\Sigma W) \\ &\xleftarrow{\cong} \tilde{H}^n(V) \oplus \tilde{H}^n(W) \end{aligned}$$

where we assume $n+1 > 0$. So $[[G]]$ determines a class in $H^n(V) \oplus H^n(W)$ up to some small ambiguity (when $n = 0$), and that class is taken to $-[[g]]$ by $e_V^* \oplus -e_W^*$. □

13.3. Cohomology of mapping cones and quotients

Proposition 13.3.1. *For a map $f\colon X \to Y$, there is a natural long exact sequence of cohomology groups*

$$\cdots \xrightarrow{j^*} H^{n-1}(Y) \xrightarrow{f^*} H^{n-1}(X) \longrightarrow \tilde{H}^n(\mathrm{cone}(f)) \xrightarrow{j^*} H^n(Y) \xrightarrow{f^*} H^n(X) \longrightarrow \cdots$$

If $f\colon X \to Y$ is the inclusion of a closed subset and a cofibration, then the projection $\mathrm{cone}(f) \to Y/X$ *is a homotopy equivalence and consequently there is another long exact sequence*

$$\cdots \longrightarrow H^{n-1}(Y) \xrightarrow{f^*} H^{n-1}(X) \longrightarrow \tilde{H}^n(Y/X) \longrightarrow H^n(Y) \xrightarrow{f^*} H^n(X) \longrightarrow \cdots$$

Proof. This can be proved like proposition 12.3.2, using the Mayer-Vietoris sequence in cohomology instead of the Mayer-Vietoris sequence in homology. □

13.4. Cohomology of CW-spaces

Definition 13.4.1. Let X be a CW-space. The *cohomological variant* of the cellular chain complex of X is the following chain complex. In degree $-n$ it has the abelian group

$$\tilde{H}^n(X^n/X^{n-1})$$

and the differential $d\colon \tilde{H}^n(X^n/X^{n-1}) \longrightarrow H^{n+1}(X^{n+1}/X^n)$ is the composition of the homomorphism $\tilde{H}^n(X^n/X^{n-1}) \to H^n(X^n)$ determined by the projection[3] and the boundary operator $H^n(X^n) \longrightarrow H^{n+1}(X^{n+1}/X^n)$ from the second long exact sequence in proposition 13.3.1.

For this cohomological variant of the cellular chain complex, we have a theory which is quite analogous to that of the cellular chain complex. Here are the most important facts.

PROPOSITION 13.4.2. *For a CW-space X, the cohomology group $H^n(X)$ is isomorphic to the $(-n)$-th homology group of the cohomological variant of the cellular chain complex of X.*

PROPOSITION 13.4.3. *For a CW-space X, the cohomological variant of the cellular chain complex of X is isomorphic to $\hom(C(X), \mathbb{Z})$, where $C(X)$ is the cellular chain complex of X.*

COROLLARY 13.4.4. *For a CW-space X, the cohomology group $H^n(X)$ is isomorphic to $H_{-n}(\hom(C(X), \mathbb{Z})$.* □

The proof of proposition 13.4.2 is very similar to that of theorem 12.4.4. But there is one little aspect which is different, and that is in the shape of the groups $\tilde{H}^n(X^n/X^{n-1})$. For this reason I think it is worthwhile to formulate the cohomological version of lemma 12.4.2 and a consequence. So let m be a fixed non-negative integer and let Q be a CW-space with a distinguished 0-cell $\star$ (base point). We want to assume that all cells of Q have dimension m, with the possible exception of the distinguished 0 cell.

LEMMA 13.4.5. *Then $\tilde{H}^m(Q)$ is a product of infinite cyclic groups, one summand for each m-cell, excluding the base point cell if $m = 0$. Moreover $\tilde{H}^n(Q) = 0$ for $n \neq m$.*

PROOF. The case $m = 0$ is easy, so we assume $m > 0$. Let Λ be an indexing set for the m-cells of Z. For each m-cell $E_\lambda \subset Q$ let K_λ be the closure of E_λ. By the axioms for a CW-space, $K_\lambda = E_\lambda \cup \star$. Therefore K_λ is homeomorphic to a sphere S^m and has a distinguished base point. (But we did not *choose* a homeomorphism of K_λ with S^m.) Now let $Y = \coprod_{\lambda\in\Lambda} K_\lambda$ and $X = \coprod_{\lambda\in\Lambda} \star$. Then we can identify Q with Y/X. This leads to a long exact sequence in cohomology

$$\cdots \longleftarrow H^n(X) \longleftarrow \qquad H^n(Y) \longleftarrow \tilde{H}^n(Q) \longleftarrow H^{n-1}(X) \longleftarrow H^{n-1}(Y) \longleftarrow \cdots$$

The maps $H^n(Y) \to H_n(X)$ are surjective because the inclusion $X \to Y$ admits a left inverse $Y \to X$. Therefore the long exact sequence breaks up into short exact sequences

$$0 \leftarrow H^n(X) \leftarrow H^n(Y) \leftarrow \tilde{H}^n(Q) \to 0.$$

In other words, $H^n(Q)$ is isomorphic to $H^n(Y)$ if $n > 0$, and zero if $n = 0$. Also $H^n(Y) = \prod_{\lambda\in\Lambda} H^n(K_\lambda)$. Because K_λ is homeomorphic to S^m, the group $H^n(K_\lambda)$ is zero if $n > 0$, $n \neq m$ and infinite cyclic if $n = m$. (Here you may object that we never took the time to calculate the cohomology of spheres. But it works like the calculation of the homology of spheres.) □

COROLLARY 13.4.6. *For CW-spaces X, there is a natural isomorphism from $\tilde{H}^m(X^m/X^{m-1})$ to $\hom(\tilde{H}_m(X^m/X^{m-1}), \mathbb{Z})$.*

[3]Think of $\tilde{H}^n(X^n/X^{n-1})$ as the kernel of the map $H^n(X^n/X^{n-1}) \to H^n(\star)$ induced by the inclusion of the base point.

PROOF. First of all, *naturality* refers to situations where we have a cellular map $X \to Y$. — The case $m = 0$ is easy and covered by earlier discussions of H_0 and H^0. For $m > 0$, by lemma 13.4.5 and a comparison with lemma 4.7, it is enough to handle the case where X has only one m-cell and one 0-cell and no other cells. Then X is homeomorphic to S^m. In particular $\tilde{H}_m(X) \cong \mathbb{Z}$ and $\tilde{H}^m(X) \cong \mathbb{Z}$, so an isomorphism from $\tilde{H}^m(X^m/X^{m-1}) = \tilde{H}^m(X)$ to $\hom(\tilde{H}_m(X^m/X^{m-1}), \mathbb{Z}) = \hom(\tilde{H}_m(X^m), \mathbb{Z})$ certainly exists. But the problem is that we have a choice of two. It is not easy to make the choice. Let's return to the definitions. Let $a \in H_m(X)$ be represented by a mapping cycle $\alpha\colon S^m \to X$ and let $b \in H^m(X)$ be represented by a mapping cycle $\beta\colon X \to S^m$. Then $\beta \circ \alpha$ is a mapping cycle $S^m \to S^m$ and so represents an element

$$\langle b, a\rangle \in [[S^n, S^n]]/[[\star, S^n]] = H_m(S^m) \cong \mathbb{Z}.$$

(The isomorphism $[[S^n, S^n]]/[[\star, S^n]] \to \mathbb{Z}$ is completely determined if we let $\mathrm{id}\colon S^m \to S^m$ correspond to $1 \in \mathbb{Z}$. See remark 7.2.7.) The map $a, b \mapsto \langle b, a\rangle$ is a bilinear map from $H_m(X) \times H^m(X)$ to $\mathbb{Z}$ and it is easy to check that the corresponding map from $H^m(X)$ to $\hom(H_m(X), \mathbb{Z})$ is an isomorphism. For this check, it does not hurt to assume that X *is* S^m. □

A few words need to be said about the proof of proposition 13.4.3. It is not a problem to formulate and prove a cohomology analogue of proposition 12.4.9. It follows then from proposition 12.4.9 and its cohomology analogue that the diagram

$$\begin{array}{ccc}
\tilde{H}^m(X^m/X^{m-1}) & \longrightarrow & \hom(\tilde{H}_m(X^m/X^{m-1}), \mathbb{Z}) \\
\downarrow & & \downarrow \\
\tilde{H}^m(X^{m+1}/X^m) & \longrightarrow & \hom(\tilde{H}_{m+1}(X^{m+1}/X^m), \mathbb{Z})
\end{array}$$

commutes (horizontal arrows as in corollary 13.4.6, left-hand vertical arrow from the cohomological variant of the cellular chain complex, right-hand vertical arrow determined by differential in $C(X)$, the cellular chain complex itself). Of course, before we can use proposition 12.4.9 and its cohomology analogue, we should choose characteristic maps for all the cells of X.

CHAPTER 14

External products and the cup product

14.1. Products in homology and cohomology

Definition 14.1.1. Given mapping cycles $f\colon X_1 \to Y_1$ and $g\colon X_2 \to Y_2$ we define $f \otimes g\colon X_1 \times X_2 \to Y_1 \times Y_2$. Idea: if the germ of f at $x_1 \in X_1$ is $\sum a_j \varphi_j$ and the germ of g at $x_2 \in X_2$ is $\sum b_k \gamma_k$, then the germ of $f \times g$ at (x_1, x_2) shall be

$$\sum (a_j b_k) \cdot (\varphi_j \times \gamma_k)$$

where $(\varphi_j \times \gamma_k)(u, v) = (\varphi_j(u), \gamma_k(v)) \in Y_1 \times Y_2$. Pass to homotopy classes:

$$[[f]] \otimes [[g]] := [[f \otimes g]] \in [[X_1 \times X_2, Y_1 \times Y_2]].$$

(Yes, it is well defined.)

Definition 14.1.2. External products in homology: given $[[f]] \in H_m(X)$ and $[[g]] \in H_n(Y)$ we think

$$f\colon \mathbb{R}^m \cup \{\infty\} \to X, \qquad g\colon \mathbb{R}^n \cup \{\infty\} \to Y$$

where $\mathbb{R}^m \cup \{\infty\}$ is the one-point compactification, etc. But we can also assume that f is zero in a neighborhood of ∞, and similarly for g. In other words we can write

$$f\colon \mathbb{R}^m \to X, \qquad g\colon \mathbb{R}^n \to Y$$

where f and g have *compact support*. Then

$$f \otimes g\colon \mathbb{R}^m \times \mathbb{R}^n \to X \times Y$$

has compact support and represents an element in $H_{m+n}(X \times Y)$. We call it $[[f]] \times [[g]]$. Indeed it depends only on $[[f]] \in H_m(X)$ and $[[g]] \in H_n(Y)$.

Example 14.1.3. Under construction: the suspension isomorphism of theorem 13.1.5 has an alternative description in which it is given by external product $z_1 \times$, where $z_1 \in H_1(S^1)$ is the standard generator.

Definition 14.1.4. External products in cohomology: given $[[f]] \in H^m(X_1)$ and $[[g]] \in H^n(X_2)$ we think $f\colon X_1 \to \mathbb{R}^m \cup \{\infty\}$ and $g\colon X_2 \to \mathbb{R}^n \cup \{\infty\}$ and we form the composition

$$X_1 \times X_2 \xrightarrow{f \otimes g} (\mathbb{R}^m \cup \{\infty\}) \times (\mathbb{R}^n \cup \{\infty\}) \xrightarrow{\mu_{m,n}} \mathbb{R}^{m+n} \cup \{\infty\}$$

where $\mu_{m,n}$ is the obvious quotient map. This represents an element

$$[[f]] \times [[g]] \in H^{m+n}(X_1 \times X_2).$$

As the notation suggests, it depends only on $[[f]] \in H_m(X)$ and $[[g]] \in H_n(Y)$.

Definition 14.1.5. Internal products in cohomology: given $[[f]] \in H^m(X)$ and $[[g]] \in H^n(X)$, form $[[f]] \times [[g]] \in H^{m+n}(X \times X)$ and apply the diagonal map $\mathrm{diag}\colon X \to X \times X$ to get

$$[[f]] \smile [[g]] := \mathrm{diag}^*([[f]] \times [[g]]) \in H^{m+n}(X).$$

This is the *cup product*.

Proposition 14.1.6. *The external products in homology and cohomology and the cup product are associative and graded commutative. The cup product on* $H^*(X)$ *has a neutral element* $1 \in H^0(X)$. *The external products also have neutral elements in* $H_0(\star)$, $H^0(\star)$.

Sketch proof. In the case of external products in cohomology, the meaning of *graded commutative* is as follows: the image of $[[f]] \times [[g]]$ under the isomorphism

$$H^{m+n}(X_1 \times X_2) \longrightarrow H^{n+m}(X_2 \times X_1)$$

is $(-1)^{mn}[[g]] \times [[f]]$. The sign comes in as the degree of the self-map of $\mathbb{R}^{m+n} \cup \infty$ given by $(x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n}) \mapsto (x_{m+1}, \ldots, x_{m+n}, x_1, \ldots, x_m)$. The proof should be clear. The case of the external product in homology is similar. The neutral element $1 \in H^0(X)$ for the cup product is given by the constant map from X to S^0 which takes all of X to the element of S^0 which is not the base point $\star$, on the understanding that $H^0(X)$ is $[[X, S^0]]/[[X, \star]]$. (I'm currently a little undecided as to which element of S^0 ought to be the base point ... this is related to the notorious sign problems.) Alternatively, if we use the calculation according to which $H^0(X)$ is the set of continuous maps from X to $\mathbb{Z}$, then the element $1 \in H^0(X)$ is given by the constant map from X to $\mathbb{Z}$ with value 1. $\square$

14.2. A defensive rant on mapping cycles

In the last section we saw that the definition of products in homology and cohomology based on mapping cycles is simple. This is in stark contrast to the cumbersome definition of products in singular homology and cohomology. See section B.3. But in some respects the products in singular homology and cohomology have better or more predictable formal properties than the products in homology and cohomology based on mapping cycles. Quite generally, singular homology and cohomology is (to me) more obscure in the definitions than mapping cycle homology and cohomology, but its formal properties seem to be more predictable. One explanation for that could be that the formal properties of singular homology and cohomology have had more time to be understood. Whatever the reason may be, the consequence is that in typical modern expositions of singular homology and cohomology, the formal properties are often emphasized while the user is encouraged to forget the definitions. I have tried until now to follow a similarly sterilized approach in setting up homology and cohomology with mapping cycles, but perhaps that was not a good idea. Every now and then we need to return to the definitions. It is important to get used to the idea that mapping cycles *behave in many ways like continuous maps*.
The difficulty here is that, in so many ways, mapping cycles do not seem to behave like continuous maps. Their definition is not as pointwise as the definition of continuous maps. Even if we think of the *value* of a mapping cycle $f\colon X \to Y$ at a point $x_0 \in X$ as a finite formal linear combination

$$\sum_i a_i f_i$$

where the a_i are integers and the f_i are germs of continuous maps from (X, x_0) to Y, then it is disturbing that the finitely many elements $f_i(x_0)$ in Y can be distinct. We cannot say *where* approximately in Y that value is located. (Another important difference

between continuous maps and mapping cycles was mentioned at the end of section 4.3.) Nevertheless, let me mention some aspects of continuous maps which generalize well to mapping cycles.

(i) If $f\colon X \to S^m$ is a continuous map, and $x_0 \in X$, then there is an open neighborhood U of x_0 such that f restricted to U is homotopic to a constant map.

(ii) Suppose that X is a normal space. Let $f\colon X \to Y$ be a continuous map, $U \subset X$ an open subset and $A \subset X$ a closed subset such that $A \subset U$. If $f_{|U}$ is homotopic to a constant map, then f is homotopic to a composition

$$X \xrightarrow{\text{quotient map}} X/A \longrightarrow Y.$$

(iii) Let X and Y be spaces with base points x_0 and y_0. Let $f\colon X \to S^m$ and $g\colon Y \to S^n$ be base-point preserving continuous maps. (Think $S^m = \mathbb{R}^m \cup \{\infty\}$ and $S^n = \mathbb{R}^n \cup \{\infty\}$, using ∞ as the base point in both cases.) Then there is an induced map of smash products (see definition 14.4.1 just below)

$$f \wedge g\colon X \wedge Y \longrightarrow S^m \wedge S^n = S^{m+n}.$$

We have already seen that property (i) does not generalize well to singular cohomology H^0. The homomorphism $H^0(X) \to H^0(U)$ can be highly nontrivial for arbitrarily small neighborhoods U of x_0. Our example was $X = \{0\} \cup \{2^{-i} \mid i = 0, 1, 2, \dots\}$, a subspace of $\mathbb{R}$. Similar examples of spaces could be given to illustrate the bad behavior of singular cohomology H^n when $n > 0$. A good example for $n = 1$ is the Hawaiian earring: the union of the circles of radius 2^{-i} and center $(2^{-i}, 0)$ in the plane $\mathbb{R}^2$, where $i = 0, 1, 2, \dots$. It is a subspace of $\mathbb{R}^2$. In this case $x_0 = (0, 0)$ is the interesting choice of base point. — Property (ii) does generalize to singular cohomology. I have nevertheless added it to the list because it combines well with property (i) in situations where property (i) holds. — I suspect that property (iii) as stated does not generalize well to singular cohomology. More precisely, I do not think that the external product $a \times b$ of elements

$$a \in \tilde{H}^m(X) = \ker[\, H^m(X) \to H^m(\{x_0\})\,]$$

and

$$b \in \tilde{H}^n(Y) = \ker[\, H^m(Y) \to H^m(\{y_0\})\,]$$

can always be promoted to an element of $\tilde{H}^{m+n}(X \wedge Y)$. An example that I would try is $X = Y =$ Hawaiian earring, $m = n = 1$ and $x_0 = y_0 = (0, 0)$. It is not easy but I think it has been well investigated by others.

Proof of (i). Choose a small open neighborhood V of $f(x_0)$ which is contractible in S^m. Then $U = f^{-1}(V)$ has the required property. □

Proof of (ii). Let $(h_t\colon U \to S^m)_{t \in [0,1]}$ be a homotopy so that $h_0 \equiv f$ on U and h_1 is constant. Choose a continuous function $\psi\colon X \to [0, 1]$ such that $\psi \equiv 1$ on A and $\operatorname{supp}(\psi) \subset U$. Define $f^\sharp\colon X \to Y$ by $f^\sharp(x) = f(x)$ for $x \notin U$ and $f^\sharp(x) = h_{\psi(x)}(x)$ for $x \in U$. Then $f^\sharp$ is homotopic to f (easy) and since it is constant on A, it can be written as a composition $X \to X/A \to Y$. □

14.3. Good news about mapping cycles

LEMMA 14.3.1. *Let X be a normal space, $U \subset X$ an open subset and $A \subset X$ a closed subset such that $A \subset U$. Let $\nu \in H^m(X)$ be a class such that the image of ν in $H^m(U)$ is zero. Then ν is in the image of the homomorphism*

$$\tilde{H}^m(X/A) \to H^m(X)$$

induced by the projection $X \to X/A$.

PROOF. Write $X/\!\!/U$ for the mapping cone of the inclusion $U \to X$. Represent the class ν by a mapping cycle $f\colon X \to S^m$. We can assume that it is traceless. Choose a homotopy

$$h\colon [0,1] \times U \longrightarrow S^m$$

such that $h_0 = f|_U$ and $h_1 \equiv 0$. We can assume that this is stationary near $\{0,1\} \times U$. Together, f and h then define a mapping cycle $\bar{f}\colon X/\!\!/U \longrightarrow S^m$ which agrees with f on $X \subset X/\!\!/U$. Now choose a continuous function $\psi\colon X \to [0,1]$ such that $\psi \equiv 1$ in a neighborhood of A and $\mathrm{supp}(\psi) \subset U$. (This exists because X is normal.) We use this to make a continuous map $e\colon X \longrightarrow X/\!\!/U$ by $e(x) = x \in X \subset X/\!\!/U$ for $x \notin U$ and $e(x) =$ element represented by $(\psi(x), x)$ in $[0,1] \times U$ for $x \in U$. (In particular $e(x)$ is the cone point if $\psi(x) = 1$.) Clearly e is homotopic to the inclusion of X in $X/\!\!/U$. Therefore the mapping cycle $\bar{f}e\colon X \to S^m$ is homotopic to $\bar{f}|_X = f$. But $\bar{f}e$ is $\equiv 0$ on a neighborhood of A by construction, and so can be viewed as a mapping cycle $X/A \to S^m$. □

Let X be a space with a base point x_0. We use the standard description of $\tilde{H}^n(X)$ as the kernel of $H^n(X) \to H^n(\{x_0\})$.

LEMMA 14.3.2. *For any $\nu \in \tilde{H}^n(X) = \ker[H^n(X) \to H^n(\{x_0\})]$ there exists an open neighborhood U of x_0 such that ν is in the kernel of the homomorphism $H^n(X) \to H^n(U)$ determined by $U \hookrightarrow X$.*

PROOF. It is instructive to begin with the case $n = 0$. In this case ν corresponds to a continuous map from X to $\mathbb{Z}$ which takes the value 0 at x_0. (Here we use an earlier description of H^0; see ... .) Because a continuous map $X \to \mathbb{Z}$ is locally constant, there must be a neighborhood U of x_0 in X such that the map is $\equiv 0$ on U.
In the case $n > 0$, choose a mapping cycle $f\colon X \to S^n$ representing ν. We can assume that f is traceless, i.e., the composition of $f\colon X \to S^m$ with the map $S^m \to \star$ is $\equiv 0$. Choose an open neighborhood U of x_0 such that f is given by a finite formal linear combination

$$\sum_i a_i f_i$$

where $a_i \in \mathbb{Z}$ and f_i is a continuous map from U to S^n, and $\sum_i a_i = 0$. Making U smaller if necessary, we can assume that $f_i(U)$ is contained in a contractible subset of S^n, for example, a metric open ball of radius ε about $f_i(x_0)$ in the standard metric of S^n. Then the image of $\nu = [[f]]$ in $H^n(U)$ is $\sum_i a_i[[f_i|_U]]$. This is zero since each $f_i|_U$ is homotopic to a constant map. □

In the proof of proposition 14.3.3 below the following method will be used. Suppose that $g\colon X \times Y \to S^n$ is a mapping cycle and that X comes with a base point x_0. Also, for simplicity, suppose that Y is compact. We want to find an open neighborhood U of $\{x_0\}$ in X such that $g|_{U \times Y}$ is homotopic to the composition $gq|_{U \times Y}$ where $q\colon X \times Y \to X \times Y$ is defined by $q(x,y) = (x_0, y)$. To do this we choose an open neighborhood U of x_0 in X and a covering of Y by open sets $V_1, \ldots, V_r$ such that on each $U \times V_j$, where

$j = 1, 2, \ldots, r$, the mapping cycle g can be written as a formal linear combination with integer coefficients of continuous functions g_{ij} ,

$$g|_{U \times V_j} = \sum_i a_{ij} g_{ij} \ .$$

Making U and the V_j sufficiently small, we can assume that $g_{ij}(U \times V_j)$ is contained in a metric open ball in S^n of radius $< \varepsilon$, where $\varepsilon > 0$ is fixed and small. Let

$$G_{ij} \colon U \times V_j \times [0,1] \to S^n$$

be defined so that $G_{ij}(x, y, t)$ is the point on the geodesic segment from $g_{ij}(x, y)$ to $g_{ij}(x_0, y)$ which divides the segment in the ratio $t : (1 - t)$. In particular $G_{ij}(x, y, 0) = g_{ij}(x, y)$ and $G_{ij}(x, y, 1) = g_{ij}(x_0, y)$. Then we can define a mapping cycle

$$G \colon U \times Y \times [0,1] \to S^n$$

in such a way that G agrees with $\sum_i a_{ij} G_{ij}$ on $U \times V_j \times [0,1]$. This G is a homotopy from $g|_{U \times Y}$ to $gq|_{U \times Y}$, as required. Homotopies obtained by this construction will be called *short geodesic homotopies*. The cases where $Y = \star$ and $Y = [0,1]$ are important.

PROPOSITION 14.3.3. *Suppose that X is a normal space with base point x_0. Then any class in $\tilde{H}^n(X)$ can be represented by a mapping cycle $X \to S^n$ which is $\equiv 0$ in an open neighborhood of x_0.*
If two mapping cycles $f, g \colon X \to S^n$ with that property represent the same element of $\tilde{H}^n(X)$, then there exists a homotopy $h \colon X \times [0,1] \to S^n$ with the following properties.

