CURVATURE AND FINITE DOMINATION

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Abstract. Upper bounds obtained by Gromov on the Betti numbers of certain closed Riemannian manifolds are shown to be upper bounds on the minimum number of cells in CW–spaces dominating such manifolds.

In [Gro1], Gromov obtains a bound on the sum of the Betti numbers of a closed Riemannian manifold V in terms of a lower bound on the sectional curvature and an upper bound on the diameter. In more detail: Fix a field F, and let $\beta_i = \beta_i(V;F)$ be the dimension over F of $H_i(V;F)$. Let $D = D(V)$ be the diameter of V.

Theorem A. [Gro1] *There exists a constant $\mathcal{C} = \mathcal{C}(n)$ such that every closed connected Riemannian n-manifold V satisfies*

$$\sum_0^n \beta_i \leq \mathcal{C}^{1+\kappa D}$$

provided the sectional curvature of V is bounded from below by $-\kappa^2$, where $\kappa \geq 0$.

Corollary. *If V has non–negative sectional curvature, then the sum of the Betti numbers is $\leq \mathcal{C}$.*

A stronger theorem can be obtained with a small change in Gromov's proof. *Terminology:* For spaces X and Y, we say that X *dominates* Y if there exist maps

$$Y \xrightarrow{\ i\ } X \xrightarrow{\ r\ } Y$$

such that ri is homotopic to the identity.

Theorem B. *There exists a constant $\mathcal{C} = \mathcal{C}(n)$ such that every closed connected Riemannian n-manifold V can be dominated by a CW-space X having at most*

$$\mathcal{C}^{1+\kappa D}$$

cells. (Assume as before that the sectional curvature of V is bounded from below by $-\kappa^2$, where $\kappa > 0$.)

Note that Theorem B implies an upper bound for the minimum number of generators of $\pi_1(V)$. This is in agreement with [Gro2], but less explicit.

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The referee has asked me to point out that Theorem B explains better than Theorem A does how a lower bound on the sectional curvatures of V restricts the topological complexity of V. It gives an upper bound on the minimum *number* of cells in a CW–space X dominating V, but there is no bound for the complexity of the attaching maps for the cells of X. For example, in dimension 3 there are infinitely many different homology types of compact Riemannian manifolds of constant sectional curvature 1 (lens spaces). In dimension 7, there are infinitely many different homology types of compact *simply connected* Riemannian manifolds having strictly positive sectional curvature (the examples of Allof and Wallach, [AlWa]).

The referee has also drawn my attention to [Abr1] and [Abr2]. Abresch extended Gromov's result to asymptotically non-negatively curved manifolds (which are complete by definition, but not always closed). He obtained more explicit bounds on the Betti numbers. His result is

$$(*) \qquad \sum_i \beta_i(V^n) \leq c(n) \cdot \exp\left(\frac{15n-13}{4} \cdot b_1(V^n)\right)$$

where $b_1(V^n)$ is a real number (not a Betti number) measuring to some extent the "amount" of negative curvature in V and, in Abresch's own words,

the function $c(n)$ can be effectively estimated by an expression which grows exponentially in n^3.

If V is closed, with diameter D and sectional curvature bounded from below by $-\kappa^2$ everywhere, then $b_1(V) \leq \kappa D$ by [Abr1, 2.3].

Again, a small change in Abresch's proof shows that inequality (*) and the estimate for $c(n)$ remain correct if the sum of the Betti numbers is replaced by the minimum number of cells in a CW–space dominating V.

The changes should be made in §2.3 of [Gro1], and in §1 of [Abr2]. This is where the *Leray spectral sequence* appears. Gromov refers to [Groth] for details. Grothendieck's account is of course "homological". A more geometric explanation of the Leray spectral sequence (using *homotopy direct limits*) is available. This is where we start.

1. The Leray spectral sequence

Let $\mathcal{A}$ be a simplicial complex. We shall regard $\mathcal{A}$ as a category: objects are the simplices of $\mathcal{A}$, and morphisms are the inclusion maps. For a contravariant functor $\mathcal{Z}$ from $\mathcal{A}$ to the category of spaces, let

$$|\mathcal{Z}| := \Big(\coprod_{s\subset\mathcal{A}} \mathcal{Z}(s) \times s\Big)/\sim$$

where the coproduct runs over all simplices $s \subset \mathcal{A}$ and $\sim$ stands for the "usual" relations, $(f^*a, b) \sim (a, b)$ whenever $a \in \mathcal{Z}(s')$, $b \in s$, and $f : s \hookrightarrow s'$. Note that $|\mathcal{Z}|$ projects to $\mathcal{A}$ by $(a, b) \mapsto b$. The inverse image of the k–skeleton of $\mathcal{A}$ under this map is the *vertical k–skeleton* of $|\mathcal{Z}|$, denoted by $|\mathcal{Z}|(k)$.

The construction $|\mathcal{Z}|$ is a special case of a *homotopy direct limit.* The notion goes back to [Se] and the standard reference is [BK].

With $\mathcal{A}