(i) *$h_0 = f$ and $h_1 = g + \iota \circ k$, where $k \colon X \to \star$ is a mapping cycle and $\iota \colon \star \to S^n$ is the inclusion of the base point.*
(ii) *h is $\equiv 0$ in an open neighborhood of $\{x_0\} \times [0,1]$.*

PROOF. The case $n = 0$ is an exercise. We now assume $n > 0$. Let $v \in \tilde{H}^n(X)$ be represented by a traceless mapping cycle $f \colon X \to S^n$. Without loss of generality f restricted to x_0 is $\equiv 0$, otherwise we can subtract a constant mapping cycle (without changing the class v). Using the method of short geodesic homotopies, we find an open neighborhood U of x_0 and a homotopy $\Phi \colon U \times [0,1] \to S^n$ from $f|_U$ to 0. Now choose a closed subset A of X which is contained in U and which is a neighborhood of x_0 in X. By lemma 14.3.1, the class v can be represented by a mapping cycle which is zero on A. Now for the second part: we can start with a homotopy $h' \colon X \times [0,1] \to S^n$ which satisfies property (i), with h' instead of h. We get this directly from the assumptions. We can also arrange that h' restricted to $\{x_0\} \times [0,1]$ is zero. By the method of short sectional homotopies, we can find an open neighborhood U of x_0 in X and a homotopy $\Phi' \colon \big(U \times [0,1]\big) \times [0,1] \to S^n$ from $h'|_{U \times [0,1]}$ to zero. In addition we may assume that $f|_U \equiv 0$ and $g|_U \equiv 0$. Then Φ' will be zero on $U \times \{0,1\} \times [0,1]$. Reparameterizing, we can improve Φ' to a homotopy Φ which is stationary near $t = 0$ and $t = 1$, so that h' and Φ together define a mapping cycle

$$h' \cup \Phi \colon (X /\!\!/ U) \times [0,1] \to S^n$$

where $X /\!\!/ U$ is the mapping cone of $U \to X$. As in the proof of lemma 14.3.1, there is a map $e \colon X \to X /\!\!/ U$ which is homotopic to the inclusion, and which is equal to the inclusion on $X \smallsetminus U$, but which takes all of A to the cone point. The composition

$$X \times [0,1] \xrightarrow{(x,t) \mapsto (e(x),t)} (X /\!\!/ U) \times [0,1] \xrightarrow{h' \cup \Phi} S^n$$

is the homotopy h that we require. $\square$

14.4. More on cup product and external product in cohomology

DEFINITION 14.4.1. For spaces X and Y with base points x_0 and y_0, the smash product $X \wedge Y$ is the quotient space

$$\frac{X \times Y}{X \times \{y_0\} \ \cup \ \{x_0\} \times Y}.$$

(Important example: if X is a sphere, $X = \mathbb{R}^m \cup \{\infty\}$ with $x_0 = \infty$ and Y is also a sphere, $Y = \mathbb{R}^n \cup \infty$ with $y_0 = \infty$, then $X \wedge Y$ is clearly identified with $(\mathbb{R}^m \cup \mathbb{R}^n) \cup \infty$ and so is again a sphere.)

COROLLARY 14.4.2. *Given* $a \in \tilde{H}^m(X)$ *and* $b \in \tilde{H}^n(Y)$*, the external product of* a *and* b *has a well-defined refinement to an element of* $\tilde{H}^{m+n}(X \wedge Y)$*.*

PROOF. Use proposition 14.3.3 to represent a by a mapping cycle $f\colon X \to S^m$ which is zero in an open neighborhood U of the base point x_0 and to represent b by a mapping cycle $g\colon Y \to S^n$ which is zero in an open neighborhood V of the base point y_0. Then the composition

$$X \times Y \xrightarrow{f \otimes g} (\mathbb{R}^m \cup \{\infty\}) \times (\mathbb{R}^n \cup \{\infty\}) \xrightarrow{\mu_{m,n}} \mathbb{R}^{m+n} \cup \{\infty\},$$

which is supposed to represent $a \times b$, is $\equiv 0$ on $X \times V$ and on $U \times Y$, and so can be viewed as a mapping cycle $X \wedge Y \to S^{m+n}$. Its class in $\tilde{H}^{m+n}(X \wedge Y)$ is independent of choices by the second part of proposition 14.3.3. $\square$

COROLLARY 14.4.3. *Let* A *and* B *be closed subsets of* X*. Given* $a \in \tilde{H}^m(X/A)$ *and* $b \in \tilde{H}^n(X/B)$*, we can write*

$$a \smile b \in \tilde{H}^{m+n}(X/(A \cup B)) \ .$$

In more detail: let us agree to view a and b as elements of $H^m(X)$ and $H^n(X)$, respectively, using the homomorphisms $H^m(X/A) \to H^m(X)$ and $H^n(X/B) \to H^n(X)$ determined by the projections from X to X/A and X/B. If we form their cup product $a \smile b \in H^{m+n}(X)$ following standard instructions, then this is in the image of the homomorphism

$$\tilde{H}^{m+n}(X/(A \cup B)) \to H^{m+n}(X)$$

determined by the projection $X \to X/(A \cup B)$. Moreover there is a preferred choice of element of $\tilde{H}^{m+n}(X/(A \cup B))$ which maps to the traditional cup product $a \smile b \in H^{m+n}(X)$.

PROOF. The external product $a \times b$ lives in $\tilde{H}^{m+n}(X/A \wedge X/B)$ by corollary 14.4.2. The composition

$$X \xrightarrow{\text{diag}} X \times X \longrightarrow X/A \wedge X/B$$

takes all of $A \cup B$ to the base point and can therefore be viewed as a map θ from $X/(A \cup B)$ to $X/A \wedge X/B$. The re-defined $a \smile b$ that we are looking for is $\theta^*(a \times b) \in \tilde{H}^{m+n}(X/(A \cup B))$. $\square$

14.5. A glimpse of Lyusternik-Schnirelmann theory

DEFINITION 14.5.1. A path-connected space X has Lyusternik-Schnirelmann (LS) invariant $\leq r$ if there exists a covering of X by open subsets

$$U_0, U_1, U_2, \ldots, U_r$$

such that the inclusion $U_i \to X$ is homotopic to a constant map for each $i = 0, 1, 2, \ldots, r$. The same X is said to have LS invariant $= r$ if it has LS invariant $\leq r$ but does not have LS invariant $\leq r - 1$.

(*Remarks*: the official terminology is LS *category*, not LS invariant. But this clashes with the use of the word *category* as in categories and functors. This is reminiscent of the concept *Baire category* in general topology, which also clashes with *category* as in categories and functors.
The above definition of LS invariant seems to be the standard in homotopy theory, but the original old definition of LS invariant (and the one I used in the lecture on Tuesday, and the one I found on Wikipedia!) differs from the above by 1. That is, LS invariant r as above would have been called LS invariant $r + 1$ by the ancients.)

EXAMPLE 14.5.2. A space X has LS invariant 0 if and only if it is contractible. The sphere S^m for $m > 0$ has LS invariant 1. The suspension ΣX of any nonempty space X has LS invariant ≤ 1 because the two open sets given by ΣX minus north pole and ΣX minus south pole make up an open covering of the type required. It is easy to show that the torus $S^1 \times S^1$ has LS invariant ≤ 3. It is slightly harder to show that it has LS invariant ≤ 2. It follows from proposition 14.5.3 below that it does not have LS invariant ≤ 1; therefore the LS invariant of $S^1 \times S^1$ is 2. It is easy to show that complex projective space $\mathbb{C}P^n$ has LS invariant $\leq n$. (There is a standard "atlas" for $\mathbb{C}P^n$ as a differentiable manifold, for example, which has $n + 1$ charts $U_1, \ldots U_{n+1}$, all contractible in their own right. It follows that the inclusions $U_i \to \mathbb{C}P^n$ are nullhomotopic.) It follows from proposition 14.5.3 below that $\mathbb{C}P^n$ does not have LS invariant $\leq n - 1$; therefore it has LS invariant $= n$.

PROPOSITION 14.5.3. *Suppose that X is a path-connected normal space which has* LS *invariant $\leq r - 1$. Then for any selection of elements $a_1, a_2, \ldots, a_r$ in the cohomology of X, where $a_i \in H^{m_i}(X)$ and $m_i > 0$ for $i = 1, 2, \ldots, r$, we have*

$$a_1 \smile a_2 \smile a_3 \smile \cdots \smile a_r = 0 \in H^{\sum_i m_i}(X) \ .$$

PROOF. Choose a covering of X by open subsets $U_1, U_2, \ldots, U_r$ such that the inclusion $U_i \to X$ is homotopic to a constant map for each $i = 1, 2, \ldots, r$. Since X is normal, it has a covering by closed subsets $A_1, A_2, \ldots, A_r$ such that $A_i \subset U_i$ for $i = 1, 2, \ldots, r$. By lemma 14.3.1, the class a_i is in the image of the homomorphism

$$H^{m_i}(X/A_i) \to H^{m_i}(X)$$

(even though we assume no special relationship between A_i and a_i). Therefore by corollary 14.4.3 the cup product $a_1 \smile a_2 \smile a_3 \smile \cdots \smile a_r$ is in the image of the homomorphism

$$H^{\sum m_i}(X/\textstyle\bigcup_i A_i) \longrightarrow H^{\sum m_i}(X)$$

determined by the projection $X \longrightarrow X/\bigcup_i A_i$. But $X/\bigcup_i A_i$ is a one-point space. $\square$

REMARK 14.5.4. The LS invariant is rightly so called because it is a homotopy invariant. Let us prove this. Suppose that $f\colon X \to Y$ is a map between path connected spaces and that f admits a homotopy inverse $g\colon Y \to X$. Suppose that Y has LS invariant $\leq r$, so Y admits a covering by open subsets $V_0, V_1, \ldots, V_r$ such that the inclusions $V_i \to Y$ are nullhomotopic. We need to show that X has LS invariant $\leq r$, too. The open sets $U_i = f^{-1}(V_i)$ form an open covering of X. The inclusion $e_i\colon U_i \to X$ is nullhomotopic because it is homotopic to $(gf)e_i = g(fe_i)$ where fe_i is already nullhomotopic because it lands in $V_i \subset Y$. □

14.6. Another glimpse of Lyusternik-Schnirelmann theory

PROPOSITION 14.6.1. *Let Y be a path-connected normal space with base point $\star$ which has* LS *invariant $\leq r-1$. Then the diagonal map*

$$Y \longrightarrow \underbrace{Y \times Y \times \cdots \times Y}_{r}$$

is homotopic to a map with image contained in $\bigcup_{i=1}^{r} Y^{i-1} \times \{\star\} \times Y^{r-i}$.

PROOF. Choose an open cover of Y with open sets $U_1, \ldots, U_r$ such that the inclusion $U_i \to Y$ is nullhomotopic for each i. For each i choose a nullhomotopy

$$(h_t^{(i)}\colon U_i \to Y)_{t\in[0,1]}$$

of the inclusion, so that $h_0^{(i)}(y) = y$ and $h_1^{(i)}(y) = \star$ for all $y \in U_i$. Let us extend the parameter interval for these homotopies by setting

$$h_t^{(i)} = h_1^{(i)} \qquad \text{if } t > 1.$$

Choose a partition of unity $(\psi_1, \psi_2, \ldots, \psi_r)$ subordinate to the open covering[1] by $U_1, \ldots, U_r$; so $\psi_i\colon Y \to [0,1]$ is continuous, $\operatorname{supp}(\psi_i) \subset U_i$ and $\sum_i \psi_i \equiv 1$. Put

$$g_{i,t}(y) = \begin{cases} h^{(i)}_{t\cdot r\cdot\psi_i(y)}(y) & \text{if } y \in U_i \\ y & \text{if } y \notin U_i \end{cases}$$

for $t \in [0,1]$. (This is continuous because $\operatorname{supp}(\psi_i)$ is closed in Y by definition and contained in U_i.) Define a homotopy

$$\big(h_t\colon Y \to Y \times Y \times \cdots \times Y\big)_{t\in[0,1]}$$

by setting $h_t(y) = (g_{1,t}(y), \ldots, g_{r,t}(y))$. Then $h_0(y) = (y, y, \ldots, y)$ for all $y \in Y$. For every $y \in Y$ there is some $i \in \{1, 2, \ldots, r\}$ such that $r\psi_i(y) = 1$; then $g_{i,1}(y) = \star$ for that i, by construction. Therefore h_1 is a map with image contained in $\bigcup_{i=1}^{r} Y^{i-1} \times \{\star\} \times Y^{r-i}$. □

ANOTHER PROOF OF PROPOSITION 14.5.3. Write $a_i \in \tilde{H}^{m_i}(X)$. Using corollary 14.4.2 we get for the external product

$$a_1 \times a_2 \times \cdots \times a_r \in \tilde{H}^{\sum_i m_i}(X \wedge X \wedge X \cdots \wedge X).$$

[1] If Y is paracompact, then every open covering of Y has a subordinate partition of unity. But I think that in the case of a finite open covering, it suffices to assume that Y is normal.

The class $a_1 \smile a_2 \smile a_3 \smile \cdots \smile a_r \in \tilde{H}^{\sum_i m_i}(X)$ is obtained from that by applying the homomorphism in (reduced) cohomology determined by the composition

$$X \xrightarrow{\text{diag}} \underbrace{X \times X \times \cdots \times X}_{r} \longrightarrow \underbrace{X \wedge X \wedge X \cdots \wedge X}_{r}.$$

But that composition is nullhomotopic by proposition 14.6.1. $\square$

CHAPTER 15

More on products ... and the cap product

15.1. Naturality of products

PROPOSITION 15.1.1. *The external products in homology and cohomology are natural.* □

To spell this out, suppose that $f\colon X_1 \to Y_1$ and $g\colon X_2 \to Y_2$ are continuous maps. Let $f \times g\colon X_1 \times X_2 \longrightarrow Y_1 \times Y_2$ be given by $(f \times g)(x_1, x_2) = (f_1(x_1), f_2(x_2))$. The following squares are claimed to be commutative.

$$\begin{tikzcd}
H_m(X_1) \times H_n(X_2) \arrow[r, "(a,b)\mapsto a\times b"] \arrow[d, "(a,b)\mapsto(f_*(a),g_*(b))"'] & H_{m+n}(X_1 \times X_2) \arrow[d, "(f\times g)_*"] \\
H_m(Y_1) \times H_n(Y_2) \arrow[r, "(u,v)\mapsto u\times v"] & H_{m+n}(Y_1 \times Y_2)
\end{tikzcd}$$

$$\begin{tikzcd}
H^m(X_1) \times H^n(X_2) \arrow[r, "(a,b)\mapsto a\times b"] & H^{m+n}(X_1 \times X_2) \\
H^m(Y_1) \times H^n(Y_2) \arrow[r, "(u,v)\mapsto u\times v"] \arrow[u, "(a,b)\mapsto(f^*(a),g^*(b))"] & H^{m+n}(Y_1 \times Y_2) \arrow[u, "(f\times g)^*"']
\end{tikzcd}$$

COROLLARY 15.1.2. *The cup product in cohomology is natural.* □

To spell this out as well, suppose that $f\colon X \to Y$ is continuous. For u in $H^m(Y)$ and v in $H^n(Y)$ we have $u \smile v$ in $H^{m+n}(Y)$ and $f^*(u)$ in $H^m(X)$ as well as $f^*(v)$ in $H^n(X)$. It is claimed that

$$f^*(u \smile v) = f^*(u) \smile g^*(v) \in H^{m+n}(X).$$

To unravel this some more we introduce the concept of a graded ring.

DEFINITION 15.1.3. A *graded ring* is a family $(R_n)_{n\in\mathbb{Z}}$ of abelian groups R_n together with bi-additive maps $R_m \times R_n \longrightarrow R_{m+n}$ for all $m, n \in \mathbb{Z}$ (for which we write $(a, b) \mapsto a \cdot b$) such that the following conditions are satisfied.

- The associative law holds:
$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \in R_{m+n+p}$$
for all $a \in R_m$, $b \in R_n$ and $c \in R_p$.
- There is an element in R_0, denoted by 1, such that $1 \cdot a = a = a \cdot 1$ for every $m \in \mathbb{Z}$ and $a \in R_m$. (This is automatically unique.)

The graded ring $R = (R_m)_{m\in\mathbb{Z}}$ is *graded commutative* if for all $m, n \in \mathbb{Z}$ and $a \in R_m$, $b \in R_n$ we have $a \cdot b = (-1)^{mn} b \cdot a$.
A *homomorphism* h from one graded ring $(R_n)_{n\in\mathbb{Z}}$ to another, $(S_n)_{n\in\mathbb{Z}}$, is a sequence $(h_n\colon R_n \to S_n)_{n\in\mathbb{Z}}$ of homomorphisms of abelian groups such that $h_{m+n}(a \cdot b) = h_m(a) \cdot h_n(b)$ holds in S_{m+n} for every $a \in R_m$ and $b \in R_n$.

EXAMPLE 15.1.4. A space X determines a graded ring $(R_n)_{n\in\mathbb{Z}}$ where R_n is $H^n(X)$ for $n \geq 0$ and $R_n = 0$ for $n < 0$. The product $R_m \times R_n \to R_{m+n}$ is the cup product, $a \cdot b := a \smile b$. This graded ring is graded commutative. Standard notation for this graded ring is probably $H^*(X)$, which is admittedly not ideal.
The message of 15.1.2 is that for a continuous map $f\colon X \to Y$, the induced maps

$$f^*\colon H^n(Y) \to H^n(X)$$

for all $n \geq 0$ make up a homomorphism of graded rings from the graded ring $H^*(Y)$ to the graded ring $H^*(X)$.

EXAMPLE 15.1.5. If you have heard about differential forms, then you will remember the following example of a graded ring. Let V be a k-dimensional vector space over $\mathbb{R}$. For an integer $n \geq 0$ let $\mathrm{alt}_n(V)$ be the vector space of alternating n-forms on V. (These are the multilinear maps

$$\omega\colon \underbrace{V \times V \times V \times \cdots \times V}_{n} \longrightarrow \mathbb{R}$$

which are insensitive to a permutation of the n variables except for a factor ± 1, the sign of the permutation.) The *shuffle product* is a bilinear map $\mathrm{alt}_m(V) \times \mathrm{alt}_n(V) \to \mathrm{alt}_{m+n}(V)$. For $n < 0$ put $\mathrm{alt}_n(V) := 0$. Then the collection of vector spaces $(\mathrm{alt}_n(V))_{n\in\mathbb{Z}}$ with the shuffle product is a graded ring. It is also graded commutative.

15.2. Products and the Mayer-Vietoris sequence

Let Y be a space with open subsets V and W such that $V \cup W = Y$. Let Z be a another space and choose a class $b \in H_p(Z)$. From the homology Mayer-Vietoris sequence of Y, V, W we have a boundary homomorphism

$$\partial\colon H_m(Y) \longrightarrow H_{m-1}(V \cap W)\ .$$

From the homology Mayer-Vietoris sequence of $Y \times Z, V \times Z, W \times Z$ we have a boundary homomorphism

$$\partial\colon H_{m+p}(Y \times Z) \longrightarrow H_{m+p-1}((V \cap W) \times Z)\ .$$

PROPOSITION 15.2.1. *The following square commutes:*

$$\begin{tikzcd}
H_m(Y) \arrow[r, "\partial"] \arrow[d, "\times b"] & H_{m-1}(V \cap W) \arrow[d, "\times b"] \\
H_{m+p}(Y \times Z) \arrow[r, "\partial"] & H_{m+p-1}((V \cap W) \times Z)
\end{tikzcd}$$

This is best done by going back to the definitions. Represent a and b by mapping cycles with compact support:

$$\alpha\colon \mathbb{R}^m \to Y, \qquad \beta\colon \mathbb{R}^n \to Z$$

so that $a \times b$ is rep by $\alpha \otimes \beta\colon \mathbb{R}^m \times \mathbb{R}^n \to Y \times Z$, which again has compact support.

COROLLARY 15.2.2. *The following square commutes up to multiplication by* $(-1)^p$ *:*

$$\begin{tikzcd}
H_m(Y) \arrow[r, "\partial"] \arrow[d, "b\times"] & H_{m-1}(V \cap W) \arrow[d, "b\times"] \\
H_{m+p}(Z \times Y) \arrow[r, "\partial"] & H_{m+p-1}(Z \times (V \cap W))
\end{tikzcd}$$

Proof. This could be done by inspection, too, but we can also deduce it from proposition 15.2.1 in the following way. There is a commutative diagram

$$\begin{tikzcd}
H_m(Y) \arrow[r, "\partial"] \arrow[d, "\times b"] & H_{m-1}(V \cap W) \arrow[d, "\times b"] \\
H_{m+p}(Y \times Z) \arrow[r, "\partial"] \arrow[d, "\cong"] & H_{m+p-1}((V \cap W) \times Z) \arrow[d, "\cong"] \\
H_{m+p}(Z \times Y) \arrow[r, "\partial"] & H_{m+p-1}(Z \times (V \cap W))
\end{tikzcd}$$

where the top square commutes by the proposition and the bottom square commutes by naturality of the Mayer-Vietoris sequence. Choose $a \in H_m(Y)$ and chase it through the diagram all the way to $H_{m+p-1}(Z \times (V \cap W))$. One way gives $\partial((-1)^{mp} b \times a)) = (-1)^{mp}\partial(b \times a)$ and the other way gives $(-1)^{(m-1)p} b \times \partial(a)$. Therefore $\partial(b \times a) = (-1)^p b \times \partial(a)$. □

The cohomology versions are as follows. We can keep Y, V, W as above but we need to make some assumptions. For simplicity suppose that Y, V, W and $V \cap W$ are paracompact, and also that $Y \times Z$, $V \times Z$, $W \times Z$ and $(V \cap W) \times Z$ are paracompact. Choose $b \in H^p(Z)$. From the cohomology Mayer-Vietoris sequence of Y, V, W we have a boundary homomorphism

$$\delta \colon H^m(V \cap W) \longrightarrow H^{m+1}(Y) \ .$$

From the cohomology Mayer-Vietoris sequence of $Y \times Z, V \times Z, W \times Z$ we have a boundary homomorphism

$$\delta \colon H^{m+p}((V \cap W) \times Z) \longrightarrow H_{m+p+1}(Y \times Z) \ .$$

Proposition 15.2.3. *The following square commutes:*

$$\begin{tikzcd}
H^m(V \cap W) \arrow[r, "\delta"] \arrow[d, "\times b"] & H^{m+1}(Y) \arrow[d, "\times b"] \\
H^{m+p}((V \cap W) \times Z) \arrow[r, "\delta"] & H^{m+p+1}(Y \times Z)
\end{tikzcd}$$

Corollary 15.2.4. *The following square commutes up to multiplication by* $(-1)^p$ *:*

$$\begin{tikzcd}
H^m(V \cap W) \arrow[r, "\delta"] \arrow[d, "b\times"] & H^{m+1}(Y) \arrow[d, "b\times"] \\
H^{m+p}(Z \times (V \cap W)) \arrow[r, "\delta"] & H^{m+p+1}(Z \times Y)
\end{tikzcd}$$

There are variants of these statements for long exact sequences associated with a map $A \to X$ and its mapping cone $\mathrm{cone}(A \to X)$. Just to give the idea, here is the homology version.

Proposition 15.2.5. *The following square commutes:*

$$\begin{tikzcd}
H_m(\mathrm{cone}(A \to X)) \arrow[r, "\partial"] \arrow[d, "\times b"] & H_{m-1}(A) \arrow[d, "\times b"] \\
H_{m+p}(\mathrm{cone}(A \times Z \to X \times Z)) \arrow[r, "\partial"] & H_{m+p-1}(A \times Z)
\end{tikzcd}$$

Note in passing that $\mathrm{cone}(A \times Z \to X \times Z)$ should not be confused with $\mathrm{cone}(A \to X) \times Z$. But it is a quotient space of $\mathrm{cone}(A \to X) \times Z$ in an obvious way, and this must be used to make sense of the left-hand column in the square.

15.3. Products and cellular chain complexes

Let X and Y be CW-spaces, and for simplicity suppose that Y is compact (so Y has only finitely many cells). Then $X \times Y$ is also a CW-space in such a way that

$$(X \times Y)^n = \bigcup_{p=0}^{n} X^p \times Y^{n-p}.$$

It was mentioned/promised earlier that we have

$$C(X) \otimes C(Y) \cong C(X \times Y)$$

for the cellular chain complexes. Now is the time to clarify this using external products in homology. We start by noting that there is an inclusion map

$$e_{p,q} \colon (X^p/X^{p-1}) \wedge (Y^q/Y^{q-1}) \longrightarrow (X \times Y)^{p+q}/(X \times Y)^{p+q-1}$$

which becomes clearer if we note

$$(X^p/X^{p-1}) \wedge (Y^q/Y^{q-1}) = \frac{X^p \times Y^q}{(X^p \times Y^{q-1}) \cup (X^{p-1} \times Y^q)}\,.$$

Therefore we obtain a homomorphism

$$\nu_{p,q} \colon C(X)_p \otimes C(Y)_q \longrightarrow C(X \otimes Y)_{p+q}$$

by composing as follows:

$$\begin{tikzcd}
C(X)_p \otimes C(Y)_q \arrow[d, equal] \\
\tilde{H}_p(X^p/X^{p-1}) \otimes \tilde{H}_q(Y^q/Y^{q-1}) \arrow[d, "\cong"', "\text{external prod.}"] \\
\tilde{H}_p((X^p/X^{p-1}) \wedge (Y^q/Y^{q-1})) \arrow[d, "\text{induced by } e_{p,q}"] \\
\tilde{H}_{p+q}((X \times Y)^{p+q}/(X \times Y)^{p+q-1}) \arrow[d, equal] \\
C(X \times Y)_{p+q}
\end{tikzcd}$$

The arrow labeled *external product* is an isomorphism because X^p/X^{p-1} is a wedge of p-spheres and Y^q/Y^{q-1} is a wedge of q-spheres, so that the smash product $(X^p/X^{p-1}) \wedge (Y^q/Y^{q-1})$ is a wedge of $(p+q)$-spheres.
Writing $n = p + q$ and $q = n - p$, we obtain a homomorphism

$$(\nu_{0,n}, \nu_{1,n-1}, \ldots, \nu_{n,0}) \colon \bigoplus_{p=0}^{n} C(X)_p \otimes C(Y)_{n-p} \longrightarrow C(X \times Y)_n\,.$$

This is an isomorphism. (The infinite cyclic summand in the target group corresponding to an n-cell E of $X \times Y$ has a counterpart in the source group which can be found by asking how that n-cell is a product of a p-cell E' of X and an $(n-p)$-cell E'' of Y. The pair of cells (E', E'') contributes an infinite cyclic summand to the source group.) Instead of $\bigoplus_{p=0}^{n} C(X)_p \otimes C(Y)_{n-p}$ we can also write $(C(X) \otimes C(Y))_n$ using definition 12.1.9 of the tensor product of chain complexes. At the same time we rename our map

$$u_n \colon (C(X) \otimes C(Y))_n \longrightarrow C(X \times Y)_n \,.$$

Proposition 15.3.1. *The isomorphisms* $u_n \colon \big(C(X) \otimes C(Y)\big)_n \longrightarrow C(X \times Y)_n$ *taken together for all* n *are compatible with the differentials. So they define an isomorphism of chain complexes from the tensor product* $C(X) \otimes C(Y)$ *of the cellular chain complexes of* X *and* Y *to* $C(X \times Y)$*, the cellular chain complex of the product* $X \times Y$*.*

Proof. (A more grown-up proof would probably rely on proposition 15.2.5, but I could not face this.) Choose a p-cell E in X and a q-cell F in Y, where $p + q = n$. Let $K_E \subset C(X)_p$ and $K_F \subset C(Y)_q$ be the corresponding infinite cyclic summands. It is enough to verify that the equation $du_n = u_{n-1}d$ holds on the infinite cyclic summand

$$K_F \otimes K_E \subset \big(C(X) \otimes C(Y)\big)_n$$

since E and F were arbitrary. Now we use the following observation (which is going to be explained below). Let $A = D^p$ and $B = D^q$, viewed as CW-spaces with the standard structure. (For example A has three cells except when $p = 0$.)

(✠) There exists a cellular map $A \to X$ such that the induced map of cellular chain complexes takes $C(A)_p$ isomorphically to the summand $K_E \subset C(X)_p$. There exists a cellular map $B \to Y$ such that the induced map of ... isomorphically to the summand $K_F \subset C(Y)_q$.

If we believe this for the moment, then the proof is reduced to showing that the diagram

$$(✠✠) \qquad \begin{tikzcd} (C(A) \otimes C(B))_n \arrow[r, "u_n"] \arrow[d] & C(A \otimes B)_n \arrow[d] \\ (C(A) \otimes C(B))_{n-1} \arrow[r, "u_{n-1}"] & C(A \otimes B)_{n-1} \end{tikzcd}$$

commutes, where the left-hand vertical arrow is the differential in the tensor product of $C(A)$ and $C(B)$, while the right-hand vertical arrow is the differential in $C(A \times B)$. One might say that this is true *by inspection*. But here are some details.[1] I am going to assume $p, q > 1$. (The case where $p = 0$ or $q = 0$ is not interesting. The cases where neither is zero but $p = 1$ or $q = 1$ should be looked at separately; they are easier than the cases where $p, q > 1$ but still interesting.) Write $A = D^p$ and $B = D^q$. The p-cell of A has a preferred orientation as a smooth manifold and we orient the $(p-1)$-cell (as a smooth manifold) according to the ONF convention, outward normal first. If we now choose characteristic maps for the p-cell and for the $(p-1)$-cell to be locally diffeomorphic away from the boundaries of the source disks and, in that sense, orientation preserving, then the differential

$$\mathbb{Z} = C(A)_p \to C(A)_{p-1} = \mathbb{Z}$$

is the identity. See remark 7.2.7. Proceed in the same way to choose characteristic maps for B. Now the cells of $A \times B$ of dimension $n = p + q$ and $n - 1$ are already equipped

[1] They are still very sketchy, but I hope they illustrate what incidence numbers are and how they can sometimes be determined.

with characteristic maps, which have the form $D^k \times D^\ell \to A \times B$ where $(k, \ell) = (p, q)$ or $(p, q-1)$ or $(p-1, q)$. (Use the characteristic maps which we selected for A and B.) These characteristic maps for the cells of $A \times B$ have some smoothness properties and so provide orientations for the cells as smooth manifolds. The incidence number for the unique n-cell and the $(n-1)$ cell which is contained in $S^{p-1} \times D^q$ is 1. This is another way of saying that the orientations of these cells are compatible in the sense of the ONF convention, which is easy to check (in a neighborhood in $A \times B$ of any point in that $(n-1)$-cell). Similarly, the incidence number for the unique n-cell and the $(n-1)$-cell which is contained in $D^p \times S^{q-1}$ is $(-1)^p$. (The sign has something to do with an outward normal which comes as number $p+1$ in a list of $p+q$ vectors instead of coming first.) This determination of incidence numbers establishes the commutativity of (✠✠) in the cases $p, q > 1$.

It remains to give an argument for (✠). We start by choosing a characteristic map $f\colon D^p \to X$ for the cell E. There is no guarantee that this is cellular; for $p > 1$, the 0-cell of D^p might not be taken to a 0-cell of X. But it does take p-skeleton to p-skeleton, and $(p-1)$-skeleton to $(p-1)$-skeleton, and so induces a map

$$\mathbb{Z} = C(D^p)_p \longrightarrow C(X)_p$$

which gives an isomorphism of $\mathbb{Z}$ with the summand $K_E \subset C(X)_p$. Next, choose a homotopy from $f|_{S^{p-1}}$ to a cellular map, in X^{p-1}. Use the homotopy extension property to extend this to a homotopy $(h_t)_{t\in[0,1]}$ from $f = h_0$ to some other map h_1, in X^p. So each h_t is a map from D^p to X^p and takes S^{p-1} to X^{p-1}. Moreover h_1 is cellular by construction. Each h_t induces a map $D^p/S^{p-1} \to X^p/X^{p-1}$. Therefore h_0 and h_1 induce the same homomorphism from $\mathbb{Z} = C(D^p)_p$ to $C(X)_p$. As we noted before in the case of $h_0 = f$, that homomorphism gives an isomorphism of $\mathbb{Z}$ with the summand $K_E \subset C(X)_p$. □

Corollary 15.3.2. *Let the classes* $a \in H^p(X) \cong H_{-p}(\hom(C(X), \mathbb{Z})$ *and* $b \in H^q(Y) \cong H_{-q}(\hom(C(Y), \mathbb{Z}))$ *be represented by* $a' \in \hom(C(X)_p, \mathbb{Z})$ *and* $b' \in \hom(C(Y)_q, \mathbb{Z})$*, respectively. Then* $a \times b \in H^{p+q}(X \times Y)$ *is represented by* $a' \otimes b'$ *, more precisely, by the composition*

$$C(X \times Y)_{p+q} \xrightarrow[15.3.1]{\cong} (C(X) \otimes C(Y))_{p+q} \xrightarrow{\text{proj}} C(X)_p \otimes C(Y)_q \xrightarrow{a' \otimes b'} \mathbb{Z}\ .$$

Proof. Let $X' = X/X^{p-1}$ and $Y' = Y/Y^{q-1}$, so that X' and Y' are CW-spaces with a base point which is a 0-cell. The projection $X \to X'$ induces a homomorphism from $\tilde{H}^p(X')$ to $H^p(X)$ which is onto (as one can see by using the description of the cohomology groups in terms of the cellular chain complexes). Similarly the projection $Y \to Y'$ induces a homomorphism from $\tilde{H}^q(Y')$ to $H^q(Y)$ which is onto. Next, let $X'' = X^p/X^{p-1}$ and $Y'' = Y^q/Y^{q-1}$. Then the inclusion $X'' \wedge Y'' \to X' \wedge Y'$ induces a homomorphism

$$\tilde{H}^{p+q}(X' \wedge Y') \longrightarrow \tilde{H}^{p+q}(X'' \wedge Y'')$$

which is injective (as one can see by using the description of the cohomology groups in terms of the cellular chain complexes). Now we have the following commutative diagram:

$$\begin{tikzcd}
H^p(X) \times H^q(Y) \arrow[r, "\text{ext. prod.}"] & H^{p+q}(X \times Y) \\
\tilde{H}^p(X') \times \tilde{H}^q(Y') \arrow[r, "\text{ext. prod.}"] \arrow[u, two heads] \arrow[d] & \tilde{H}^{p+q}(X' \wedge Y') \arrow[u] \arrow[d, tail] \\
\tilde{H}^p(X'') \times \tilde{H}^q(Y'') \arrow[r, "\text{ext. prod.}"] & \tilde{H}^{p+q}(X'' \wedge Y'')
\end{tikzcd}$$

(where we use corollary 14.4.2). Therefore it is enough to prove the following. If X'' is a wedge $\bigvee_\alpha S^p$ and Y'' is a wedge $\bigvee_\beta S^q$, so that $X'' \wedge Y'' = \bigvee_{\alpha,\beta} S^{p+q}$, then the external product of a class

$$a = (a_\alpha) \in \tilde{H}^p(X'') \cong \prod_\alpha \mathbb{Z}$$

and a class

$$b = (b_\beta) \in \tilde{H}^q(Y'') \cong \prod_\beta \mathbb{Z}$$

is $(a_\alpha \cdot b_\beta) \in \tilde{H}^q(X'' \wedge Y'') \cong \prod_{\alpha,\beta} \mathbb{Z}$. By naturality we can further reduce this to the case where $X'' = S^p$ and $Y'' = S^q$ and

$$a = 1 \in \mathbb{Z} \cong \tilde{H}^p(S^p), \qquad b = 1 \in \mathbb{Z} \cong \tilde{H}^q(S^q).$$

The task is then to show that $a \times b = 1 \in \tilde{H}^{p+q}(S^{p+q}) \cong \mathbb{Z})$. In other words, if a is represented by the mapping cycle $\mathrm{id}\colon S^p \to S^p$ and b is represented by $\mathrm{id}\colon S^q \to S^q$, then $a \times b$ is also represented by $\mathrm{id}\colon S^{p+q} \to S^{p+q}$. This should be straightforward. □

15.4. The cap product

Lemma 15.4.1. *For $q \geq 0$, the external product with the standard generator $z_q \in \tilde{H}_q(S^q)$ determines natural isomorphisms*

$$H_m(X) \longrightarrow \tilde{H}_{m+q}\Big(\frac{S^q \times X}{\star \times X}\Big), \qquad H_m(X) \longrightarrow \tilde{H}_{m+q}\Big(\frac{X \times S^q}{X \times \star}\Big).$$

Proof. The inclusion $\star \to S^q$ is certainly a cofibration. It follows that the inclusion $\star \times X \to S^q \times X$ is also a cofibration (by the retraction criterion ... reference???). Therefore we have a long exact sequence

$$\cdots \to H_k(\star \times X) \to H_k(S^q \times X) \to \tilde{H}_k\Big(\frac{S^q \times X}{\star \times X}\Big) \to H_{k-1}(\star \times X) \to \cdots$$

Since the homomorphisms $H_k(\star \times X) \to H_k(S^q \times X)$ in this sequence are clearly injective, the long exact sequence breaks up into short exact sequences and we can write informally

$$\tilde{H}_k\Big(\frac{S^q \times X}{\star \times X}\Big) = \frac{H_k(S^q \times X)}{H_k(\star \times X)}.$$

Now we write $S^q = V \cup W$ where V is S^q minus the north pole and W is S^q minus the south pole. Then there is a long exact Mayer-Vietoris sequence

$$\cdots \to H_k((V \cap W) \times X) \to H_k(V \times X) \oplus H_k(W \times X) \to H_k(S^q \times X) \xrightarrow{\partial} H_{k-1}((V \cap W) \times X) \to \cdots$$

which we can also write in the form

$$\cdots \to H_k(S^{q-1} \times X) \to H_k(X) \oplus H_k(X) \to H_k(S^q \times X) \xrightarrow{\partial} H_{k-1}(S^{q-1} \times X) \to \cdots$$

because $V, W \simeq \star$ and $V \cap W \simeq S^{q-1}$. It follows easily that the arrow ∂ in the last long exact sequence induces an isomorphism

$$\frac{H_k(S^q \times X)}{H_k(\star \times X)} \longrightarrow \frac{H_{k-1}(S^{q-1} \times X)}{H_{k-1}(\star \times X)}.$$

If $k = m + q$ and $a \in H_m(X)$, then $\partial(z_q \times a) = \pm z_{q-1} \times a$ by proposition 15.2.1, because in the Mayer-Vietoris sequence for S^q, V, W we have $\partial(z_q) = \pm z_{q-1}$. Therefore the statement that we want to prove is true by induction on q. The induction beginning, case $q = 0$, is easy. □

Corollary 15.4.2. *For $q \geq 0$, the external product with the standard generator z_q in $\tilde{H}_q(S^q)$ determines natural isomorphisms*

$$H_{m+p}\Big(\frac{X \times S^p}{X \times \star}\Big) \longrightarrow \tilde{H}_{m+p+q}\Big(\frac{X \times S^{p+q}}{X \times \star}\Big).$$

Proof. There is a commutative triangle

$$\begin{tikzcd}
 & H_m(X) \arrow[dl, "\times z_p", "\cong"'] \arrow[dr, "\times z_{p+q}", "\cong"'] & \\
H_{m+p}\Big(\frac{X \times S^p}{X \times \star}\Big) \arrow[rr, "\times z_q"] & & \tilde{H}_{m+p+q}\Big(\frac{X \times S^{p+q}}{X \times \star}\Big).
\end{tikzcd}$$

If more clarification is needed, write

$$\frac{X \times S^p}{X \times \star} = X_+ \wedge S^p$$

where X_+ means $X \sqcup \{\infty\}$, viewed as a space with base point ∞. Then

$$\frac{X \times S^{p+q}}{X \times \star} = X_+ \wedge S^{p+q} \cong X_+ \wedge S^p \wedge S^q.$$

□

Definition 15.4.3. Let $[[g]] \in H_p(X)$ and $[[f]] \in H^q(X)$. The *cap product*

$$[[f]] \frown [[g]] \in H_{p-q}(X)$$

is the class represented by the composition

$$S^p \xrightarrow{g} X \xrightarrow{\text{diag}} X \times X \xrightarrow{\text{id}\otimes f} X \times S^q \xrightarrow{\text{quot}} \frac{X \times S^q}{X \times \star}$$

where we use lemma 15.4.1: $\tilde{H}_p\left(\frac{X \times S^q}{X \times \star}\right) \cong H_{p-q}(X)$.

Lemma 15.4.4. *The cap product is associative:* $(a \smile b) \frown c = a \frown (b \frown c)$.

Proof. Let the degrees of a and b be p and q, respectively. Denote by K_a the (grade-shifting) endomorphism of the homology of X given by cap product with a; similarly with b and $a \smile b$ instead of a. Write σ_p for external multiplication $\times z_p$; similarly with q and $p+q$. The challenge is to show

$$K_a K_b = K_{a \smile b} \ .$$

By lemma 15.4.1 and by the definition of the cap product, it is enough to show

$$\sigma_{p+q} K_a K_b = \sigma_{p+q} K_{a \smile b}.$$

Choose mapping cycles α and β representing a and b respectively. We get a commutative diagram of maps and mapping cycles

$$\begin{tikzcd}
X \arrow[r, "\mathrm{diag}"] \arrow[d, "\mathrm{diag}"] & X \times X \times X \arrow[d, "\mathrm{id}\otimes\mathrm{id}\otimes\beta"] \arrow[dr, "\mathrm{id}\otimes\alpha\otimes\beta"] & \\
X \times X \arrow[d, "\mathrm{id}\otimes\beta"] & & \\
X \times S^q \arrow[r, "\mathrm{diag}\otimes\mathrm{id}"] \arrow[d, "\mathrm{quot.}"] & X \times X \times S^q \arrow[r, "\mathrm{id}\otimes\alpha\otimes\mathrm{id}"] \arrow[d, "\mathrm{quot.}"] & X \times S^p \times S^q \arrow[d, "\mathrm{quot.}"] \\
\dfrac{X \times S^q}{X \times \star} \arrow[r, "\mathrm{diag}\otimes\mathrm{id}"] & \dfrac{X \times X \times S^q}{(u,v,\star) \sim (u',v,\star)} \arrow[r, "\mathrm{id}\otimes\alpha\otimes\mathrm{id}"] & \dfrac{X \times S^{p+q}}{X \times \star}
\end{tikzcd}$$

(The term in the middle of the lower row is a product of two factors, one of which is a copy of X while the other is a copy $(X \times S^q)/(X \times \star)$. But to make the diagram work we need to write it in this confusing way.) Going along the top from X to the bottom right term gives the mapping cycle which induces $\sigma_{p+q} K_{a \smile b}$. Going along the left-hand column gives the mapping cycle which induces $\sigma_q K_b$. Going along the bottom row we get the mapping cycle which induces $\sigma_q(\sigma_p K_a)\sigma_q^{-1}$. (Use corollary 15.4.2.) Therefore

$$\sigma_{p+q} K_{a \smile b} = (\sigma_q \sigma_p K_a \sigma_q^{-1})(\sigma_q K_b) = \sigma_{p+q} K_a K_b. \quad \square$$

The cap product leads us to the notion of *graded module* over a graded ring.

Definition 15.4.5. A *graded module* over a graded ring $R = (R_n)_{n \in \mathbb{Z}}$ is a sequence $W = (W_m)_{m \in \mathbb{Z}}$ of abelian groups W_m, together with bi-additive maps $R_n \times W_m \longrightarrow W_{m+n}$ (for which we write $(a, x) \mapsto a \cdot x$) such that the following conditions are satisfied.

- The associative law holds: $a \cdot (b \cdot x) = (a \cdot b) \cdot x$ for $a \in R_p$, $b \in R_q$ and $x \in W_m$, where p, q, m are arbitrary.
- For every $m \in \mathbb{Z}$ and $x \in W_m$ we have $1 \cdot x = x$, where $1 \in R_0$ is the multiplicative unit.

The obvious example is: $R = H^*(X)$ and $W = H_*(X)$, using the cup product as the multiplication in R and the cap product for the graded module structure on W. More precisely, let R_n be $H^{-n}(X)$ and let W_m be $H_m(X)$ and let the product $R_{-n} \times W_m \to W_{m-n}$ be the cap product,

$$H^n(X) \times H_m(X) \to H_{m-n}(X).$$

(In an earlier edition of this chapter I gave a slightly different definition of graded module which led me to define $\mathsf{R}_n = \mathsf{H}^n(\mathsf{X})$ rather than $\mathsf{R}_n = \mathsf{H}^{-n}(\mathsf{X})$. A more honest way to resolve the matter would be to write $\mathsf{H}^{-n}(\mathsf{X})$ consistently for what we have until now called $\mathsf{H}^n(\mathsf{X})$, but that would obviously cause a lot of confusion.)

CHAPTER 16

Orientations and fundamental classes of manifolds

16.1. Local homology groups and orientations of manifolds

Let X be a space and A a closed subspace. I will use the notation $X/\!\!/A$ for the mapping cone of the inclusion $A \to X$.

PROPOSITION 16.1.1. *Taking X and $A \subset X$ as above, suppose that X is a normal space. Let L be a subset of A such that the closure of L in X is contained in the interior of A. Then the inclusion*

$$(X \smallsetminus L)/\!\!/(A \smallsetminus L) \longrightarrow X/\!\!/A$$

is a homotopy equivalence.

PROOF. Exercise. □

REMARK 16.1.2. The point of this proposition is that it allows us to import some concepts from singular homology theory (definition B.1.2). Singular homology has the concept of *relative homology groups* $H_n(X, A)$ for a space X and subspace A. By construction they fit into a long exact sequence

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to H_{n-1}(X) \to \cdots$$

Moreover singular homology theory has the following theorem famously called *excision*: if $L \subset A \subset X$ are such that the closure of L in X is contained in the interior of A, then the inclusion-induced homomorphism

$$H_n(X \smallsetminus L, A \smallsetminus L) \longrightarrow H_n(X, A)$$

is an isomorphism. The long exact sequence above and the excision theorem, and homotopy invariance, are the standard tools used to calculate singular homology groups of spaces. (They are more standard than the Mayer-Vietoris sequences which I have emphasized, although the Mayer-Vietoris sequences exist, too, in singular homology.)
But it turns out that $H_n(X, A)$ is always isomorphic to $\tilde{H}_n(X/\!\!/A)$, in singular homology theory. The excision theorem can therefore be obtained as a corollary of proposition 16.1.1 if X happens to be a normal space.
Therefore we can survive rather well without homology of pairs $H_n(X, A)$ (no matter whether it is singular homology or homology based on mapping cycles) by using the reduced homology of mapping cones $\tilde{H}_n(X/\!\!/A)$ instead. Or we can *define* $H_n(X, A)$ to mean $\tilde{H}_n(X/\!\!/A)$. We have the long exact sequence of proposition 12.3.2. The same applies *mutatis mutandis* in cohomology. (There is the long exact sequence of proposition 13.3.1.)
Because of proposition 16.1.1 we get isomorphisms

$$\tilde{H}_n((X \smallsetminus L)/\!\!/(A \smallsetminus L)) \longrightarrow \tilde{H}_n(X/\!\!/A),$$

$$\tilde{H}^n((X \smallsetminus L)/\!\!/(A \smallsetminus L)) \longleftarrow \tilde{H}^n(X/\!\!/A)$$

in homology and cohomology based on mapping cycles, if X is normal. In the homology case, the hypothesis that X be normal turns out to be unnecessary, but no proof of that will be given here.

Definition 16.1.3. The local homology groups of a normal space X at a point $x \in X$ are the groups $\tilde{H}_n(X/\!\!/(X \smallsetminus \{x\}))$, for $n \in \mathbb{Z}$. (I take the liberty to write $\tilde{H}_n(X/\!\!/(X \smallsetminus x))$ in the following.)

By proposition 16.1.1, if U is any neighborhood of x in X, then the inclusion-induced homomorphism

$$\tilde{H}_n(U/\!\!/(U \smallsetminus x)) \longrightarrow \tilde{H}_n(X/\!\!/(X \smallsetminus x))$$

is an isomorphism. This is the locality property of the local homology groups.

Example 16.1.4. Let M be an n-dimensional manifold, $x \in M$. Let U be an open neighborhood of x which is homeomorphic to $\mathbb{R}^n$. Then it is easy to see that $U/\!\!/(U \smallsetminus x)$ is homotopy equivalent to a sphere S^n. In this way we get a calculation of the local homology groups of M at any point $x \in M$:

$$\tilde{H}_k(M/\!\!/(M \smallsetminus x)) \cong \begin{cases} \mathbb{Z} & \text{if } k = n \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 16.1.5. *The local homology groups* $\tilde{H}_n(M/\!\!/(M \smallsetminus x))$ *of an* n*-dimensional manifold* M *form a fiber bundle* $M_\omega \to M$ *with fibers homeomorphic to* $\mathbb{Z}$. (Each fiber is also equipped with a structure of abelian group, etc.).

Proof. (This lemma should perhaps be called a definition.) We can define the fiber bundle using fiber bundle charts. Choose a covering of M by open subsets U_α satisfying the following condition. For each α there exists an open set V_α in M which contains U_α and a homeomorphism $V_\alpha \to \mathbb{R}^n$ which takes U_α homeomorphically to the open unit ball in $\mathbb{R}^n$. Then $\tilde{H}_n(M/\!\!/(M \smallsetminus U_\alpha))$ is isomorphic to $\mathbb{Z}$. We choose specific isomorphisms from $\tilde{H}_n(M/\!\!/(M \smallsetminus U_\alpha))$ to $\mathbb{Z}$. For $x \in U_\alpha$ we have isomorphisms

$$\mathbb{Z} \cong \tilde{H}_n(M/\!\!/(M \smallsetminus U_\alpha)) \to \tilde{H}_n(M/\!\!/(M \smallsetminus x))$$

induced by the inclusion of $M \smallsetminus U_\alpha$ in $M \smallsetminus x$. As x runs through U_α, this gives us a bundle chart. In other words, it allows us to make a bijection φ_α from the disjoint union of the local homology groups $\tilde{H}_n(M/\!\!/(M \smallsetminus x))$ for $x \in U_\alpha$ to a product

$$U_\alpha \times \mathbb{Z}.$$

We need to show that the changes-of-charts $(\varphi_\beta)^{-1}\varphi_\alpha$ are continuous at every $x \in U_\alpha \cap U_\beta$. For that, choose an open subset $W \subset U_\alpha \cap U_\beta$ containing x with the usual good properties. (Namely, there exist another open subset W' of M and a homeomorphism $W' \to \mathbb{R}^n$ which takes W homeomorphically to the open unit ball.) Then the homomorphisms in the diagram

$$\begin{tikzcd} \tilde{H}_n(M/\!\!/(M \smallsetminus U_\alpha)) \arrow[r] & \tilde{H}_n(M/\!\!/(M \smallsetminus W)) \arrow[d] & \tilde{H}_n(M/\!\!/(M \smallsetminus U_\beta)) \arrow[l] \\ & \tilde{H}_n(M/\!\!/(M \smallsetminus x)) & \end{tikzcd}$$

are all isomorphisms, for $x \in W$. It follows that the $\mathbb{Z}$-coordinate of $(\varphi_\beta)^{-1}\varphi_\alpha$ is constant on W.

Definition 16.1.6. An *orientation* of an n-dimensional manifold M at a point $x \in M$ is a choice of generator of the local homology group $\tilde{H}_n(M/\!\!/(M \smallsetminus x))$ (which is infinite cyclic). An *orientation* of an n-dimensional manifold M is a choice of $s(x) \in \tilde{H}_n(M/\!\!/(M \smallsetminus x))$, for every $x \in M$, such that $s(x)$ is a generator of $\tilde{H}_n(M/\!\!/(M \smallsetminus x))$ and the map $x \mapsto s(x)$ from M to M_ω is continuous. (In that case $s\colon M \to M_\omega$ is a continuous section of the fiber bundle $M_\omega \to M$.)
The manifold M is said to be *orientable* if it admits an orientation.

Remark 16.1.7. An orientation s of M gives rise to a homeomorphism

$$M \times \mathbb{Z} \to M_\omega$$

given by $(x, z) \mapsto z \cdot s(x) \in \tilde{H}_n(M/\!\!/(M \smallsetminus x))$. That amounts to a trivialization of the fiber bundle $M_\omega \to M$, respecting the abelian group structure. Conversely, etc.
Instead of focusing on the fiber bundle $M_\omega \to M$ we can also focus on the sub-bundle $M_\omega^\times \to M$ which selects the two generators in each fiber (where the fiber is viewed as an infinite cyclic group). This is then a fiber bundle on M where each fiber has exactly two elements, in other words, a two-sheeted covering. An orientation of M can also be defined as a section of

$$M_\omega^\times \to M\ .$$

16.2. Fundamental classes

Definition 16.2.1. A *fundamental class* for an n-dimensional manifold M is an element $z \in H_n(M)$ such that, for each $x \in M$, the image of z under the inclusion-induced homomorphism

$$H_n(M) \longrightarrow \tilde{H}_n(M/\!\!/(M \smallsetminus x))$$

is a generator of the infinite cyclic group $\tilde{H}_n(M/\!\!/(M \smallsetminus x))$.
More generally, for an open subset U of M, an element $z \in \tilde{H}_n(M/\!\!/U)$ will be called a *fundamental class relative to* U if, for each $x \in M \smallsetminus U$, the image of z in each local homology group $\tilde{H}_n(M/\!\!/(M \smallsetminus x))$ is a generator of $\tilde{H}_n(M/\!\!/(M \smallsetminus x))$.

Remark 16.2.2. A fundamental class z for M defines an orientation s of M by $s(x) =$ image of z in $\tilde{H}_n(M/\!\!/(M \smallsetminus x))$. Therefore a fundamental class can only exist of M is orientable.
Suppose that M is not compact; then there is no fundamental class. Indeed, we know that for any mapping cycle $f\colon S^n \to M$ representing a class in $H_n(M)$, there exists a compact subset $K \subset M$ such that f can be viewed as a mapping cycle from S^n to K. Then, for $x \in M \smallsetminus K$, the image of the class of f in $\tilde{H}_n(M/\!\!/(M \smallsetminus x))$ is zero (therefore not a generator) because K is contained in $M \smallsetminus x$.
By a similar argument, if U is an open subset of M whose complement is noncompact, then there cannot be a fundamental class relative to U.

Let $U \subset M$ be open with complement A. We write $M_\omega|_A \to A$ for the restriction of the fiber bundle $M_\omega \to M$ to A. Let $\Gamma(M_\omega|_A \to A)$ be the set of *continuous* sections of $M_\omega|_A \to A$ (maps from A to $M_\omega|_A$ which are right inverse to the bundle projection). It is an abelian group by pointwise addition. Briefly, $\Gamma(M_\omega|_A \to A)$ is the set of functions s which for every $x \in A$ select continuously

$$s(x) \in \tilde{H}_n(M/\!\!/(M \smallsetminus x)).$$

A homomorphism Φ_A of abelian groups from $\tilde{H}_n(M/\!\!/U)$ to $\Gamma(M_\omega|_A \to A)$ is defined by

$$z \mapsto \big(A \ni x \mapsto \text{image of } z \text{ in } \tilde{H}_n(M/\!\!/(M \smallsetminus x))\big).$$

THEOREM 16.2.3. *Suppose that* A *is compact. Then this homomorphism*

$$\Phi_A \colon \tilde{H}_n(M/\!\!/U) \longrightarrow \Gamma(M_\omega|_A \to A)$$

is an isomorphism. Moreover $\tilde{H}_k(M/\!\!/U)$ *is zero for* $k > n$.

PROOF. We are going to prove this by a process reminiscent of induction. There are two "induction beginnings" like this:

(i) Both statements hold for A if A is a point or if $A = \emptyset$.
(ii) Both statements hold for A if there exist a neighborhood V of A in M and a homeomorphism $h \colon V \to \mathbb{R}^n$ taking A to the cube $[0,1]^n$.

There are two types of "induction steps" as follows.

(iii) If the two statements hold for $A = A_1$ and $A = A_2$ and $A = A_1 \cap A_2$, then they hold for $A = A_1 \cup A_2$.
(iv) Suppose that $A_0 \supset A_1 \supset A_2 \supset \cdots$ is a descending sequence of compact subsets of M. If the two statements hold for $A = A_i$, where $i = 0, 1, 2, \ldots$, then they hold for $A = \bigcap_i A_i$.

Proof of (i): clear. Proof of (ii): choose $x \in A$. In the commutative square

$$\begin{tikzcd}
\tilde{H}_n(M/\!\!/(M \smallsetminus A)) \arrow[r, "(\text{incl.})_*"] \arrow[d, "\Phi_A"] & \tilde{H}_n(M/\!\!/(M \smallsetminus x)) \arrow[d, "\Phi_x"] \\
\Gamma(M_\omega|_A \to A) \arrow[r, "\text{restr.}"] & \Gamma(M_\omega|_x \to x)
\end{tikzcd}$$

the upper horizontal arrow is an isomorphism by inspection. The lower horizontal arrow is an isomorphism because A is contractible (which implies that any fiber bundle over A is a trivial fiber bundle, here: isomorphic to the projection $A \times \mathbb{Z} \to A$). The right-hand vertical arrow is an isomorphism by (i). Therefore the left-hand vertical arrow is an isomorphism. For $k > n$, we also have isomorphisms

$$\tilde{H}_k(M/\!\!/(M \smallsetminus A)) \cong \tilde{H}_k(M/\!\!/(M \smallsetminus x)) = 0.$$

Proof of (iii): Let $X_1 = M/\!\!/(M \smallsetminus A_1)$ and $X_2 = M/\!\!/(M \smallsetminus A_2)$, so that $X_1 \cup X_2 = M/\!\!/(M \smallsetminus (A_1 \cap A_2))$ and $X_1 \cap X_2 = M/\!\!/(M \smallsetminus (A_1 \cup A_2))$. It is not quite true that X_1 and X_2 are open in $X_1 \cup X_2$, but nevertheless there is a long exact Mayer-Vietoris sequence

$$\cdots \to H_k(X_1 \cap X_2) \to H_k(X_1) \oplus H_k(X_2) \to H_k(X_1 \cup X_2) \to H_{k-1}(X_1 \cap X_2) \to \cdots$$

Reason: it is easy to find open neighborhoods Y_1 and Y_2 of X_1 and X_2 respectively in $X_1 \cup X_2$ such that the inclusions $X_1 \to Y_1$ and $X_2 \to Y_2$ and $X_1 \cup X_2 \to Y_1 \cup Y_2$ are homotopy equivalences. (Let Y_1 be the union of X_1 and a standard neighborhood of the cone tip in $X_1 \cup X_2$, consisting of all points represented by pairs (t, x) where $t > 1/2$; remember that the cone tip corresponds to $t = 1$. See remark 16.2.5 for details.) Since $H_k(X_1$, $H_k(X_2)$ and $H_k(X_1 \cup X_2)$ are zero by assumption if $k > n$, exactness of the sequence implies that $H_k(X_1 \cap X_2)$ is zero for all $k > n$. For $k = n$ we can extract an exact subsequence

$$0 \longrightarrow H_n(X_1 \cap X_2) \longrightarrow H_n(X_1) \oplus H_n(X_2) \longrightarrow H_n(X_1 \cup X_2)$$

which easily implies an exact sequence

$$0 \longrightarrow \tilde{H}_n(X_1 \cap X_2) \longrightarrow \tilde{H}_n(X_1) \oplus \tilde{H}_n(X_2) \longrightarrow \tilde{H}_n(X_1 \cup X_2).$$

That exact sequence is part of a commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \tilde{H}_n(X_1 \cap X_2) & \longrightarrow & \tilde{H}_n(X_1) \oplus \tilde{H}_n(X_2) & \longrightarrow & \tilde{H}_n(X_1 \cup X_2) \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \to & \Gamma(M_\omega|_{A_1 \cup A_2} \to A_1 \cup A_2) \to & & \begin{array}{c}\Gamma(M_\omega|_{A_1} \to A_1) \\ \oplus \Gamma(M_\omega|_{A_2} \to A_2)\end{array} & \to & \Gamma(M_\omega|_{A_1 \cap A_2} \to A_1 \cap A_2)
\end{array}$$

where the vertical arrows are given by $\Phi_{A_1 \cup A_2}$, $\Phi_{A_1} \oplus \Phi_{A_2}$ and $\Phi_{A_1 \cap A_2}$. The lower row of this diagram is clearly also exact. (If continuous sections of $M_\omega \to M$ are specified over A_1 and A_2, and they agree over $A_1 \cap A_2$, then they glue uniquely to a continuous section over $A_1 \cup A_2$.) Therefore, since the middle and right-hand vertical arrows are isomorphisms by assumption, the left-hand vertical arrow is an isomorphism.

Proof of (iv). Let $X_i = M/\!\!/(M \smallsetminus A_i)$ and $X_\infty = M/\!\!/(M \smallsetminus \bigcap_i A_i) = \bigcup_i X_i$. It is unfortunate that the X_i are not open in X_∞ , but as in the proof of (iii) it is easy to construct open neighborhoods Y_i of X_i in X_∞ such that the inclusions $X_i \to Y_i$ are homotopy equivalences. We also require $Y_i \subset Y_{i+1}$ for all i. Now we can say that X_∞ is the monotone union of the open subset Y_i for $i = 0, 1, 2, \ldots$. By lemma 12.4.5, the inclusions $Y_i \to X_\infty$ induce an isomorphism

$$\operatorname{colim}_{i \geq 0} \tilde{H}_k(Y_i) \longrightarrow \tilde{H}_k(X_\infty) \ .$$

(This may look like undefined notation or vocabulary. The symbol colim means *direct limit*, a notion from category theory. Here is a translation. It follows from 12.4.5. that every element of $\tilde{H}_k(X_\infty)$ comes from $\tilde{H}_k(Y_i)$ for some i, and that if two elements of $\tilde{H}_k(Y_i)$ have the same image in $\tilde{H}_k(X_\infty)$, then there is $j > i$ such that they already have the same image in $\tilde{H}_k(Y_\ell)$. See also remark **??** for more details.) Therefore we can also say that the inclusions $X_i \to X_\infty$ induce an isomorphism

$$\operatorname{colim}_{i \geq 0} \tilde{H}_k(X_i) \longrightarrow \tilde{H}_k(X_\infty) \ .$$

In the case where $k > n$, this implies that $\tilde{H}_k(X_\infty) = 0$ because $\tilde{H}_k(X_i) = 0$ for $i = 0, 1, 2, \ldots$ by assumption. In the case $k = n$ we note that restriction of sections from A_i to $A_\infty = \bigcap_i A_i$ leads to an isomorphism

$$\operatorname{colim}_{i \geq 0} \Gamma(M_\omega|_{A_i} \to A_i) \longrightarrow \Gamma(M_\omega|_{A_\infty} \to A_\infty).$$

(Translation: it is claimed that every section of $M_\omega \to M$ over A_∞ can be extended to a section over A_i for some i, and any two such extensions to A_i agree on A_j for some $j > i$. See also remark rem-detfight for more details.) Therefore we can complete the argument using the commutative diagram

$$\begin{array}{ccc}
\operatorname{colim}_{i \geq 0} \tilde{H}_n(X_i) & \xrightarrow{\cong} & \tilde{H}_n(X_\infty) \\
\downarrow \cong & & \downarrow \\
\operatorname{colim}_{i \geq 0} \Gamma(M_\omega|_{A_i} \to A_i) & \xrightarrow{\cong} & \Gamma(M_\omega|_{A_\infty} \to A_\infty).
\end{array}$$

Now all the tools are in place and we can get the induction machinery going. In case we need this again, here is an abstract formulation. Suppose that $\mathcal{K}$ is a collection of compact subsets of M which satisfies the following conditions.

(a) $\emptyset \in \mathcal{K}$.
(b) If A is a compact subset of M and there exist an open neighborhood V of A and a homeomorphism $V \to \mathbb{R}^n$ taking A to the cube $[0,1]^n$, then $A \in \mathcal{K}$.
(c) If $(A_i)_{i=0,1,\dots}$ is a descending sequence of compact subsets of M such that $A_i \in \mathcal{K}$ for all i, then $\bigcap_i A_i \in \mathcal{K}$.
(d) If $A_1 \in \mathcal{K}$ and $A_2 \in \mathcal{K}$ and $A_1 \cap A_2 \in \mathcal{K}$, then $A_1 \cup A_2 \in \mathcal{K}$.

Then $\mathcal{K}$ is the collection of all compact subsets of M. I leave this as an exercise. □

Corollary 16.2.4. *If M is a compact n-manifold (without boundary), then*

(i) $H_n(M)$ *is isomorphic to the abelian group of sections of* $M_\omega \to M$*;*
(ii) *there is a bijection between the set of fundamental classes for M and the set of orientations of M.*

If M is also connected, then $H_n(M) \cong \mathbb{Z}$ (in the orientable case), or $H_n(M) = 0$ (in the non-orientable case).

Proof. Statements (i) and (ii) follow directly from the special case $A = M$ of theorem 16.2.3. Now suppose that M is connected. If M is orientable, then the bundle $M_\omega \to M$ is isomorphic to a trivial bundle $M \times \mathbb{Z} \to M$. Therefore sections of it correspond to (continuous) maps from M to $\mathbb{Z}$. Such maps are constant. Therefore $H_n(M) \cong \mathbb{Z}$ by theorem 16.2.3. — For the converse, we note that every fiber of $M_\omega \to M$ is an abelian group isomorphic to $\mathbb{Z}$, and although we have a choice of two isomorphisms, the two differ only by a sign. So the absolute value of an element in any fiber of $M_\omega \to M$ is a well-defined non-negative integer. If M is connected and $H_n(M) \neq 0$, then by theorem 16.2.3, the fiber bundle $M_\omega \to M$ has a section $\sigma\colon M \to M_\omega$ such that $\sigma(x_0) \neq 0 \in \tilde{H}_n(M/\!\!/(M \smallsetminus x_0))$ for some $x_0 \in M$. Then $|\sigma(x_0)| > 0$ and we have $|\sigma(x)| = |\sigma(x_0)| > 0$ for all $x \in M$, by continuity. Divide σ by the number $|\sigma(x_0)|$ to obtain a continuous section of the fiber bundle $M_\omega \to M$ which qualifies as an orientation. Therefore M is orientable. □

Remark 16.2.5. In this remark, some (I hope all) of the missing details in the proof of theorem 16.2.3 are supplied.
(a) Let U be an open subset of a space X. Then $X/\!\!/U = \mathrm{cone}(U \to X)$ is a subset of $X/\!\!/X = \mathrm{cone}(X)$. It will hardly ever be open in $X/\!\!/X$. But let $W \subset X/\!\!/X$ consist of all points represented by pairs (t,x) where $t > 1/2$. (The cone tip corresponds to $t = 1$.) Then $(X/\!\!/U) \cup W$ is open in $X/\!\!/X$ and the inclusion

$$e\colon X/\!\!/U \longrightarrow (X/\!\!/U) \cup W$$

is a homotopy equivalence. Therefore $(X/\!\!/U) \cup W$ is a good substitute for $X/\!\!/U$ in many cases. Here is a proof of the claim that e is a homotopy equivalence. Choose a monotone continuous function $\psi\colon [0,1] \to [0,1]$ which has $\psi(0) = 0$ and $\psi(t) = 1$ for $t \geq 1/2$. Define $g\colon X/\!\!/X \to X/\!\!/X$ by $(t,x) \mapsto (\psi(t),x)$. The map g is homotopic to the identity by the obvious homotopy $h_s(t,x) := (st+(1-t)\psi(x),x)$ where $s \in [0,1]$. The map g restricts to a map $g_0\colon (X/\!\!/U) \cup W \to X/\!\!/U$. The homotopy $(h_s)_{s\in[0,1]}$ restricts to a homotopy from $g_0 e$ to the identity on $X/\!\!/U$. Similarly, $(h_{1-s})_{s\in[0,1]}$ restricts to a homotopy from $e g_0$ to the identity.

(b) For a diagram of abelian groups and homomorphisms

$$B_0 \xrightarrow{f_0} B_1 \xrightarrow{f_1} B_2 \xrightarrow{f_2} B_3 \xrightarrow{f_3} \cdots$$

the colimit of the diagram is an abelian group B_∞ defined as follows. The elements of B_∞ are equivalence classes of pairs (i, x) where $i \in \mathbb{N}$ and $x \in B_i$. Two such pairs (i, x) and (j, y) are considered equivalent if for some $k \geq i, j$ the images of $x \in B_i$ and $y \in B_j$ in B_k are equal. Addition of equivalence classes is defined as follows. Given representatives (i, x) and (j, y) choose any $k \geq j, i$ and let x', y' be the images of $x \in B_i$ and $y \in B_j$, respectively, in B_k. Define $[(i, x)] + [(j, y)] = [(k, x' + y')]$. This construction leads to a commutative diagram of abelian groups and homorphisms

$$\begin{tikzcd}
B_0 \arrow[r, "f_0"] \arrow[drrrr] & B_1 \arrow[r, "f_1"] \arrow[drrr] & B_2 \arrow[r, "f_2"] \arrow[drr] & B_3 \arrow[r, "f_3"] \arrow[dr] & \cdots \\
& & & & B_\infty
\end{tikzcd}$$

where the homomorphism from B_i to B_∞ takes an element x of B_i to the equivalence class of the pair (i, x).

(c) Let $p\colon E \to B$ be a fiber bundle with discrete fibers (also known as *covering map*). Let $A_0 \supset A_1 \supset A_2 \supset A_3 \supset \dots$ be a descending sequence of compact subsets of B and put $A_\infty = \bigcap_{i \geq 0} A_i$. Write

$$p_i\colon E|_{A_i} \to A_i$$

for the restricted fiber bundle (where we allow $i \in \mathbb{N}$ but also $i = \infty$). We need to show: any continuous section of p_∞ can be extended to a continuous section of p_i for some $i \in \mathbb{N}$; and if two sections of p_i determine the same section of p_∞ by restriction, then they determine the same section of p_j for some $j \geq i$, by restriction. Let us start with a continuous section σ of p_∞. Because the fiber bundle is locally trivial and because a continuous map from any space to a discrete space is locally constant, it is easy to find (finitely many) open subsets $U_1, \dots U_r$ of B such that their union contains A_∞ and such that σ restricted to $A_\infty \cap U_s$ extends to a continuous section τ_s on all of U_s, for $s = 1, 2, \dots, k$. By the same reasoning, the subset V of $\bigcup_{s=1}^r U_s$ consisting of all x where $\tau_s(x)$ is independent of s if it is defined is open in B. Therefore we have found an open neighborhood V of A_∞ and an extension of σ to a section of $p_V\colon E|_V \to V$. One of the A_i must be contained in V, otherwise we have a strange open covering of the compact set A_0 by $A_0 \cap V$ and the sets $A_0 \smallsetminus A_i$ where $i \geq 1$. Therefore we have extended σ to a section of p_i. Next, if we have two extensions of σ to sections ρ_1 and ρ_2 of p_i, then the subset W of A_i where they agree is open in A_i (by the *locally constant* argument) and contains A_∞. Therefore there exists $j \geq i$ such that $A_j \subset W$, and this implies that ρ_1 and ρ_2 agree on A_j. $\square$

CHAPTER 17

Poincaré duality

17.1. The duality statement

The goal of the chapter is to prove the following.

THEOREM 17.1.1. *Let M be an oriented compact n-dimensional manifold (without boundary). Let $\varphi \in H_n(M)$ be the fundamental class. Then for every $k \in \mathbb{Z}$ the cap product with φ is an isomorphism*

$$H^k(M) \longrightarrow H_{n-k}(M) \ ; \ a \mapsto a \frown \varphi \ .$$

This is called Poincaré duality. Comment: recall cor. 16.2.4 in cumulative lecture notes. Short summary: in the previous section we introduced the fiber bundle $M_\omega \to M$ such that the fiber over $x \in M$ is the local homology group $\tilde{H}_n(M/\!\!/(M \smallsetminus x))$. An orientation of M is a continuous section s of that such that $s(x) \in \tilde{H}_n(M/\!\!/(M \smallsetminus x))$ is a generator of that local homology group, for every $x \in M$. Then we showed that there is a unique $\varphi \in H_n(M)$ such that the image of φ in $\tilde{H}_n(M/\!\!/(M \smallsetminus x))$ agrees with $s(x)$, for every $x \in M$.

In order to prove this theorem by some kind of induction (similar to the induction seen in the previous chapter) we need to formulate a stronger statement. In order to formulate a stronger statement we need a stronger form of cap product. It is a good opportunity to introduce some refinements of cap product and cup product.

17.2. Various refinements of cup product and cap product

REMARK 17.2.1. Let X be a normal space with a closed subset A. For $a \in \tilde{H}^m(X/A)$ and $b \in H^n(X \smallsetminus A)$, the product $a \smile b \in \tilde{H}^{m+n}(X/A)$ is defined.
Idea/proof/definition: it is easy to reduce to the case where A is a point, denoted $\star$. Represent a by a mapping cycle $\alpha\colon X \to S^m$ which is zero in an open neighborhood U of $\star$. This is possible by prop 14.3.3. cumulative lecture notes. Represent b by a mapping cycle $\beta\colon X \smallsetminus \star \to S^n$. The composition

$$X \smallsetminus \star \xrightarrow{\text{diag}} X \times (X \smallsetminus \star) \xrightarrow{\alpha \otimes \beta} S^m \times S^n \xrightarrow{\mu_{m,n}} S^{m+n}$$

is a mapping cycle which is $\equiv 0$ on $U \smallsetminus \star$ and which can therefore be extended to a mapping cycle on all of X (which is zero on all of U). $\square$

REMARK 17.2.2. Let X be a normal space with a closed subset A. For $a \in \tilde{H}^q(X/A)$ and $b \in H_p(X)$, the cap product $a \frown b \in H_{p-q}(X \smallsetminus A)$ is defined.
Idea/proof/definition: Let a be represented by a mapping cycle α from X/A to S^q and let b be represented by a mapping cycle β from S^p to X. By proposition 14.3.3., cumulative

lecture notes, we can assume that $\alpha \equiv 0$ in an open neighborhood U of A. Now $a \frown b$ as an element of $H_{p-q}(X)$ was defined to be the class represented by the composition

$$S^p \xrightarrow{\beta} X \xrightarrow{\text{diag}} X \times X \xrightarrow{\text{id}\otimes\alpha} X \times S^q \xrightarrow{\text{quot}} \frac{X \times S^q}{X \times \star}$$

where we use the isomorphism

$$\tilde{H}_p\left(\frac{X \times S^q}{X \times \star}\right) \cong H_{p-q}(X)$$

given by external product with $z_q \in H_q(S^q)$. In the above composition, the (sub)composition

$$X \xrightarrow{\text{diag}} X \times X \xrightarrow{\text{id}\otimes\alpha} X \times S^q$$

is a mapping cycle which factors through $(X \smallsetminus A) \times S^q \subset X \times S^q$. (In case you don't believe it, here is an argument. Suppose that the germ of the mapping cycle α at some $x \in X$ is

$$\sum_i b_i \cdot \alpha_{i,x}$$

where $b_i \in \mathbb{Z}$ and $\alpha_{i,x}\colon (X, x) \to S^q$ is a continuous function germ. Then the composition $X \to X \times X \to X \times S^q$ above has the following germ at x:

$$\lambda \;=\; \sum_i b_i \cdot \Big(y \mapsto (y, \alpha_{i,x}(y)) \in X \times S^q\Big)$$

where y is a variable in some small neighborhood of $x \in X$. If $x \in U$, then we know that already $\sum_i b_i \cdot \alpha_{i,x} \equiv 0$ and we get $\lambda = 0$. If $x \notin U$, then we can assume $y \notin A$ since $X \smallsetminus A$ is a neighborhood of x, and so λ as a germ certainly lands in $(X \smallsetminus A) \times S^q$.)

Remark 17.2.3. The refined cap/cup products in the previous remarks satisfy an associativity formula, as follows. Let X be a normal space with closed subset A. Let $b \in \tilde{H}^q(X/A)$ and $a \in H^r(X \smallsetminus A)$ and $c \in H_p(X)$, so that $b \frown c \in H_{p-q}(X \smallsetminus A)$ and $a \frown (b \frown c) \in H_{p-q-r}(X \smallsetminus A)$ and $a \smile b \in H^{q+r}(X/A)$ and $(a \smile b) \frown c \in H_{p-q-r}(X \smallsetminus A)$. Then

$$a \frown (b \frown c) = (a \smile b) \frown c \in H_{p-q-r}(X \smallsetminus A).$$

Remark 17.2.4. The refined cap product in the previous remarks satisfies a complicated naturality formula as follows. Let $f\colon X \to Y$ be a map, $B \subset Y$ closed, $A := f^{-1}(B)$. Let $a \in H^q(Y/B)$ and $b \in H_p(X)$, so that we have $f_*(b) \in H_p(Y)$ and $f^*(a) \in H^q(X/A)$. Then

$$a \frown f_*(b) = f_*(f^*(a) \frown b) \;\in H_{p-q}(Y \smallsetminus B)\;.$$

Example 17.2.5. Let X be a normal space with two open subsets V and W such that $X = V \cup W$. Then $X \smallsetminus V$ and $X \smallsetminus W$ are disjoint closed subsets of X and we can find a continuous function $\psi\colon X \to [0,1]$ such that $\psi \equiv 0$ on $X \smallsetminus W$ and $\psi \equiv 1$ on $X \smallsetminus V$. This induces a map

$$f_{V,W}\colon X/A \to [0,1]/\{0,1\} \cong S^1$$

where $A = X \smallsetminus (V \cap W)$. In $\tilde{H}^1(S^1) \cong \mathbb{Z}$ we choose the standard generator $[[\text{id}]]$ and we form

$$\eta_{V,W} = f^*_{V,W}[[\text{id}]] \in \tilde{H}^1(X/A).$$

Now the refined cap product with $\eta_{V,W}$ is a homomorphism from $H_k(X)$ to $H_{k-1}(X \smallsetminus A) = H_{k-1}(V \cap W)$. Similarly the refined cup product with $\eta_{V,W}$ is a homomorphism from $H^k(X \smallsetminus A) = H^k(V \cap W)$ to $\tilde{H}^{k+1}(X/A)$ which we can compose with the homomorphism

$$\tilde{H}^{k+1}(X/A) \to H^{k+1}(X)$$

induced by the quotient map $X \to X/A$. Claim: Up to sign ± 1, these two homomorphisms (from $H_k(X)$ to $H_{k-1}(V \cap W)$ and from $H^k(V \cap W)$ to $H^{k+1}(X)$) are the boundary operators from the Mayer-Vietoris sequences associated with $X = V \cup W$. (In the cohomology case, we should assume that X, V, W and $V \cap W$ are paracompact in order to have a Mayer-Vietoris sequence.) *Proof still under construction.*

17.3. A stronger duality statement

THEOREM 17.3.1. *Let M be an oriented compact n-dimensional manifold (without boundary) and let $\varphi \in H_n(M)$ be the fundamental class. Then for every closed subset A of M and every $k \in \mathbb{Z}$ the cap product with φ is an isomorphism*

$$\tilde{H}^k(M/A) \longrightarrow H_{n-k}(M \smallsetminus A) \ ; \ a \mapsto a \frown \varphi \ .$$

Note that the special case $A = \emptyset$ of theorem 17.3.1 is theorem 17.1.1. — Now it is easy to imagine how we can prove this by some form of induction. Let $\mathcal{K}$ be the collection of all closed subsets A of M such that the cap product with φ is an isomorphism

$$\tilde{H}^k(M/A) \longrightarrow H_{n-k}(M \smallsetminus A)$$

for all $k \in \mathbb{Z}$. We ought to show the following.

(a) $M \in \mathcal{K}$.
(b) If A is a closed subset of M and $M \smallsetminus A$ is homeomorphic to $\mathbb{R}^n$, then $A \in \mathcal{K}$.
(c) If $(A_i)_{i=0,1,\ldots}$ is a descending sequence of closed subsets of M such that $A_i \in \mathcal{K}$ for all i, then $\bigcap_i A_i \in \mathcal{K}$.
(d) If $A_1 \in \mathcal{K}$ and $A_2 \in \mathcal{K}$ and $A_1 \cup A_2 \in \mathcal{K}$, then $A_1 \cap A_2 \in \mathcal{K}$.

Then it is an exercise (as usual) to show that $\mathcal{K}$ is the set of *all* closed subsets of M. Note that the induction scheme is *downwards*: the easiest case is $A = M$ and the case that we want most is $A = \emptyset$.

17.4. Yet another Mayer-Vietoris sequence

Proving (a),(b) and (c) does not require any special ideas or tools but (d) is interesting. Let $V_1 = M \smallsetminus A_1$ and $V_2 = M \smallsetminus A_2$. Then we have a long exact Mayer-Vietoris sequence relating the homology groups of V_1, V_2, $V_1 \cup V_2$ and $V_1 \cap V_2$. It seems therefore that we also need a long exact Mayer-Vietoris sequence relating the reduced *cohomology* groups of M/A_1, M/A_2, $M/(A_1 \cup A_2)$ and $M/(A_1 \cap A_2)$.

LEMMA 17.4.1. *Let X be a compact metrizable space with closed subsets A_1 and A_2. There is a long exact Mayer-Vietoris sequence relating the reduced cohomology groups of X/A_1, X/A_2, $X/(A_1 \cup A_2)$ and $X/(A_1 \cap A_2)$.*

Proof. Choose a metric on X. Write $A_1 = \bigcap_{i=0}^{\infty} V_i$ where $V_i \subset X$ consists of all $x \in X$ whose distance from A_1 is $< 2^{-i}$. Then we can make a commutative diagram

$$\begin{tikzcd}
\cdots & \tilde{H}^k(X/\!\!/V_4) \arrow[l] \arrow[dr] & \tilde{H}^k(X/\!\!/V_3) \arrow[l] \arrow[dr] & \tilde{H}^k(X/\!\!/V_2) \arrow[l] \arrow[d] & \tilde{H}^k(X/\!\!/V_1) \arrow[l] \arrow[dl] & \tilde{H}^k(X/\!\!/V_0) \arrow[l] \arrow[dll] \\
 & & & \tilde{H}^k(X/A_1) & &
\end{tikzcd}$$

using maps $X/A_i \to X/\!\!/V_i$ as in proposition 14.3.3. More precisely, we choose a continuous function $\psi_i \colon X \to [0,1]$ which is $\equiv 1$ on A_1 and $\equiv 0$ outside V_i and obtain a continuous map $X \to X/\!\!/V_i$ which is given by $x \mapsto x \in X \subset X/\!\!/V_i$ for x outside V_i and by $x \mapsto (\psi_i(x), x) \in V_i/\!\!/V_i \subset X/\!\!/V_i$ if $x \in V_i$. This continuous map takes A_1 to the base point of $X/\!\!/V_i$ and so can be thought of as a map from X/A_i to $X/\!\!/V_i$. It is easy to show that for each i, the diagram

$$\begin{tikzcd}
X/\!\!/V_{i+1} \arrow[r, "\text{incl.}"] & X/\!\!/V_i \\
X/A_1 \arrow[u] \arrow[ur] &
\end{tikzcd}$$

commutes up to homotopy. Therefore we obtain a commutative diagram of reduced cohomology groups as above. Referring to this diagram of reduced cohomology groups, we can say that

$$\tilde{H}^k(X/A_1) \cong \operatorname{colim}_i \tilde{H}^k(X/\!\!/V_i).$$

In down-to earth language, this means the following. Every element of $\tilde{H}^k(X/A_1)$ is the image of some element in $\tilde{H}^k(X/\!\!/V_i)$ under the homomorphism

$$\tilde{H}^k(X/\!\!/V_i) \longrightarrow \tilde{H}^k(X/A_1)$$

above, for some i; and if two elements of $\tilde{H}^k(X/\!\!/V_i)$ have the same image under that same homomorphism, then then there exists $j \geq i$ such that they already have the same image under

$$\tilde{H}^k(X/\!\!/V_i) \longrightarrow \tilde{H}^k(X/\!\!/V_j).$$

The argument for that was given in (the proof of) proposition 14.3.3. Note that it does not matter much here whether we work with X and closed subset A_1 and a neighborhood of that, or with the based space X/A_1 and the closed subset $\{\star\}$ and a neighborhood of the base point.

Similarly we write $A_2 = \bigcap_{i=0}^{\infty} W_i$ where $W_i \subset X$ consists of all $x \in X$ whose distance from A_2 is $< 2^{-i}$. Then there are is an isomorphism

$$\tilde{H}^k(X/A_2) \cong \operatorname{colim}_i \tilde{H}^k(X/\!\!/V_2)$$

and there are isomorphisms

$$\tilde{H}^k(X/(A_1 \cap A_2)) \cong \operatorname{colim}_i \tilde{H}^k(X/\!\!/(V_i \cap W_i))$$
$$\tilde{H}^k(X/(A_1 \cup A_2)) \cong \operatorname{colim}_i \tilde{H}^k(X/\!\!/(V_i \cup W_i)).$$

Therefore it suffices to show that for each fixed i the reduced cohomology groups of $H^k(X/\!\!/V_i)$, $H^k(X/\!\!/W_i)$, $H^k(X/\!\!/(V_i \cap W_i))$ and $H^k(X/\!\!/(V_i \cup W_i))$ are related by a long exact Mayer-Vietoris sequence (and that there is enough compatibility as i varies). This would be immediately clear if we could say that $X/\!\!/V_i$ and $X/\!\!/W_i$ are open subsets of

$X/\!\!/(V_i \cup W_i)$ with intersection $X/\!\!/(V_i \cap W_i)$. The problem is that $X/\!\!/V_i$ and $X/\!\!/W_i$ are not quite open in $X/\!\!/(V_i \cup W_i)$ because of a problem at the cone tip. But we know from section 16 cumulative lecture notes (proof of thm 16.2.3.(iii)) how to solve this: $X/\!\!/V_i$ has an open neighborhood Y_1 and $X/\!\!/W_i$ has an open neighborhood Y_2 such that the inclusions

$$X/\!\!/V_i \to Y_1 \ , \quad X/\!\!/W_i \to Y_2 \ , \quad X/\!\!/(V_1 \cap V_2) \to Y_1 \cap Y_2 \ , \quad X/\!\!/(V_1 \cup V_2) \to Y_1 \cup Y_2$$

are homotopy equivalences. □

17.5. The Mayer-Vietoris-plus-five-lemma argument

Here we do the most interesting type of induction step in the induction scheme: the one with label (d). Recall that $\mathcal{K}$ is the set of all closed subsets of M such that the cap product with φ is an isomorphism

$$\tilde{H}^k(M/A) \longrightarrow H_{n-k}(M \smallsetminus A)$$

for all $k \in \mathbb{Z}$.

Lemma 17.5.1. *If $A_1 \in \mathcal{K}$ and $A_2 \in \mathcal{K}$ and $A_1 \cup A_2 \in \mathcal{K}$, then $A_1 \cap A_2 \in \mathcal{K}$.*

Proof. Write $V_1 = M \smallsetminus A_1$ and $V_2 = M \smallsetminus A_2$. We have a diagram where the columns are long exact Mayer-Vietoris sequences:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\tilde{H}^k(M/(A_1 \cup A_2)) \arrow[r] \arrow[d] & H_{n-k}(V_1 \cap V_2) \arrow[d] \\
\tilde{H}^k(M/A_1) \oplus \tilde{H}^k(M/A_2) \arrow[r] \arrow[d] & H_{n-k}(V_1) \oplus H_{n-k}(V_2) \arrow[d] \\
\tilde{H}^k(M/(A_1 \cap A_2)) \arrow[r, "?"] \arrow[d, "\delta"] & H_{n-k}(V_1 \cup V_2) \arrow[d, "\partial"] \\
\tilde{H}^{k+1}(M/(A_1 \cup A_2)) \arrow[r] \arrow[d] & H_{n-k-1}(V_1 \cap V_2) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

The horizontal arrows are given by cap product (in the refined sense) with φ. Therefore two out of three are isomorphisms by our assumptions; those with a question mark label are not yet known to be isomorphisms. If we can show that the diagram commutes, then we can use the five lemma and conclude that the horizontal arrows with the question mark label are also isomorphisms. (Commutativity up to a factor ± 1 is also enough.) The only place where commutativity is not obvious, up to a factor ± 1 perhaps, is the square(s) involving the boundary operators δ and ∂. But we have the interpretation of δ and ∂ as a cup product, respectively cap product, with a class in $H^1((V_1 \cup V_2)/(V_1 \cup V_2) \smallsetminus (V_1 \cap V_2))$.

See example 17.2.5. And we have the associativity formula of remark 17.2.3. Together they establish the commutativity of that square up to possibly a factor ± 1. $\square$

17.6. Completion of proof

We continue with the less exciting parts of the proof of Poincaré duality: establishing properties (a), (b) and (c).

Property (a) is trivial because the reduced cohomology of M/M is zero (in all dimensions) and the homology of $M \smallsetminus M = \emptyset$ is also zero in all dimensions.

We turn to the proof of (b). The case where $n = 0$ reduces to the assertion that cap product with the generator of $H_0(\star) = \mathbb{Z}$ is an isomorphism from $H^0(\star) = \mathbb{Z}$ to $H_0(\star) = \mathbb{Z}$. This is easily verified with mapping cycles. Now we assume $n > 0$. Here the reduced cohomology of M/A is zero in all dimensions except n, and the homology of $M \smallsetminus A$ is also zero in all dimensions except 0. Therefore we can assume $k = n$, that is, we only need to show that cap product with the fundamental class φ gives an isomorphism

$$\tilde{H}^n(M/A) \longrightarrow H_0(M \smallsetminus A).$$

Put $M_1 := M/A$. Then M_1 is homeomorphic to S^n since it is the one-point compactification of $M \smallsetminus A \cong \mathbb{R}^n$. Let $\varphi_1 \in H_n(M_1)$ be the image of the fundamental class $\varphi \in H_n(M)$ under the homomorphism induced by the quotient map $M \to M/A = M_1$. Almost by definition we have a commutative diagram

$$\begin{tikzcd}
\tilde{H}^n(M/A) \arrow[r, "\frown\varphi"] \arrow[d, equal] & H_0(M \smallsetminus A) \arrow[d, equal] \\
\tilde{H}^n(M_1) \arrow[r, "\frown\varphi_1"] \arrow[d, "\cong"] & H_0(M_1 \smallsetminus \star) \arrow[d, "\cong"'] \\
H^n(M_1) \arrow[r, "\frown\varphi_1"] & H_0(M_1)
\end{tikzcd}$$

Moreover, φ_1 is a fundamental class for $M_1 \cong S^n$. To show this, select $x \in M \smallsetminus A$. The commutative diagram

$$\begin{tikzcd}
H_n(M) \arrow[r, "\text{induced by incl.}"] \arrow[d] & H_n(M /\!\!/ (M \smallsetminus x)) \arrow[d, "\cong"] \\
H_n(M_1) \arrow[r, "\text{induced by incl.}"] & H_n(M_1 /\!\!/ (M_1 \smallsetminus x))
\end{tikzcd}$$

shows that $\phi_1 \in H_n(M_1)$ maps to a generator of $H_n(M_1 /\!\!/ (M_1 \smallsetminus x))$, so it passes the test for a fundamental class. (It is enough to test at one $x \in M_1$ because M_1 is connected ... because $n > 0$.) Therefore we have reduced the proof to the following claim: cap product with the/a fundamental class of S^n gives an isomorphism from $H^n(S^n)$ to $H_0(S^n)$. Again, this is easy to verify with mapping cycles (since we know a good mapping cycle representing the fundamental class of S^n).

For the proof of (c) we write $A_\infty = \bigcap_i A_i$ and $U_i = M \smallsetminus A_i$, $U_\infty = M \smallsetminus A_\infty$. The naturality property of the cap product (remark 17.2.4) implies a commutative diagram

$$\begin{tikzcd}
\cdots & H^k(M/A_{i+1}) \arrow[l] \arrow[d, "\frown\varphi"] & H^k(M/A_i) \arrow[l] \arrow[d, "\frown\varphi"] & H^k(M/A_{i-1}) \arrow[l] \arrow[d, "\frown\varphi"] & \cdots \arrow[l] \\
\cdots & H_{n-k}(U_{i+1}) \arrow[l] & H_{n-k}(U_i) \arrow[l] & H_{n-k}(U_{i-1}) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

The inclusions $U_i \to U_\infty$ for all i determine a commutative square

$$\begin{tikzcd}
\operatorname{colim}_i H_{n-k}(U_i) \arrow[r] \arrow[d, "\frown\varphi"] & H_{n-k}(U_\infty) \arrow[d, "\frown\varphi"] \\
\operatorname{colim}_i H^k(M/A_i) \arrow[r] & H^k(M/A_\infty).
\end{tikzcd}$$

In the first (ladder-shaped) diagram, all the vertical arrows are isomorphisms; therefore, in the square, the left-hand vertical arrow is an isomorphism. We know that the horizontal arrows in the square are also isomorphisms. Therefore the right-hand vertical arrow is an isomorphism, too. $\square$

APPENDIX A

The fundamental group

The next few (sub)sections contain a sketchy account of the fundamental group, often from the point of view of category theory. To hammer this in right from the start, I start by introducing (once again) the category $\mathcal{T}op_*$ whose objects are spaces X with a selected (base) point, often denoted $\star \in X$. A morphism is then a continuous map from one space with base point to another space with base point, taking base point to base point. We often say *based space* or *pointed space* (for an object of $\mathcal{T}op_*$) and *based map* or *pointed map* (for a morphism in $\mathcal{T}op_*$). In the category $\mathcal{T}op$, we also have a concept of *based homotopy*: two based maps $f, g\colon X \to Y$ are *based homotopic* if there exists a homotopy $(h_t\colon X \to Y)_{t\in[0,1]}$ where $h_0 = f$, $h_1 = g$ and each h_t is a based map. *Based homotopic* is an equivalence relation on the set of based maps from X to Y. The set of equivalence classes is usually denoted by $[X, Y]_*$. As in the case of unbased homotopy, the based homotopy relation is compatible with composition of maps, so that we can construct a based homotopy category $\mathcal{H}o\mathcal{T}op_*$. The objects of $\mathcal{H}o\mathcal{T}op_*$ are still the based spaces $X, Y, \ldots$, but the set of morphisms $\mathrm{mor}_{\mathcal{H}o\mathcal{T}op_*}(X, Y)$ is $[X, Y]_*$.

A.1. The fundamental group as a functor

The *fundamental group* is a covariant functor π_1 from $\mathcal{T}op$ to the category of groups. We write $\pi_1(X)$ or $\pi_1(X, \star)$ for the value of that functor on a based space X. (It is a group, and we call it *the fundamental group of* X. The functor π_1 is also *based homotopy invariant*. This can be expressed in one of three equivalent ways.

(i) If X and Y are based spaces, and $f, g\colon X \to Y$ are based maps which are based homotopic, then the homomorphisms $\pi_1(f)$ and $\pi_1(g)$, both from $\pi_1(X, \star)$ to $\pi_1(Y, \star)$, agree.

(ii) If $f\colon X \to Y$ is a based homotopy equivalence of based spaces, then

$$\pi_1(f)\colon \pi_1(X, \star) \to \pi_1(Y, \star)$$

is an isomorphism of groups.

(iii) The functor π_1 can be written as a composition of functors

$$\mathcal{T}op_* \to \mathcal{H}o\mathcal{T}op_* \to \mathcal{G}roups$$

where the first functor from $\mathcal{T}op_*$ to $\mathcal{H}o\mathcal{T}op_*$ is the obvious one (passing from based maps to based homotopy classes).

This is somewhat reminiscent of the properties of homology and cohomology groups. But there are a few important differences. First of all, these fundamental groups really can be non-commutative groups. Secondly, the definition/construction of the fundamental group is much, much more elementary than the definition of the homology and cohomology groups. Thirdly, the *determination* of the fundamental group of a particular space X can be hard, often harder than the determination of the homology groups and cohomology

groups of X.
When I was young, even younger than today, the study of high-dimensional manifolds with the method(s) of homology and cohomology etc. was all the rage. But already in the early 1980s interest shifted towards low-dimensional topology. In low-dimensional topology, the fundamental group tends to be more important than homology and cohomology, so this shift also meant a shift away from homology and cohomology and towards fundamental groups. By the year 2000, the ratio of "number of pages of published research in low-dimensional topology" versus "number of pages of published research in high-dimensional topology" was approximately 10:1. (Today the balance is a little more even, I believe.)

Enough introductory chat for now ... let us see the definition. (There will be some more chat after the definition.)

Definition A.1.1. As a set, the fundamental group $\pi_1(X,\star)$ of a based space X is the set of based homotopy classes from S^1 to X.

Here it is often convenient to think of S^1 as a quotient space of the interval $[0,1]$, obtained by identifying the points 0 and 1. Equivalently, we say $t \in [0,1]$ and we mean $\exp(2\pi i t) \in S^1$. Therefore every element $a \in \pi_1(X,\star)$ can be represented by a *path* $\alpha\colon [0,1] \to X$ such that $\alpha(0) = \alpha(1)$, and every path α satisfying $\alpha(0) = \alpha(1) = \star$ represents an element of $\pi_1(X,\star)$. Two such paths α, β present the same element of $\pi_1(X,\star)$ if and only if they are homotopic by a homotopy $(h_t\colon [0,1] \to X)_{t\in[0,1]}$ which satisfies the condition $h_t(0) = \star = h_t(1)$ for all $t \in [0,1]$. This brings us to the definition of the group structure in $\pi_1(X,\star)$.

Definition A.1.2. Let $a, b \in \pi_1(X,\star)$ be represented by paths

$$\alpha\colon [0,1] \to X, \qquad \beta\colon [0,1] \to X,$$

respectively, so that $\alpha(0) = \alpha(1) = \beta(0) = \beta(1) = \star \in X$. The product $a \cdot b \in \pi_1(X)$ is represented by the path $\gamma\colon [0,1] \to X$ where $\gamma(t) = \beta(2t)$ if $t \le 1/2$ and $\gamma(t) = \alpha(2t-1)$ if $t \ge 1/2$.

This definition rather calls for at least one verification. Because we have defined $a \cdot b$ using representatives α and β, it is necessary to verify that the outcome does not depend on the choice of representatives α and β, but only on a and b. (This is left to the reader.)

Now we would like to say: $\pi_1(X,\star)$ with the multiplication defined just above is actually a group. This is not very hard, and it is again mostly left to the reader, but it is also a good opportunity for me to introduce some more notation. It is convenient to say: every path $\alpha\colon [0,c] \to X$ where $c \in \mathbb{R}$, $c \ge 0$, determines an element of $\pi_1(X,\star)$ provided $\alpha(0) = \alpha(c)$. (If you want to convert such an α to the standard form, pre-compose it with any continuous map $u\colon [0,1] \to [0,c]$ taking 0 to 0 and 1 to c. It does not matter which u you choose.) If we have $\alpha\colon [0,c] \to X$ and $\beta\colon [0,d] \to X$ such that $\alpha(0) = \star = \alpha(c)$ and $\beta(0) = \star = \beta(d)$, then we can define

$$\alpha \circ \beta\colon [0, c+d] \to X$$

by $t \mapsto \beta(t)$ if $t \le c$ and $t \mapsto \beta(t-c)$ if $t \ge c$. This kind of composition is obviously associative. You can use it as an alternative definition of the product in $\pi_1(X,\star)$.
With this notation it is easy to verify that

- the neutral element of $\pi_1(X,\star)$ is represented by the unique map $[0,0] \to X$ taking 0 to $\star$;

- the inverse of an element of $\pi_1(X, \star)$ represented by a path

$$\alpha\colon [0, c] \to X$$

with $\alpha(0) = \star = \alpha(c)$ is the element represented by the path α^{-1} where $\alpha^{-1}\colon [0, c] \to X$ is defined by $\alpha^{-1}(t) = \alpha(c - t)$.

EXAMPLE A.1.3. The fundamental group of S^1 (with base point $1 \in S^1 \subset \mathbb{C}$) is isomorphic to $\mathbb{Z}$, by the isomorphism taking $b \in \mathbb{Z}$ to the path $[0, 1] \to S^1$ given by $t \mapsto \exp(2\pi i b t)$. (The bijection is already known to us from section 1.2; only the claim that it is a homomorphism should be verified.)

Who invented the fundamental group? Surprisingly, or unsurprisingly, it was again our hero Henri Poincaré who is also responsible for homology and cohomology. He discovered it after making an interesting mistake. I believe the mistake was the following. He thought he could prove that a map $f\colon X \to Y$ of spaces which induces an isomorphism

$$f_*\colon H_k(X) \to H_k(Y)$$

for all $k \in \mathbb{Z}$ is a homotopy equivalence. (There were some mild conditions on X and Y; in modern language, he would have assumed that X and Y are CW-spaces.) Having published his argument for that, he found a counterexample. It was the 3-dimensional manifold $SO(3)/J$ where $SO(3)$ is the group of orthogonal real 3×3-matrices with determinant $+1$ (also known as the group of orientation-preserving linear isometries of $\mathbb{R}^3$) and J is the subgroup of the orientation-preserving linear symmetries of the icosahedron (which has 60 elements and is isomorphic to the alternating group A_5). Here $SO(3)/J$ denotes the space of left cosets of J in $SO(3)$, not the quotient space obtained by collapsing J to a single point. More precisely, Poincaré was able show that there is a map $SO(3)/J \to S^3$ which does induce isomorphisms in H_k for all $k \in \mathbb{Z}$, but somehow he knew that it was not a homotopy equivalence. In the process he developed the language enabling him to say why not: $SO(3)/J$ has a fundamental group which is finite of order 120, while S^3 has a trivial fundamental group. *Therefore* the two are not homotopy equivalent (as pointed spaces or otherwise).

A.2. The Seifert-VanKampen theorem

THEOREM A.2.1. *Let X be a space with base point $\star$. Suppose that X is the union of two open subsets V and W, where V, W and $V \cap W$ are path connected and $\star \in V \cap W$. Let G be a group and let $p_V\colon \pi_1(V) \to G$ and $p_W\colon \pi_1(W) \to G$ be homomorphisms such that the diagram*

$$(✠) \qquad \begin{tikzcd} \pi_1(V \cap W) \arrow[r, "\text{incl. induced}"] \arrow[d, "\text{incl. induced}"'] & \pi_1(V) \arrow[d, "p_V"] \\ \pi_1(W) \arrow[r, "p_W"'] & G \end{tikzcd}$$

of groups and homomorphisms commutes. Then there is a unique homomorphism z *from* $\pi_1(X)$ *to* G *making the following diagram commutative:*

$$\begin{tikzcd}
\pi_1(V \cap W) \arrow[r, "\text{incl. induced}"] \arrow[d, "\text{incl. induced}"'] & \pi_1(V) \arrow[d, "\text{incl. induced}"'] \arrow[ddr, "p_V"] & \\
\pi_1(W) \arrow[r, "\text{incl. induced}"] \arrow[drr, "p_W"'] & \pi_1(X) \arrow[dr, dotted, "z"] & \\
& & G
\end{tikzcd}$$

Proof. The following notation will be used in this proof.

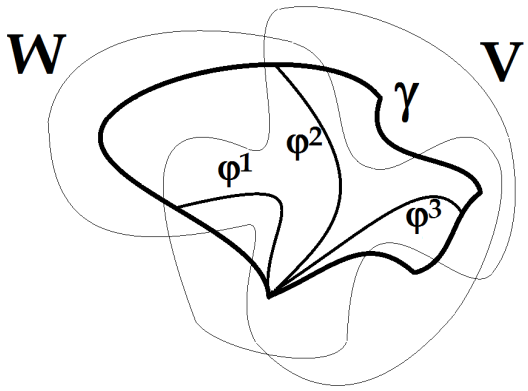
- For a path $\gamma\colon [a,b] \to X$ where $a \le b \in \mathbb{R}$, we denote by $\bar{\gamma}$ the path $[-b,-a] \to X$ given by $t \mapsto \gamma(-t)$.
- We try not to make a great distinction between a path $\gamma\colon [a,b] \to X$ and the path $[0, b-a] \to X$ given by $t \mapsto \gamma(t+a)$.
- In particular, for two paths $\gamma\colon [a,b] \to X$ and $\zeta\colon [c,d] \to X$ where $\gamma(b) = \zeta(c)$, we denote by $\zeta \circ \gamma$ the path $[a, b-c+d] \to X$ given by γ on $[a,b]$ and by $t \mapsto \zeta(t-b+c)$ on $[b, b-c+d]$.
- For two paths $\beta\colon [0,c] \to X$ and $\gamma\colon [0,d] \to X$ satisfying $\beta(0) = \beta(c) = \star$ and $\gamma(0) = \gamma(d) = \star$, we write $\beta \simeq_* \gamma$ if there exists a homotopy $(h_t\colon [0,1] \to X)_{t\in[0,1]}$ such that $h_0(s) = \beta(cs)$ and $h_1(s) = \gamma(ds)$, where $h_t(0) = \star = h_t(1)$ for all $t \in [0,1]$.

Now suppose that an element of $\pi_1(X) = \pi_1(X,\star)$ is represented by a path $\gamma\colon [0,1] \to X$ where $\gamma(0) = \star = \gamma(1)$. Claim: there exist elements

$$0 = a(0) \le a(1) \cdots \le a(r) = 1$$

such that each $a(j)$ is mapped to $V \cap W$ by γ and each interval $[a(j), a(j+1)]$ is mapped either to V or to W. The proof is easy.[1] Since $V \cap W$ is path connected, we can choose paths $\varphi^j\colon [0,1] \to V \cap W$ such that $\varphi^j(0) = \star$ and $\varphi^j(1) = \gamma(a(j))$ for $j = 1, 2, \ldots, r-1$. For $j = 1, 2, \ldots, r$ let γ^j be the restriction of γ to the interval $[a(j-1), a(j)]$. (Warning: these superscripts j are not to be read as exponents.)

[1] *Lebesgue's lemma*: for any covering of $[0,1]$ by open sets, there exists $\varepsilon > 0$ such that every subinterval of $[0,1]$ of length $< \varepsilon$ is contained in one of the open sets of the covering. Apply this to the covering of $[0,1]$ by the open sets $\gamma^{-1}(V)$ and $\gamma^{-1}(W)$. Get your ε; without loss of generality $\varepsilon = 1/n$ for a positive integer n. Then you can choose the $a(j)$ to have the form k/n for this n and a selection of $k \in \{0, 1, 2, \ldots, n\}$.



Then

$$\begin{aligned}\gamma &= \gamma^r \circ \gamma^{r-1} \circ \cdots \circ \gamma^1 \\ &\simeq_* \gamma^r \circ \varphi^{r-1} \circ \bar{\varphi}^{r-1} \circ \gamma^{r-1} \circ \varphi^{r-2} \circ \bar{\varphi}^{r-2} \circ \cdots \circ \gamma^2 \circ \varphi^1 \circ \bar{\varphi}^1 \circ \gamma^1 \\ &= \left(\gamma^r \circ \varphi^{r-1}\right) \circ \left(\bar{\varphi}^{r-1} \circ \gamma^{r-1} \circ \varphi^{r-2}\right) \circ \cdots \circ \left(\bar{\varphi}^2 \circ \gamma^2 \circ \varphi^1\right) \circ \left(\bar{\varphi}^1 \circ \gamma^1\right) \\ &=: \beta^r \circ \beta^{r-1} \circ \beta^{r-1} \cdots\cdots \beta^1 \ .\end{aligned}$$

So $\gamma \simeq_* \beta^r \circ \beta^{r-1} \circ \beta^{r-1} \cdots\cdots \beta^1$ where $\beta^r, \beta^{r-1}, \ldots, \beta^1$ are paths, beginning and ending at $\star$, which run either in V or in W. For each $j = 1, 2, \ldots, r$ choose V or W such that β^j runs in that open set, and write p_j to mean p_V or p_W accordingly. Therefore we *must* define

$$z([\gamma]) = p_r([\beta^r]) \cdot p_{r-1}([\beta^{r-1}]) \cdots p_2([\beta^2]) \cdot p_1([\beta^1]) \in G$$

if we want to ensure that the above diagram with the dotted arrow z commutes. What remains to be done? Mainly we have to show that the above formula for $z([\gamma])$ does not depend on the many choices we made.

(i) Let's begin with the very last choices that we made. For each $j = 1, 2, \ldots, r$ we selected an element of V or W such that β^j runs in that open set, and we defined p_j to be p_V or p_W accordingly. What if β^j runs in $V \cap W$? Then we have a choice ... but since (✠) commutes it does not matter for our proposed value of $z([\gamma]) \in G$ which choice we make.

(ii) We could choose a different subdivision of the interval $[0, 1]$ by points $a(1), \ldots, a(r)$ and different paths φ^j. One way to show that it doesn't matter is like this. Suppose that we have selected $a(1), \ldots, a(r)$ and paths φ^j as above, for $j = 1, 2, \ldots, r-1$. Let somebody select an additional element $b \in [0, 1]$ such that $\gamma(b) \in V \cap W$, and a path $\psi\colon [0, 1] \to V \cap W$ where $\psi(0) = \star$ and $\psi(1) = \gamma(b)$, and k so that $a(k-1) \le b \le a(k)$. Repeat the whole process with this new subdivision

$$a(0), a(1), \ldots, a(k-1), b, a(k), \ldots, a(r).$$

Choose the new p_k and p_{k+1} so that they agree with the old p_k . It is easy to see that the proposed value of $z([\gamma]) \in G$ does not change.

(iii) We need to show that the proposed value $z([\gamma]) \in G$ is not sensitive to the choice of a representative γ. Let us write informally $z(\gamma) \in G$ for the proposed value; now we know at least that it depends only on $\gamma\colon [0, 1] \to X$ with $\gamma(0) = \gamma(1) = \star$. Let $(\gamma_t\colon [0, 1] \to X)_{t \in [0,1]}$ be a homotopy where $\gamma_t(0) = \gamma_t(1) = \star$ for all $t \in [0, 1]$. It is enough to show that $t \mapsto z(\gamma_t) \in G$ is a locally constant function of the variable

$t \in [0,1]$. So choose $s \in [0,1]$. Choose a subdvision $a(0), a(1), \ldots, a(r)$ of the interval $[0,1]$ as above, using γ_s in place of γ, and choose paths φ^j as above, to get

$$\gamma_s \simeq_* \beta_s^r \circ \beta_s^{r-1} \circ \cdots \circ \beta_s^1$$

as above. (The subscripts s in the right-hand side will be useful for distinction in a moment.) For $u \in [0,1]$ sufficiently close to s, and $u \geq s$, the subdivision $a(0), a(1), \ldots, a(r)$ is still a suitable subdivision for γ_u and instead of φ^j we can use the path $\zeta^j \circ \varphi^j$ where ζ^j is defined by $t \mapsto \gamma_t(a(j))$ for $t \in [s,u]$. Similarly for $u \in [0,1]$ sufficiently close to s, and $u \leq s$, the subdivision $a(0), a(1), \ldots, a(r)$ is still a suitable subdivision for γ_u and instead of φ^j we can use the path $\bar{\zeta}^j \circ \varphi^j$ where ζ^j is defined by $t \mapsto \gamma_t(a(j))$ for $t \in [u,s]$. Then we get

$$\gamma_u \simeq_* \beta_u^r \circ \beta_u^{r-1} \circ \cdots \circ \beta_u^1$$

with the same r. With these choices, it is easy to see that $[\beta_s^j] = [\beta_u^j]$ in $\pi_1(V)$ or $\pi_1(W)$, as appropriate. That implies $z(\gamma_s) = z(\gamma_u)$. Therefore $t \mapsto z(\gamma_t)$ is constant in a neighborhood of $s \in [0,1]$. $\square$

The formulation of the Seifert-vanKampen theorem above is in category language. If we wanted to use some more category language, we could also say that the commutative square

$$\begin{tikzcd}
\pi_1(V \cap W) \arrow[r, "\text{incl. induced}"] \arrow[d, "\text{incl. induced}"'] & \pi_1(V) \arrow[d, "\text{incl. induced}"'] \\
\pi_1(W) \arrow[r, "\text{incl. induced}"] & \pi_1(X)
\end{tikzcd}$$

is a *pushout square*, or even better, that $\pi_1(X)$ is the *direct limit* of the diagram

$$\begin{tikzcd}
\pi_1(V \cap W) \arrow[r, "\text{incl. induced}"] \arrow[d, "\text{incl. induced}"'] & \pi_1(V) \\
\pi_1(W) &
\end{tikzcd} \qquad (✠✠)$$

Stating the theorem in this way makes it easier to prove. Turning the statement (as above) into an explicit description of $\pi_1(X)$ in terms of the diagram is the business of group theory! When this is done the outcome is as follows.

- Elements of $\pi_1(X)$ can be imagined as words $x_r x_{r-1} \cdots x_2 x_1$ where each letter is taken from $\pi_1(V)$ or from $\pi_1(W)$; strictly speaking the letters are elements of the disjoint union $\pi_1(V) \sqcup \pi_1(W)$.
- Two such words $x_r x_{r-1} \cdots x_2 x_1$ and $y_s y_{s-1} \cdots y_2 y_1$ describe the same element of $\pi_1(X)$ if and only if one can be transformed into the other by some simple operations. These are as follows.
  - Delete letter given by trivial element of $\pi_1(V)$ or $\pi_1(W)$.
  - If two adjacent letters in such a word are both in $\pi_1(V)$, replace them by a single letter which is their product in $\pi_1(V)$.
  - If two adjacent letters are both in $\pi_1(W)$, replace them by a single letter which is their product in $\pi_1(W)$.
  - If a letter is the image of some $y \in \pi_1(V \cap W)$ under the homomorphism $\pi_1(V \cap W) \to \pi_1(V)$, replace by the image of the same y under the homomorphism $\pi_1(V \cap W) \to \pi_1(W)$.

– Operations inverse to the above (for example: *insert* letter given by trivial element of $\pi_1(V)$ or $\pi_1(W)$, etc.).

EXAMPLE A.2.2. It follows from Seifert-van Kampen that $\pi_1(S^n, \star)$ is trivial for $n > 1$. Proof: choose distinct points $x, y \in S^n$ which are also distinct from the chosen base point $\star$. Write $S^n = V \cup W$ where V is S^n minus x and W is S^n minus y. Since V and W are contractible, $\pi_1(V, \star)$ and $\pi_1(W, \star)$ are both trivial, and so Seifert-van Kampen implies that $\pi_1(S^n, \star)$ is trivial. Note in passing that Seifert-van Kampen is applicable; in particular $V \cap W$ is path connected since we assumed $n > 1$.

A.3. Changing the base point and forgetting the base point

PROPOSITION A.3.1. *Let X be a space and $x, y \in X$. If x and y are in the same path component of X, then $\pi_1(X, x)$ is isomorphic to $\pi_1(X, y)$.*

PROOF. Choose a path $\alpha\colon [0,1] \to X$ such that $\alpha(0) = x$ and $\alpha(1) = 1$. (We write $\bar{\alpha}\colon [0,1] \to X$ for the reverse path, $\bar{\alpha}(t) = \alpha(1-t)$, as in the previous section.) The path α can be used to define a homomorphism Φ_α from $\pi_1(X, x)$ to $\pi_1(X, y)$. Namely, for an element g of $\pi_1(X, x)$ represented by a path $\gamma\colon [0,1] \to X$ where $\gamma(0) = x = \gamma(1)$, we let

$$\Phi_\alpha(g) := [\alpha \circ \gamma \circ \bar{\alpha}.]$$

In words: use $\bar{\alpha}$, the reverse of α, to travel from y to x, then run through γ, then use α to travel back from x to y. It is easy to see that Φ_α is a homomorphism. Namely, if f in $\pi_1(X, x)$ is represented by a path φ where $\varphi(0) = x = \varphi(1)$, then

$$[\alpha \circ \varphi \circ \bar{\alpha}] \cdot [\alpha \circ \gamma \circ \bar{\alpha}] = [\alpha \circ \varphi \circ \bar{\alpha} \circ \alpha \circ \gamma \circ \bar{\alpha}] = [\alpha \circ \varphi \circ \gamma \circ \bar{\alpha}]$$

which means $\Phi_\alpha(f) \cdot \Phi_\alpha(f) = \Phi_\alpha(f \cdot g)$. Next, let's note that $\Phi_{\bar{\alpha}}$ is a homomorphism from $\pi_1(X, y)$ to $\pi_1(X, x)$. It is easy to see that $\Phi_{\bar{\alpha}} \circ \Phi_\alpha$ is the identity homomorphism on $\pi_1(X, s)$, and similarly, $\Phi_\alpha \circ \Phi_{\bar{\alpha}}$ is the identity homomorphism on $\pi_1(X, y)$. Therefore Φ_α is an isomorphism. □

REMARK A.3.2. The isomorphism Φ_α in the proof above depends very much on α. This is the reason why we need base points to define fundamental groups, despite proposition A.3.1. Indeed, let $\beta\colon [0,1] \to X$ be another path such that $\beta(0) = x$ and $\beta(1) = y$. Let $g \in \pi_1(X, x)$ be represented by a path $\gamma\colon [0,1] \to X$ where $\gamma(0) = x = \gamma(1)$, as above. Then

$$(\Phi_\beta)^{-1}(\Phi_\alpha(g)) = [\bar{\beta} \circ \alpha \circ \gamma \circ \bar{\alpha} \circ \beta] = [\bar{\beta} \circ \alpha] \cdot [\gamma] \cdot [\bar{\alpha} \circ \beta] = c^{-1} \cdot g \cdot c$$

where $c \in \pi_1(X, x)$ is represented by the path $\bar{\alpha}\beta$ which begins and ends at x. Therefore, if c is not in the center of $\pi_1(X)$, then $\Phi_\beta \neq \Phi_\alpha$.

EXAMPLE A.3.3. In the same spirit, we show that for a path connected space X with base point, the set of homotopy classes of maps from S^1 to X (not required to preserve base points) is in bijection with the set of *conjugacy classes* of the group $\pi_1(X, \star)$. As usual it is convenient to view S^1 as a quotient space of $[0,1]$. So let $\gamma\colon [0,1] \to X$ be a map satisfying $\gamma(0) = \gamma(1)$. We do not require $\gamma(0) = \star$, but since X is path connected, we can choose a path $\alpha\colon [0,1] \to X$ such that $\alpha(0) = \star$ and $\alpha(1) = \gamma(0) = \gamma(1)$. Then $[\bar{\alpha} \circ \gamma \circ \alpha]$ is an element of $\pi_1(X, \star)$. As such it depends on our choice α, but the conjugacy class does not depend on our choice of α. Furthermore, by a continuity argument, the conjugacy class depends only on the homotopy class $[\gamma] \in [S^1, X]$. In this way we have constructed a map from $[S^1, X]$ to the set of conjugacy classes of $\pi_1(X)$. In order to get a map in the opposite direction we note first that there is a forgetful map from $[S^1, X]_\star = \pi_1(X)$

to $[S^1, X]$. We need to show that this takes elements in $\pi_1(X, \star)$ which are in the same conjugacy class to the same element of $[S^1, X]$. In other words, given $\alpha, \beta \colon [0,1] \to X$ where $\alpha(0) = \alpha(1) = \beta(0) = \beta(1) = \star$, we need to show that the maps $\bar{\beta} \circ \alpha \circ \beta$ and α, viewed as maps $S^1 \to X$, are (unbased) homotopic. This is an easy exercise.

A.4. Fundamental group of a CW-space

Let X be a path connected space (not required to be a CW-space) with base point $\star$. Let $\gamma \colon S^{n-1} \to X$ be a map, where $n > 1$, and let Y be the pushout of

$$\begin{tikzcd} S^{n-1} \arrow[r, "\gamma"] \arrow[d, "\text{incl.}"'] & X \\ D^n & \end{tikzcd}$$

(so that $Y = X \sqcup D^n / \sim$, where $\sim$ means that each $z \in S^{n-1} \subset D^n$ gets identified with $\gamma(z) \in X$). Now we have an inclusion $X \to Y$ and we can use $\star \in X$ as base point for Y, too. We are interested in a comparison of the fundamental groups of X and Y.

Proposition A.4.1. *The inclusion $X \to Y$ induces a homomorphism*

$$\pi_1(X, \star) \to \pi_1(Y, \star)$$

which is an isomorphism when $n > 2$ and surjective when $n = 2$. In the case $n = 2$ the kernel is the smallest normal subgroup of $\pi_1(X, \star)$ containing the conjugacy class determined by $\gamma \colon S^1 \to X$ according to example A.3.3.

Proof. Let y_0 and z be two distinct points in $Y \smallsetminus X$. Let $V = Y \smallsetminus \{z\}$ and $W = Y \smallsetminus X$. Then $V \cup W = Y$. The inclusion $X \to V$ is a homotopy equivalence (and also a homotopy equivalence of based spaces, if we take $\star$ as the base point). Therefore it is enough to show that the inclusion $V \to Y$ induces a homomorphism

$$\pi_1(V, \star) \to \pi_1(Y, \star)$$

which is an isomorphism for $n > 2$, and surjective for $n = 2$ with kernel equal to the smallest normal subgroup of $\pi_1(X, \star)$ containing the conjugacy class determined by γ. This formulation is not totally convenient for us because the base point $\star$ is not contained in W. Therefore we try y_0 as an alternative base point. In view of proposition A.3.1, it is enough to show that the inclusion $V \to Y$ induces a homomorphism

$$\pi_1(V, y_0) \to \pi_1(Y, y_0)$$

which is an isomorphism for $n > 2$, and surjective for $n = 2$ with kernel equal to the smallest normal subgroup of $\pi_1(X, \star)$ containing the conjugacy class determined by γ. Now we can use the Seifert-van Kampen theorem to prove it. The result is that we have a pushout square of groups

$$\begin{tikzcd} \pi_1(V \cap W, y_0) \arrow[r] \arrow[d] & \pi_1(V, y_0) \arrow[d] \\ \pi_1(W, y_0) \arrow[r] & \pi_1(Y, y_0). \end{tikzcd}$$

Since W is contractible, $\pi_1(W, y_0)$ is a trivial group. The pushout square property then means that the vertical arrow from $\pi_1(V, y_0)$ to $\pi_1(Y, y_0)$ is onto, with kernel equal to the smallest normal subgroup containing the image of $\pi_1(V \cap W, y_0) \to \pi_1(V, y_0)$. But we

have $V \cap W \simeq S^{n-1}$. If $n > 2$, this has trivial fundamental group, and so the arrow from $\pi_1(V, y_0)$ to $\pi_1(Y, y_0)$ is an isomorphism. If $n = 2$ then $V \cap W \simeq S^1$ has fundamental group $\cong \mathbb{Z}$ and it is easy to see that the image of

$$\pi_1(V \cap W, y_0) \to \pi_1(V, y_0)$$

is the cyclic subgroup generated by the element corresponding to a certain closed curve in W which surrounds and at the same time avoids z_0. That curve is (unbased) homotopic to γ. So the smallest normal subgroup of $\pi_1(V, y_0)$ which contains the image of

$$\pi_1(V \cap W, y_0) \to \pi_1(V, y_0)$$

is the smallest normal subgroup of $\pi_1(V, y_0)$ containing the conjugacy class determined by γ. □

Lemma A.4.2. *Let X be a CW-space with base point $\star$ which is a 0-cell.*

- *Every element of $\pi_1(X, \star)$ is in the image of the inclusion-induced homomorphism $\pi_1(K, \star) \to \pi_1(X, \star)$ for some compact CW-subspace K of X which contains $\star$.*
- *If K is such a compact CW-subspace and two elements a, b of $\pi_1(K, \star)$ determine the same element of $\pi_1(X, \star)$, then there exists another compact CW-subspace $L \subset X$ such that $K \subset L \subset X$ and a, b determine the same element of $\pi_1(L, \star)$.*

(Reformulation in category language: *the inclusions $K_\alpha \to X$, for compact CW-subspaces K_α of X which contain $\star$, induce an isomorphism of groups*

$$\operatorname{colim}_\alpha \pi_1(K_\alpha, \star) \longrightarrow \pi_1(X, \star).$$

Proof. This is an easy consequence of the important fact that every compact subset of X is contained in a compact CW-subspace of X. □

Corollary A.4.3. *Let X be a CW-space with a single 0-cell $\star$. Choose characteristic maps $\varphi_\alpha \colon D^1 \to X^1$ for the 1-cells E_α and $\varphi_\lambda \colon D^2 \to X^2$ for the 2-cells E_λ. The maps φ_α can also be viewed as based maps φ'_α from $D^1/\partial D^1 \cong S^1$ to X^1. Then*

(i) *the inclusion $X^1 \to X^2$ induces a surjection $\pi_1(X^1, \star) \to \pi_1(X^2, \star)$;*
(ii) *the inclusion $X^2 \to X$ induces an isomorphism $\pi_1(X^2, \star) \to \pi_1(X, \star)$;*
(iii) *the group $\pi_1(X^1, \star)$ is a free group with generators $[\varphi'_\alpha]$ corresponding to the 1-cells E_α;*
(iv) *the kernel of the homomorphism $\pi_1(X^1, \star) \to \pi_1(X^2, \star)$ induced by the inclusion is the smallest normal subgroup containing the conjugacy classes determined by the maps $\varphi_\lambda|_{S^1} \colon S^1 \to X^1$ corresponding to the 2-cells E_λ.*

Proof. It follows from lemma A.4.2 that if the statement holds for every nonempty compact CW-subspace of X, then it holds for X itself. Therefore we may assume that X is a compact CW-space. Suppose that X has k cells of dimension 1 and ℓ cells of dimension 2. Constructing X^1 in k steps from the 0-cell, we obtain (iii) using the Seifert-van Kampen theorem. Constructing X^2 in ℓ steps from X^1, we obtain (i) and (iv) from proposition A.4.1. Constructing X from X^2 in finitely many steps, attaching cells of dimension > 2 only, we obtain (ii) from proposition A.4.1. □

Remark A.4.4. A connected CW-space X which has more than one 0-cell can always be replaced by a connected CW-space Y which has exactly one 0-cell and is homotopy equivalent to X. The standard procedure is as follows. The skeleton X^1 is a connected 1-dimensional CW-space, also known as a *connected graph*. It is an exercise or a result in graph theory that the connected graph X^1 contains a maximal tree: a CW-subspace

Z which is contractible but not contained in a larger contractible CW-subspace. Furthermore, such a maximal tree must contain all the 0-cells of X^1. Now let Y be the quotient X/Z. Then clearly Y has only one 0-cell. The quotient map $p\colon X \to Y = X/Z$ is a homotopy equivalence for the following reason. We have a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & X/\!\!/Z \arrow[d, "\simeq"] \\ & X/Z \end{tikzcd}$$

so that it suffices to show that the inclusion of X in the mapping cone $X/\!\!/Z$ is a homotopy equivalence. The contractibility of Z means that there exists a homotopy $(h_t\colon Z/\!\!/Z \to Z/\!\!/Z)_{t\in[0,1]}$ such that $h_0 = \mathrm{id}$, each h_t agrees with the identity on $Z \subset Z/\!\!/Z$, and h_1 has image equal to $Z \subset Z/\!\!/Z$. (Exercise.) Now write $X/\!\!/Z = X \cup (Z/\!\!/Z)$. It is easy to show that the map $X/\!\!/Z \to X$ given by id on X and by h_1 on $Z/\!\!/Z$ is a homotopy inverse for the inclusion $X \to X/\!\!/Z$.

Remark A.4.5. Let $f\colon X \to Y$ be a base-point preserving map of spaces with base point which is an ordinary homotopy equivalence. Suppose that the base point inclusions $\star \hookrightarrow X$ and $\star \hookrightarrow Y$ are cofibrations. Then f is a based homotopy equivalence, i.e., there exists a based map $g\colon Y \to X$ and base-point preserving homotopies from gf to id_X and from fg to id_Y.

The proof is not easy, but not unpleasant either. Using the homotopy extension property for $\star \to Y$, we can easily find a base-point preserving $g^\natural\colon Y \to X$ such that $g^\natural f$ and $fg^\natural$ are homotopic (by homotopies which may not be base-point preserving) to the respective identity maps. Now we can think about $g^\natural f$ and $fg^\natural$ instead of f. That is to say, we have reduced the general case to the following problem. Given a based map $e\colon X \to X$ which is homotopic to id_X in the unbased sense; then we want to know that f is a based homotopy equivalence.

This brings us to the subset $K \subset [X,X]_*$ consisting of the based homotopy classes of based maps $X \to X$ which are homotopic in the unbased sense to the identity. The subset K is a sub-monoid, i.e., if $[e_1] \in K$ and $[e_2] \in K$, then $[e_1] \circ [e_2] = [e_1 e_2] \in K$. We want to show that as a monoid in its own right, it is a group, i.e., every element has an inverse. To show it we introduce a map

$$\nu\colon \pi_1(X,\star) \to K$$

as follows. Given an element of $\pi_1(X)$ represented by $\gamma\colon [0,1] \to X$ with $\gamma(0) = \star = \gamma(1)$, the homotopy extension property for $\star \to X$ allows us to construct a homotopy

$$(h_t\colon X \to X)_{t\in[0,1]}$$

such that $h_t(\star) = \gamma(t)$ and $h_0 = \mathrm{id}$. Then $h_1\colon X \to X$ is a based map and it is homotopic to id_X in the unbased sense. We try $\nu[\gamma] := [h_1]$. It is not obvious that this is well defined, because it might seem to depend on our choice of a homotopy (h_t), but one can use the homotopy extension property for $\star \times [0,1] \hookrightarrow X \times [0,1]$ to show that it is indeed well defined.

Next we show that ν is a homomorphism. Given paths $\gamma\colon [0,1] \to X$ and $\lambda\colon [0,1] \to X$ such that $\gamma(0) = \gamma(1) = \lambda(0) = \lambda(1) = \star$, and a homotopy $(h_t\colon X \to X)_{t\in[0,1]}$ such that $h_t(\star) = \gamma(t)$ and $h_0 = \mathrm{id}$, and a homotopy $(h'_t\colon X \to X)_{t\in[0,1]}$ such that $h'_t(\star) = \lambda(t)$ and $h'_0 = \mathrm{id}$, we get a homotopy $(h''_t\colon X \to X)_{t\in[0,2]}$ where $h''_t = h_t$ for $t \in [0,1]$ and

$h''_t = h'_{t-1} \circ h_1$ for $t \in [1,2]$. This has the property that $h''_t(\star) = \gamma(t)$ for $t \in [0,1]$ and $h''_t = \lambda(t-1)$ for $t \in [1,2]$. Therefore

$$\nu([\lambda] \cdot [\gamma]) = [h''_2] = [h'_1 \circ h_1] = \nu[\lambda] \circ \nu[\gamma] \in K.$$

Finally, we note that ν is surjective. For if an element of K is represented by a based map $e\colon X \to X$ such that there is a homotopy $(h_t)_{t\in[0,1]}$ from id_X to e, then we have an element of $\pi_1(X,\star)$ represented by the path $t \mapsto h_t(\star)$. Now, since ν is a surjective homomorphism, the monoid K must be a group. — One more observation: ν is not always injective. For example, when $X = S^1$ it is the trivial homomorphism, therefore not injective. The kernel of ν is known as the *Gottlieb* subgroup of $\pi_1(X,\star)$.

For another corollary, let X be a space with base point $\star$. There is an important map

$$u\colon \pi_1(X,\star) \longrightarrow H_1(X)$$

which can be described in two equivalent ways. An element of $\pi_1(X,\star)$ is a homotopy class of based maps $S^1 \to X$ and this can also be viewed as a homotopy class of mapping cycles $S^1 \to X$, and so it determines an element of $H_1(X)$. Alternative description: An element of $\pi_1(X,\star)$ is a homotopy class of based maps $\gamma\colon S^1 \to X$ and this induces a homomorphism $\gamma_\star\colon H_1(S^1) \to H_1(X)$ which we evaluate on the element $1 \in \mathbb{Z} \cong H_1(S^1)$ to get an element in $H_1(X)$.

Corollary A.4.6. *The map u is a homomorphism. If X is a path-connected CW-space, then u is surjective and the kernel of u is the commutator subgroup*[2] *of $\pi_1(X,\star)$.*

Proof. Showing that the map is a homomorphism: this is the hardest bit. We try the following special case first: X is $S^1 \vee S^1$ with the standard base point (where the two circles are wedged together) and we take two elements a, b of $\pi_1(X,\star)$ given by the inclusion of the first wedge summand (for a) and the inclusion of the second wedge summand (for b). Let $p, q\colon X \to S^1$ be the map given by collapse of the second wedge summand (for p) and collapse of the first wedge summand (for q). Then it is easy to verify directly that $p_*(u(a \cdot b)) = 1 \in H_1(S^1) = \mathbb{Z}$ and $q_*(u(a \cdot b)) = 1 \in H_1(S^1,\star)$. It follows that $u(a \cdot b) = (1,1) \in \mathbb{Z} \times \mathbb{Z} = H_1(S^1) \times H_1(S^1) = H_1(X)$. This agrees with $(1,0)+(0,1) = u(a)+u(b) \in \mathbb{Z}\times\mathbb{Z} = H_1(S^1)\times H_1(S^1) = H_1(X)$. The case of a general X follows from this special case by naturality. Indeed if we have two elements a, b of $\pi_1(X,\star)$ represented by based maps $\alpha, \beta\colon S^1 \to X$, then we can make a map $g\colon S^1 \vee S^1 \to X$ given by α on the first wedge summand and by β on the second. Then there is a commutative diagram

$$\begin{tikzcd}
\pi_1(S^1 \vee S^1,\star) \arrow[r, "g_*"] \arrow[d, "u"] & \pi_1(X,\star) \arrow[d, "u"] \\
H_1(S^1 \vee S^1) \arrow[r, "g_*"] & H_1(X)
\end{tikzcd}$$

where the horizontal arrows are certainly homomorphisms. Since a and b are in the image of g_* , top horizontal arrow, this settles the matter.
Showing that u is surjective and $\ker(u)$ is the commutator subgroup if X is a path connected CW-space: by remark A.4.4 and a naturality argument, we can reduce to the

[2]The commutator subgroup of a group G is the subgroup K of G generated by all expressions of the form $aba^{-1}b^{-1}$, where $a, b \in G$. It is a normal subgroup and the quotient group G/K is clearly an abelian group. Every homomorphism from G to an abelian group A is trivial on K and can therefore be written in a unique way as a composition $G \to G/K \to A$, where $G \to G/K$ is the projection.

case where X has only one 0-cell. Namely, if Z is a maximal tree in X^1 , then we have a commutative diagram

$$\begin{tikzcd} \pi_1(X,\star) \arrow[r, "\cong"] \arrow[d, "u"] & \pi_1(X/Z,\star) \arrow[d, "u"] \\ H_1(X) \arrow[r, "\cong"] & H_1(X/Z). \end{tikzcd}$$

(Perhaps I am using remark A.4.5 here to justify the claim that the top horizontal arrow is an isomorphism.) Furthermore, the choice of base point in X does not matter, because if we have one choice of base point $\star_1$ and another $\star_2$, then there is a commutative diagram

$$\begin{tikzcd} \pi_1(X,\star_1) \arrow[r, "\cong"] \arrow[d, "u"] & \pi_1(X/Z,\star_2) \arrow[d, "u"] \\ H_1(X) \arrow[r, "\cong"] & H_1(X/Z) \end{tikzcd}$$

by proposition A.3.1. So we can assume that X has only one 0-cell and that the 0-cell is the base point. Now we get the result about the kernel of u by comparing the description of $\pi_1(X,\star)$ in corollary A.4.3 with the description of $H_1(X)$ in terms of the cellular chain complex. (In the latter description, $H_1(X)$ is an *abelian* group with a presentation which has one generator for every 1-cell and one relation for every two-cell.) □

Remark A.4.7. The assumption in corollary A.4.6 that X is a CW-space is not really necessary, but the proof would be harder without it.

A.5. Covering spaces

A covering space is simply a fiber bundle $p\colon E \to X$ where the fibres $p^{-1}(x)$ for $x \in X$ are discrete spaces. In more detail: let $p\colon E \to X$ be continuous map of spaces. We say that p is a covering space if for every $x \in X$ there exist an open neighborhood U of x in X and a set S and a homeomorphism $h\colon p^{-1}(U) \to U \times S$ such that h followed by the projection to U agrees with p. You can also write $\coprod_{s\in S} U$ instead of $U \times S$; perhaps this makes the topology clearer.

If X is path connected and $p\colon E \to X$ is a covering space, then the fibers $p^{-1}(x)$ and $p^{-1}(y)$ over distinct elements $x, y \in X$ have the same cardinality. This is a special case of a statement for fiber bundles (prop. 2.1.3. in cumulative lecture notes).

Two very well known examples of fiber bundles are as follows: the map $p\colon \mathbb{R} \to S^1$ where $p(t) = \exp(2\pi i t)$; the quotient map $q\colon S^n \to \mathbb{R}P^n$. In the first example, the fibers are infinite sets; in the second example, they are evidently sets of cardinality 2.

Lemma A.5.1. *Let G be a topological group and let H be a finite subgroup. Let G/H be the set of left cosets, viewed as a space with the quotient topology. Then the projection $q\colon G \to G/H$ is a covering space.*

Proof. It is part of the *topological group* assumption that G is Hausdorff. Therefore, given $x \in G$, we can find an open neighborhood U of x in G such that the translates $U \cdot h$ for $h \in H$ are pairwise disjoint. This means that $q|_U$ is a homeomorphism from U to $q(U)$. Also, $q(U)$ is an open neighborhood of xH in G/H such that $q^{-1}(q(U)) = \bigcup_{h\in H} U \cdot h \cong U \times H$. □

Fiber bundles have the homotopy lifting property (HLP) for maps from paracompact spaces (sections 2.5. and 2.6. of cumulative lecture notes). That is, if $p\colon E \to X$ is a fiber bundle and $(h_t\colon A \to X)_{t\in[0,1]}$ is a homotopy where A is paracompact, and $f\colon A \to E$ is a map such that $pf = h_0$, then there exists a homotopy

$$(\bar{h}_t\colon A \to E)_{t\in[0,1]}$$

such that $\bar{h}_0 = f$ and $p\bar{h}_t = h_t$ for all $t \in [0,1]$.

Lemma A.5.2. *If the fiber bundle is a covering space, then the UHLP holds, unique homotopy lifting property: the homotopy* $(\bar{h}_t\colon A \to E)_{t\in[0,1]}$ *is uniquely determined by these conditions.*

Proof. It is enough to establish the case where A is a point. Indeed, a counterexample to uniqueness with some A implies a counterexample to uniqueness for some subspace of A which has just one element. In the case where A is a point, we are looking at the following assertion. Let two paths $\gamma, \lambda\colon [0,1] \to E$ be given such that $p\gamma = p\lambda$ and $\gamma(0) = \lambda(0) \in E$. Then $\gamma = \lambda$. Proof of this: let K be the subset of $[0,1]$ consisting of all t where $\gamma(t) = \lambda(t)$. Since K is nonempty and $[0,1]$ is connected, it is enough to show that K is open and closed in $[0,1]$. For that, choose an open covering of X by subsets U_i such that $p^{-1}(U_i) \to U_i$ is a trivial covering space, i.e., looks like the projection from a product to a factor: $U_i \times S_i \to U_i$. Let V_i be the preimage of U_i under $p\gamma = p\lambda$. Then $K \cap V_i$ is open and closed in V_i (because it can be described as the set of points in V_i where two continuous maps from V_i to the discrete space S_i agree). Since the union of the V_i is all of $[0,1]$, it follows that K is open and closed in $[0,1]$. □

A.6. Covering spaces and the fundamental group

Let X be a path connected space with base point. We write π_1 for the fundamental group $\pi_1(X,\star)$ in this section. Let $p\colon E \to X$ be a covering space and put $F = p^{-1}(\star)$. We use these data to construct a (left) action of π_1 on F.

Let $y \in F$ be given and let $g \in \pi_1$ represented by a path $\gamma\colon [0,1] \to X$ where $\gamma(0) = \gamma(1) = \star$. By the unique homotopy lifting property (proposition 2.5.2) there exists a unique path

$$\tilde{\gamma}\colon [0,1] \to E$$

such that $\tilde{\gamma}(0) = y$ and $p\tilde{\gamma} = \gamma$. Then we have $p\tilde{\gamma}(1) = \gamma(1) = \star$. We want to define

$$g \cdot y := \tilde{\gamma}(1) \in F\,.$$

It is necessary to show that this is well defined. Let $(\gamma_t\colon [0,1] \to X)_{t\in[0,1]}$ be a homotopy such that $\gamma_t(0) = \gamma_t(1) = \star$ for all $t \in [0,1]$. By the (unique) homotopy lifting property there is a unique *continuous* map

$$(s,t) \mapsto \tilde{\gamma}_t(s)$$

from $[0,1] \times [0,1]$ to E such that $\tilde{\gamma}_t(0) = y$ for all $t \in [0,1]$ and $p\tilde{\gamma}(t,s) = \gamma_t(s)$ for all $t, s \in [0,1]$. The continuity implies that

$$\tilde{\gamma}_t(1) \in F$$

does not depend on $t \in [0,1]$. That is what we needed to know. The standard properties of an action (the associativity property $g_1 \cdot (g_2 \cdot y) = (g_1 g_2) \cdot y$ and the property $1 \cdot y = 1$ for the neutral element 1 of π_1) are almost obvious.

Now it is easy to decide what we are going to do next. The covering spaces of X are the objects of a category. A morphism from $p\colon E_0 \to X$ to $q\colon E_1 \to X$ is a map $f\colon E_0 \to E_1$ such that $qf = p$. The sets with an action of π_1 are also the objects of a category. A morphism $F_0 \to F_1$ in that category is a map e (of sets) from F_0 to F_1 which intertwines the actions of π_1 , so that $e(g \cdot z) = g \cdot e(z)$ for all $z \in F_0$ and $g \in \pi_1$.

PROPOSITION A.6.1. *The above rule which to a covering space* $p\colon E \to X$ *assigns the fiber* $p^{-1}(\star)$ *is a functor from the category of covering spaces of* X *to the category of sets with an action of* π_1 *.*

PROOF. Let $p\colon E_0 \to X$ and $q\colon E_1 \to X$ be covering spaces and let $f\colon E_0 \to E_1$ be a morphism, so that $qf = p$. By restricting f we obtain a map of sets

$$p^{-1}(\star) \longrightarrow q^{-1}(\star).$$

Since this is the map that we want to assign to the morphism f, we need to show that it intertwines the actions of π_1 defined above. So let $y \in p^{-1}(\star)$ and let $g \in \pi_1$ be represented by $\gamma\colon [0,1] \to X$. We have the unique path

$$\tilde{\gamma}\colon [0,1] \to E_0$$

where $p\tilde{\gamma} = \gamma$ and $\tilde{\gamma}(0) = y$. Then $f\tilde{\gamma}$ is a lifted path for γ, too, but it is a lift to E_1 rather than E_0. It follows that

$$g \cdot f(y) = (f\tilde{\gamma})(1) = f(\tilde{\gamma}(1)) = f(g \cdot y)$$

which is what we had to show. □

A.7. Constructing maps between covering spaces

THEOREM A.7.1. *The functor of proposition A.6.1 is fully faithful. That is to say, for any two covering spaces* $p\colon E_0 \to X$ *and* $q\colon E_1 \to X$*, where* X *is path connected, it gives a bijection from the set of maps* $f\colon E_0 \to E_1$ *such that* $qf = p$ *to the set of maps* $p^{-1}(\star) \to q^{-1}(\star)$ *respecting the actions of* π_1 *.*

PROOF. We show injectivity first. Using notation as in the statement, suppose that u and v are two maps from E_0 to E_1 which are both over X, so that $qu = p$ and $qv = p$. Suppose that u and v agree on the subset $q^{-1}(\star)$. We need to show that $u = v$. Let $y \in E$. Choose a path $\alpha\colon [0,1] \to X$ such that $\alpha(0) = p(y)$ and $\alpha(1) = \star$. Lift to a path

$$\tilde{\alpha}\colon [0,1] \to E_0$$

such that $\tilde{\alpha}(0) = y$ and $p\tilde{\alpha} = \alpha$. Now $u\tilde{\alpha}$ and $v\tilde{\alpha}$ are two paths in E_1 which cover the same path α in X. They also have the same endpoint,

$$u\tilde{\alpha}(1) = v\tilde{\alpha}(1)$$

by our assumption that u and v agree on $p^{-1}(\star)$. Therefore they must agree by the UHLP, and so

$$u(y) = u\tilde{\alpha}(0) = v\tilde{\alpha}(0) = v(y).$$

Now we show surjectivity. So we begin with w from $F_0 = p^{-1}(\star)$ to $F_1 = q^{-1}(\star)$. Here it helps to begin with the following observation. Let α be a path in X from $\star$ to $x \in X$. Then α determines a bijection b_α from $p^{-1}(\star)$ to $p^{-1}(x)$. This is obtained by looking at the various lifts of α to E_0. Similarly, α determines a bijection c_α from $q^{-1}(\star)$ to $q^{-1}(x)$. Therefore we obtain a map

$$c_\alpha \circ w \circ b_\alpha^{-1}\colon p^{-1}(x) \longrightarrow q^{-1}(x).$$

If we use this for every $x \in X$, then we have a map from $E_0 o$ to E_1. But we must show that something is well defined: $c_\alpha \circ w \circ b_\alpha^{-1}$ doe not depend on α. So suppose that β is a path competing with α. Then we have $\bar{\beta} \circ \alpha$, representing an element g of π_1. We find that $b_\alpha = b_\beta \circ \mu_g$ where μ_g is multiplication on g (on the left), and similarly $c_\alpha = c_\beta \circ \mu_g$. Therefore

$$c_\alpha \circ w \circ b_\alpha^{-1} = c_\beta \circ \mu_g \circ w \circ \mu_g^{-1} \circ b_\beta^{-1} = c_\beta \circ w \circ b_\beta^{-1} \ .$$

Therefore we have a well defined map $p^{-1}(x) \to q^{-1}(x)$ for every $x \in X$, determined by w. If $x = \star$, this map is exactly w. Therefore we have constructed a map $f\colon E_0 \to E_1$ which satisfies $qf = p$ and which extends w. (Some more work should be done to show that this is continuous ... but this is not hard.) □

A.8. Constructing covering spaces

Definition A.8.1. *A space* X *is* locally path connected *if for every* $x \in X$ *and neighborhood* V *of* x *in* X*, there exists a neighborhood* U *of* x *in* V *such that any two points in* U *can be connected by a path in* V*.*

Theorem A.8.2. *Let* X *be a based space which is path connected and locally path connected and admits a covering by open subsets* U_i *such that for every* i *and every map* $S^1 \to U_i$ *, the composition* $S^1 \to U_i \to X$ *is nullhomotopic. Then the functor of proposition A.6.1 is an equivalence of categories.*

Proof. Let F be a set equipped with an action of $\pi_1 = \pi_1(X, \star)$. We need to construct a covering space $p\colon E \to X$ such that the set $p^{-1}(\star)$, as a set with action to π_1 , is isomorphic to F.
First we describe/construct E as a set and we construct $p\colon E \to X$ as a map of sets. This does not use any special properties of X. An element of E is an equivalence class of pairs (α, z) where $\alpha\colon [0,1] \to X$ is a path satisfying $\alpha(0) = \star$ and where $z \in F$. We say $(\alpha, z_0) \sim (\beta, z_1)$ if $\alpha(1) = \beta(1)$ and $[\bar{\beta} \circ \alpha] \cdot z_0 = z_1$ holds in F. Here $\bar{\beta}$ is the reverse path of β (as usual) and consequently $\bar{\beta} \circ \alpha$ represents an element $[\bar{\beta} \circ \alpha]$ of π_1. Define

$$p\colon E \to X$$

by taking the equivalence class of (α, z) to $\alpha(1) \in X$.
Now we need to define a topology on E making p into a covering projection. In general this is hard or impossible. But with our assumptions on X we can do it. Choose a covering of X by open subsets U_i such that for every i and every map $S^1 \to U_i$, the composition $S^1 \to U_i \to X$ is nullhomotopic. For each U_i choose a covering of U_i by open subsets V_{ij} such that any two points in V_{ij} can be connected by a path in U_i. Let E_{ij} be the preimage of V_{ij} under $p\colon E \to X$. So E_{ij} consists of equivalence classes of pairs (α, z) as above, where in addition $\alpha(1) \in V_{ij}$. Choose a point $x_{ij} \in V_{ij}$. It turns out that for $y \in V_{ij}$ we can make a preferred bijection

$$b_y\colon p^{-1}(x_{ij}) \longrightarrow p^{-1}(y).$$

This works as follows. Choose a path $\gamma\colon [0,1] \to U_i$ from x_{ij} to y. This exists by assumption. It determines a bijection from $p^{-1}(x_{ij})$ to $p^{-1}(y)$ which takes the equivalence class of a pair (α, z) where $\alpha(1) = x_{ij}$ to the equivalence class of $(\gamma \circ \alpha, z)$ where, obviously, $\gamma \circ \alpha(1) = y$. (Reparameterization of $\gamma \circ \alpha$ is understood.) A different choice of path, say $\gamma'\colon [0,1] \to U_i$ from x_{ij} to y, determines *the same* bijection from $p^{-1}(x_{ij})$ to $p^{-1}(y)$ due to the fact that there exists a homotopy $(\gamma_t\colon [0,1] \to X)_{t \in [0,1]}$ where $\gamma_0 = \gamma$ and

$\gamma_1 = \gamma'$. (This is true by our assumption on the inclusion $U_i \to X$.) Therefore we get a preferred bijection

$$h_{ij} \colon E_{ij} \to V_{ij} \times p^{-1}(x_{ij}).$$

We use that to define a topology on E_{ij} so that it is the product of V_{ij} and the discrete space or set $p^{-1}(x_{ij})$.

Now it is very important, but not completely obvious, that the topologies on E_{ij} that we have defined agree on the intersections $E_{ij} \cap E_{k\ell}$. More precisely, $E_{ij} \cap E_{k\ell}$ can be viewed as an open subspace of E_{ij} and also as an open subspace of $E_{k\ell}$, and we need to know that the identity map $E_{ij} \cap E_{k\ell} \to E_{ij} \cap E_{k\ell}$ is a homeomorphism for these two topologies. What could be the problem? For a point $y \in V_{ij} \cap V_{k\ell}$ and chosen paths γ from x_{ij} to y in U_{ij} as well as λ from $x_{k\ell}$ to y in $U_{k\ell}$, we obtain a path $\bar{\lambda} \circ \gamma$ from x_{ij} to $x_{k\ell}$, hence a bijection c_y from $p^{-1}(x_{ij})$ to $p^{-1}(x_{k\ell})$ by fiber transport. We understand already that this bijection does not depend on the choice of γ and λ. But it could depend on $y \in V_{ij} \cap V_{k\ell}$. Fortunately though, we can choose a neighborhood W of y in $V_{ij} \cap V_{k\ell}$ such that any two points in W can be connected by a path in $V_{ij} \cap V_{k\ell}$. The it is easy to verify that $c_y = c_z$ for all $z \in W$. This is good enough for us, i.e., it shows that the topology on $E_{ij} \cap E_{k\ell}$ is unambiguously defined, whether we view it as an open subspace of E_{ij} or as an open subspace of $E_{k\ell}$.

Therefore, at last, we can define a topology on E by saying that a subset of E is open if and only if its intersection with each of the E_{ij} is open in E_{ij}. Then E_{ij} is an open subspace of E and as before homeomorphic to $V_{ij} \times p^{-1}(x_{ij})$ by means of the map h_{ij}. It follows that $p \colon E \to X$ is a fiber bundle, and even a covering space, with bundle charts h_{ij}.

It remains to be shown that $p^{-1}(\star)$ is isomorphic, as a set with action of π_1, to F. A map u from $p^{-1}(\star)$ to F can be defined by taking the equivalence class of (α, z) to $[\alpha] \cdot z \in F$. Here $\alpha \colon [0,1] \to X$ is a path where $\alpha(0) = \alpha(1) = \star$. It is not hard to see that u is a bijection. In order to show that u intertwines the actions of π_1, we show this: if $\beta \colon [0,1] \to X$ is a path where $\beta(0) = \beta(1) = \star$, and if

$$\tilde{\beta} \colon [0,1] \longrightarrow E$$

is the unique path satisfying $p\tilde{\beta} = \beta$ and $\tilde{\beta}(0) =$ equivalence class of (α, z), then $\tilde{\beta}(1)$ is the equivalence class of the pair $(\beta \circ \alpha, z)$. We show it by noting that we can *define*

$$\tilde{\beta}(t) = \text{ equivalence class of the pair } (\beta|_{[0,t]} \circ \alpha, z)\ .$$

This formula defines a *continuous* map because the map is continuous in each of the open sets $\beta^{-1}(V_{ij}) \subset [0,1]$. □

A.9. Path components and fundamental group of covering spaces

It is surprising that there is something left to do after theorems A.7.1 and A.8.2, but there is. Let $p \colon E \to X$ be a covering space, where X is a based path connected space with base point $\star$. Put $F = p^{-1}(\star)$. As in proposition A.6.1 we regard F as a set with an action of $\pi_1 = \pi_1(X, \star)$.

Proposition A.9.1. *For $y \in F \subset E$, the homomorphism $\pi_1(E, y) \to \pi_1(X, \star)$ induced by $p \colon E \to X$ is injective and its image is the stabilizer group*[3] *of $y \in F$ for the action of $\pi_1(X, \star)$ on F.*

[3]Also known as isotropy group.

Proof. Let $\gamma\colon [0,1] \to E$ be a path such that $\gamma(0) = \gamma(1) = y$. Then $p\gamma$ represents $p_*[\gamma] \in \pi_1(X, \star)$. Since γ is a lift of $p\gamma$ satisfying $\gamma(0) = y$, we have $[p\gamma] \cdot y = \gamma(1) = y$, which shows that $p_*[\gamma]$ acts trivially on $y \in F$. Conversely, suppose given an element in $\pi_1(X, \star)$ which is in the stabilizer subgroup for $y \in F$. Represent the element by a path $\lambda\colon [0,1] \to X$ such that $\lambda(0) = \lambda(1) = \star$. Choose a lift

$$\tilde{\lambda}\colon [0,1] \to E$$

such that $\tilde{\lambda}(0) = y$. We have $\tilde{\lambda}(1) = [\lambda] \cdot y$ by definition of the right-hand side. Since we are assuming $[\lambda] \cdot y = y$ this means that $\tilde{\lambda}$ is a path in E from y to y, and so represents an element of $\pi_1(E, y)$. So we have a homomorphism from the stabilizer group (for $y \in F$ and the action of $\pi_1(X, \star)$ on F) to $\pi_1(E, y)$ given by

$$[\lambda] \mapsto [\tilde{\lambda}]$$

where $\tilde{\lambda}$ is the unique lift of λ satisfying $\tilde{\lambda}(0) = y$. It is fairly clear that this is well defined and inverse to p_*. □

Let $\pi_0(E)$ be the set of path components of E.

Proposition A.9.2. *The map $F \to \pi_0(E)$ taking $y \in F$ to its path component is surjective. Two elements $y, z \in F$ have the same image in $\pi_0(E)$ if and only if they are in the same orbit for the action of $\pi_1(X, \star)$ on F.*

Proof. Surjectivity: for an element $w \in E$, choose a path $\gamma\colon [0,1] \to X$ from $p(y')$ to $\star$. This is possible because we are still assuming that X is path connected. There is a unique

$$\tilde{\gamma}\colon [0,1] \to E$$

such that $p\tilde{\gamma} = \gamma$ and $\tilde{\gamma}(0) = w$. Then $\tilde{\gamma}(1) \in F$. This shows that the path component of E containing w has nonempty intersection with F.
Now let $y, z \in F$. Then

$$\begin{aligned} & y \text{ and } z \text{ are in the same orbit} \\ \Leftrightarrow\ & \exists \text{ path } \gamma \text{ from } \star \text{ to } \star \text{ in } X \text{ such that } [\gamma] \cdot y = z \\ \Leftrightarrow\ & \exists \text{ path } \tilde{\gamma}\colon [0,1] \to E \text{ such that } \tilde{\gamma}(0) = y \text{ and } \tilde{\gamma}(1) = z \\ \Leftrightarrow\ & y \text{ and } z \text{ are in the same path component of } E. \end{aligned}$$

□

Example A.9.3. Suppose that X satisfies the conditions of theorem A.8.2. Then there exists a covering space $p\colon E \to X$ such that $F = p^{-1}(\star)$, with the action of $\pi_1 = \pi_1(X, \star)$ of proposition A.6.1, is a *free transitive* π_1-set. In other words, for $y, z \in F$ there is exactly one $g \in \pi_1$ such that $g \cdot y = z$. In this case the action of π_1 on F has only one orbit, so E must be path connected by proposition A.9.2. Moreover, for any $y \in F$ the stabilizer subgroup for y and the action of π_1 on F is the trivial subgroup, so that $\pi_1(E, y)$ is trivial (has only one element).
Such a covering space $p\colon E \to X$ is then called a *universal covering space* of X. By proposition A.6.1, it is unique in the same way that sets with a free transitive action of $\pi_1 = \pi_1(X, \star)$ are unique. So if $p\colon E_0 \to X$ and $q\colon E_1 \to X$ are two universal covering spaces of the same X, then there exists a homeomorphism $u\colon E_0 \to E_1$ satisfying $qu = p$. But such a homeomorphism u need not be unique. (It is a good exercise to say how and why it can fail to be unique.)

A.10. The lifting lemma

Imagine two path-connected spaces X and Y with base points $\star_X$ and $\star_Y$, respectively. Let $q\colon E \to Y$ be a covering space where E is also path-connected, with base point $\star_E \in q^{-1}(\star_Y)$. Note that $q_*\colon \pi_1(E, \star_E) \to \pi_1(Y, \star_Y)$ is an *injective* homomorphism (see previous section). Let $f\colon X \to Y$ be a based map.

LEMMA A.10.1. *Suppose that the image of* $f_*\colon \pi_1(X, \star_X) \to \pi_1(Y, \star_Y)$ *is contained in the image of* $q_*\colon \pi_1(E, \star_E) \longrightarrow \pi_1(Y, \star_Y)$. *Then there exists at most one based map* $u\colon X \to E$ *making the following diagram commutative:*

$$\begin{tikzcd} & E \arrow[d, "q"] \\ X \arrow[ur, dotted, "u"] \arrow[r, "f"] & Y \end{tikzcd}$$

If X *is locally path connected, then there exists exactly one such map* u.

PROOF. Select $x_0 \in X$. Let us try to determine $u(x_0)$. Choose a path $\alpha\colon [0,1] \to X$ from $\star_X$ to x_0 . By the unique path lifting property of $q\colon E \to X$, there exists a unique path $\alpha^\sharp\colon [0,1] \to E$ such that $q \circ \alpha^\sharp = f \circ \alpha$ and $\alpha^\sharp(0) = \star_E$. If u is continuous, which we assume, then $u \circ \alpha$ is a path which satisfies $q \circ (u \circ \alpha) = f \circ \alpha$ and $u(\alpha(0)) = \star_E$. Therefore $u \circ \alpha = \alpha^\sharp$ and

$$u(x_0) = \alpha^\sharp(1) \ .$$

This looks like a determination of x_0 , but we need to show that it is unambiguous. So let $\beta\colon [0,1] \to X$ be another path from $\star_X$ to x_0 . Is it true that $\beta^\sharp(1) = \alpha^\sharp(1)$? The answer is yes, because the concatenation

$$\bar{\beta} \circ \alpha$$

(α followed by reverse of β) is a closed path (loop) in X representing an element of $\pi_1(X, \star_X)$. By assumption the loop $f \circ (\bar{\beta} \circ \alpha)$ in Y can be lifted to a loop in E based at $\star_E$, and it is easy to see that this must be $\alpha^\sharp$ followed by the reverse of $\beta^\sharp$. Therefore $\beta^\sharp(1) = \alpha^\sharp(1)$.

Consequently we have an unambiguous definition of u. This proves the first part of the lemma. But it is not clear that u is a *continuous* map. On the other hand, we constructed u in such a way that $u \circ \gamma$ is continuous for every path γ in X, and we can exploit this. Fix $x_0 \in X$ as before. Choose an open neighborhood W of $f(x_0)$ in Y such that $q^{-1}(W) \subset E$ is homeomorphic to a product $F \times W$ for some set F (by a homeomorphism h from $q^{-1}(W)$ to $F \times W$ such that h followed by projection to W agrees with q). Now $h \circ u$ is defined on $f^{-1}(W)$, and it suffices to show that it is continuous at x_0. For that it suffices to show that $h_F \circ u$ is constant in a neighborhood of x_0 , where $h_F\colon q^{-1}(W) \to F$ is the first coordinate of h. If X is locally path connected then there exists a neighborhood V of x_0 in $f^{-1}(W) \subset X$ such that every $x_1 \in V$ can be connected to x_0 by a path γ in $f^{-1}(W)$. Since $u \circ \gamma$ is continuous, $h_F \circ u \circ \gamma$ must be constant. So $h_F(u(x_1)) = h_F(u(x_0))$, showing that $h_F \circ u$ is constant on V. $\square$

APPENDIX B

An overview of singular homology and cohomology

B.1. The singular chain complex

This section is an attempt to outline the *standard* definitions of homology and cohomology of topological spaces, and to state the most important theorems about them without proofs. The technical names are *singular homology* and *singular cohomology*. Reason for this attempt: (i) educational, (ii) facilitates communication, (iii) some readers of this may be more familiar with singular homology/cohomology and may want to see a comparison. It is not obvious from the definitions that the singular homology groups of a space X are isomorphic to the homology groups that I have championed, based on mapping cycles. (I believe they are, but I am not planning to prove it here.) Also, it is not obvious that the singular cohomology groups of a space X are isomorphic to the cohomology groups $H^n(X)$ based on mapping cycles. Indeed this is not always the case, and it is easy to give counterexamples. I suspect that the cohomology groups $H^n(X)$ based on mapping cycles are isomorphic to the *Čech* cohomology groups of X, a better known variant of cohomology, at least when X is paracompact. The good news: for a CW-space X, or a space X which is homotopy equivalent to a CW-space, the singular homology/cohomology groups of X are certainly isomorphic to the homology/cohomology groups of X based on mapping cycles. In view of that, I shall not introduce separate notation for singular (co-)homology groups, except in cases where the distinction matters and it is not obvious what is meant.

Let $\Delta^n = \{(t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \forall i : t_i \geq 0,\ \Sigma_i t_i = 1\}$ be the geometric n-simplex, a subspace of $\mathbb{R}^{n+1}$ with the subspace topology. This is incredibly important in singular homology. (We met it previously in connection with semi-simplicial sets.) There are continuous maps

$$e_i \colon \Delta^{n-1} \to \Delta^n$$

defined by inserting a zero in position i; that is,

$$e_i(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}).$$

Definition B.1.1. The singular chain complex $\mathrm{Sg}(X)$ of a space X is defined as follows. The abelian group $\mathrm{Sg}(X)_n$ is the free abelian group generated by the set of all continuous maps $\sigma \colon \Delta^n \to X$. The differential d from $\mathrm{Sg}(X)_n$ to $\mathrm{Sg}(X)_{n-1}$ is defined (on the specified generators) by

$$\sigma \mapsto \sum_{i=0}^{n} (-1)^i \sigma \circ e_i \ .$$

Therefore an element in $\mathrm{Sg}(X)_n$ can be described in the form $\Sigma_\sigma a_\sigma \cdot \sigma$, a formal sum indexed by all continuous maps $\sigma \colon \Delta^n \to X$. The numbers a_σ belong to $\mathbb{Z}$, but only finitely many of them are nonzero. (It is an exercise to show that $dd = 0$ in $\mathrm{Sg}(X)$.) A

continuous map $\Delta^n \to X$ is sometimes called a *singular* n*-simplex* of X, for complicated historical reasons.[1]

Definition B.1.2. The singular homology group $H_n(X)$ of a space X is defined to be $H_n(\mathrm{Sg}(X))$, the n-th homology group of the chain complex $\mathrm{Sg}(X)$.

Example B.1.3. Suppose that X is a point. Then there is exactly one continuous map $\sigma\colon \Delta^n \to X$, for every $n \geq 0$. Therefore $\mathrm{Sg}(X)_n = \mathbb{Z}$ for all $n \geq 0$, whereas $\mathrm{Sg}(X)_n = 0$ for $n < 0$. The differential $d\colon \mathrm{Sg}(X)_n \to \mathrm{Sg}(X)_{n-1}$ is multiplication with $\sum_{i=0}^{n}(-1)^i$. That number simplifies to 0 if n is odd and positive. It simplifies to 1 if n is even and positive. From there, it is easy to deduce that $H_0(\mathrm{Sg}(X)) \cong \mathbb{Z}$ and all other homology groups are zero.

For a chain complex C with differentials $d\colon C_n \to C_{n-1}$ we can define another chain complex $\hom(C,\mathbb{Z})$ as follows. We set $\hom(C,\mathbb{Z})_k = \hom(C_{-k},\mathbb{Z})$ and define the differential $d^*\colon \hom(C,\mathbb{Z})_k \to \hom(C,\mathbb{Z})_{k-1}$ by pre-composition with the differential $d\colon C_{-k+1} \to C_{-k}$. Note that if $C_k = 0$ for $k < 0$, which is often the case, then $\hom(C,\mathbb{Z})_\ell = 0$ for $\ell > 0$, which looks a little strange.[2]

Definition B.1.4. The singular cohomology group $H^n(X)$ of a space X is defined to be $H_{-n}(\hom(\mathrm{Sg}(X),\mathbb{Z}))$.

These definitions may look very mysterious. To make them seem less so, let me suggest that people who teach homology/cohomology in this way believe that there is an important analogy going on between the category of topological spaces and the category of chain complexes. They want to express this as quickly as possible. It is important to them that

$$X \mapsto \mathrm{Sg}(X)$$

is a covariant functor from the category of topological spaces to the category of chain complexes. Indeed, a continuous map $f\colon X \to Y$ induces homomorphisms $f_*\colon \mathrm{Sg}(X)_n \to \mathrm{Sg}(Y)_n$ by $\Sigma_\sigma a_\sigma \cdot \sigma \mapsto \Sigma_\sigma a_\sigma \cdot (\sigma \circ f)$. Letting n vary, these define a chain map from $\mathrm{Sg}(X)$ to $\mathrm{Sg}(Y)$, still denoted by f_*. It follows that singular homology is also a covariant functor, $X \mapsto H_n(X)$, and singular cohomology is a functor, $X \mapsto H^n(X)$.

Theorem B.1.5. *Suppose that* $f, g\colon X \to Y$ *are homotopic maps. Then the chain maps* $f_*, g_*\colon \mathrm{Sg}(X) \to \mathrm{Sg}(Y)$ *are chain homotopic.*

The proof is quite technical.

Corollary B.1.6. *If* f *and* g *are homotopic maps from* X *to* Y*, then they induce the same homomorphisms in singular homology,*

$$f_* = g_*\colon H_n(X) \to H_n(Y).$$

They also induce the same homomorphisms in singular cohomology,

$$f^* = g^*\colon H^n(Y) \to H^n(X).$$

[1] In the case where X has the structure of a simplicial complex there are some distinguished *injective* continuous maps $\Delta^n \to X$, and these would probably have qualified as nonsingular n-simplices in the language of the ancients.

[2] For this reason some people prefer another convention according to which $\hom(C,\mathbb{Z})_k = \hom(C_k,\mathbb{Z})$. These people must live with the consequence that d^* is a homomorphism from $\hom(C,\mathbb{Z})_k$ to $\hom(C,\mathbb{Z})_{k+1}$. As a sign of their acceptance they say that $\hom(C,\mathbb{Z})$ is a *cochain complex*, not a chain complex.

Another important topic is the Mayer-Vietoris sequence. In singular homology/cohomology this can be handled as follows. Let X be a space with open subsets V and W such that $V \cup W = X$. We introduce a chain subcomplex

$$\mathrm{Sg}^{V,W}(X) \subset \mathrm{Sg}(X)$$

as follows. An element $\Sigma_\sigma a_\sigma \cdot \sigma$ of $\mathrm{Sg}(X)_n$ belongs to the subcomplex if every $\sigma\colon \Delta^n \to X$ which appears with a nonzero coefficient $a_\sigma \in \mathbb{Z}$ in the sum lands in either V or W, or both. To put it differently, $\mathrm{Sg}^{V,W}(X)$ is the free abelian group generated by the continuous maps $\sigma\colon \Delta^n \to X$ for which $\sigma(\Delta^n)$ is contained in V or in W.

THEOREM B.1.7. *The inclusion* $\mathrm{Sg}^{V,W}(X) \subset \mathrm{Sg}(X)$ *is a chain homotopy equivalence.*

The proof of this is also quite technical.

COROLLARY B.1.8. *In the circumstances of theorem B.1.7, there is a long exact sequence of singular homology groups*

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(X) \arrow[d, "\partial"] & & & \\
& H_n(V \cap W) \arrow[r] & H_n(V) \oplus H_n(W) \arrow[r] & H_n(X) \arrow[d, "\partial"] & \\
& & & H_{n-1}(V \cap W) \arrow[r] & \cdots
\end{tikzcd}$$

PROOF. This is the long exact sequence of homology groups associated with a certain short exact sequence of chain complexes

$$0 \longrightarrow \mathrm{Sg}(V \cap W) \longrightarrow \mathrm{Sg}(V) \oplus \mathrm{Sg}(W) \longrightarrow \mathrm{Sg}^{V,W}(X) \longrightarrow 0\,.$$

Here the chain map $\mathrm{Sg}(V \cap W) \longrightarrow \mathrm{Sg}(V) \oplus \mathrm{Sg}(W)$ is the formal difference of the two inclusion maps $\mathrm{Sg}(V \cap W) \to \mathrm{Sg}(V)$ and $\mathrm{Sg}(V \cap W) \to \mathrm{Sg}(W)$, and the chain map $\mathrm{Sg}(V) \oplus \mathrm{Sg}(W) \longrightarrow \mathrm{Sg}^{V,W}(X)$ is equal to the inclusion on each of the summands $\mathrm{Sg}(V)$ and $\mathrm{Sg}(W)$. We use theorem B.1.7 as a license for writing $H_n(\mathrm{Sg}^{V,W}(X)) \cong H_n(\mathrm{Sg}(X)) = H_n(X)$. □

COROLLARY B.1.9. *In the circumstances of theorem B.1.7, there is a long exact sequence of singular cohomology groups*

$$\begin{tikzcd}
\cdots & H^{n+1}(X) \arrow[l] & & & \\
& H^n(V \cap W) \arrow[u, "\delta"] & H^n(V) \oplus H^n(W) \arrow[l] & H^n(X) \arrow[l] & \\
& & & H^{n-1}(V \cap W) \arrow[u, "\delta"] & \cdots \arrow[l]
\end{tikzcd}$$

PROOF. Apply $\hom(-, \mathbb{Z})$ to the short exact sequence of chain complexes in the proof of the previous corollary. The result is another short exact sequence of chain complexes. (Some small checks are required here.) The associated long exact sequence of homology groups is the one we require. □

Note that in the last corollary we didn't need any special conditions, such as X must be paracompact.

Example B.1.10. Let X be any space. It is relatively easy to show that the singular homology group $H_0(X)$ is isomorphic to the free abelian group generated by the path components of X. (This is in agreement with $H_0(X)$ defined in terms of mapping cycles.) More specifically, given $x \in X$ there is a unique continuous map $\Delta^0 \to X$ with image $\{x\}$, and this determines an element $[x]$ of $H_0(\mathrm{Sg}(X)) = H_0(X)$ for obvious reasons. We have $[x] = [y]$ if and only if x and y are in the same path component of X. By choosing one x in each path component of X, we obtain a set of free generators for $H_0(X)$.
It is also relatively easy to show that the singular cohomology group $H^0(X)$ is isomorphic to $\hom(H_0(X), \mathbb{Z})$. We can also say: elements of the singular cohomology group $H^0(X)$ are functions from X to $\mathbb{Z}$ which are constant on each path component. This is in contrast to $H^0(X)$ defined in terms of mapping cycles, where we had the following description: elements of $H^0(X)$ are *continuous* functions from X to $\mathbb{Z}$. (A continuous function from X to $\mathbb{Z}$ is certainly also constant on path components of X, but there are cases when that is not enough.) To be more specific, let us try

$$X = \{0\} \cup \{2^{-i} \mid i = 0, 1, 2, 3, \dots\},$$

a subspace of $\mathbb{R}$. Then $H^0(X)$ defined with mapping cycles is a free abelian group with countably many generators (exercise), and so it is countable as a set. But the singular cohomology group $H^0(X)$ is a product of copies of $\mathbb{Z}$, one copy for each $x \in X$, and so it is uncountable as a set. *Nota bene*: this X is not a CW-space and it is not even homotopy equivalent to a CW-space. So this example does not disprove the claim that singular cohomology and mapping cycle cohomology agree for CW-spaces.

How did the singular chain complex of X come to prominence? I assume that Poincaré in the late 19th century worked with simplicial complexes (see lecture notes WS13-14) and knew how to associate a chain complex with such a thing. This was already close to the definition of $\mathrm{Sg}(X)$, but as indicated above it did not use all the continuous maps $\Delta^n \to X$. Instead it used only one for each standard inclusion of a simplex in the simplicial complex. Later, when the definition of topological spaces emerged, there was a need for a definition of *chain complex of* X which did not depend on a simplicial complex structure on X, especially in cases where X was not homeomorphic to a simplicial complex. Maybe topologists then came to a gradual agreement to think big and to incorporate all the continuous maps $\Delta^n \to X$ for all $n \geq 0$ into the definition of $\mathrm{Sg}(X)$.

There is another justification for $\mathrm{Sg}(X)$ which is probably not among the reasons why it was created. I mentioned this in section 11.4 of lecture notes WS13-14. Perhaps it came too early. Let $\mathcal{H}o\mathcal{T}op$ be the homotopy category of topological spaces. Let $\mathcal{H}o\mathcal{T}op_{CW}$ be the homotopy category of CW-spaces. (The objects are the CW-spaces, and the morphisms from X to Y are homotopy classes of continuous maps $X \to Y$.)

Theorem B.1.11. *The inclusion functor* $\mathcal{H}o\mathcal{T}op_{CW} \to \mathcal{H}o\mathcal{T}op$ *has a right adjoint.*

Apologies for the abstract formulation; the meaning is as follows. For any topological space Y we can find a CW-space $Y^\natural$ and a map $u\colon Y^\natural \to Y$ such that the map

$$[X, Y^\natural] \longrightarrow [X, Y]$$

given by composition with u is a bijection *whenever* X *is a CW-space.* The square brackets denote sets of homotopy classes of maps. It is not claimed that $Y^\natural$ is uniquely determined

by Y, but it is easy to see that it must be unique up to homotopy equivalence. In particular, if Y is already a CW-space, then $Y^\natural \simeq Y$. The construction of such a space $Y^\natural$ in general takes a bit of thought. One solution is as follows. Starting with a space Y, form the semi-simplicial set SY where SY_n is the set of continuous maps from Δ^n to Y. The face operator $f^*\colon SY_n \to SY_n$ corresponding to a monotone injective map $f\colon \{0,\dots,m\} \to \{0,\dots,n\}$ is given by composition with $f_*\colon \Delta^m \to \Delta^n$, a so-called *linear* map; in coordinates, insert zeros in the positions j where $j \notin \mathrm{im}(f)$. The geometric realization $|SY|$ is then a CW-space and it comes with a canonical continuous map $|SY| \to X$. That map can be taken as $u\colon Y^\natural \to Y$.

Lemma B.1.12. *The singular chain complex* $\mathrm{Sg}(Y)$ *is naturally isomorphic to the cellular chain complex of the CW-space* $|SY|$.

The proof is not meant to be difficult. Note that $|SY|$ has one n-cell for each element of SY_n, that is, for each continuous map σ from Δ^n to Y. This contributes a direct summand $\mathbb{Z}$ to $C(|SY|)_n$, the n-th chain group of the cellular chain complex of $|SY|$. The same σ contributes a direct summand $\mathbb{Z}$ to the n-th chain group of the singular chain complex $\mathrm{Sg}(Y)_n$. In this way it becomes clear what the isomorphism should look like.

B.2. Singular homology and cohomology of pairs

Under construction.

B.3. Products in singular homology and singular cohomology

Under construction.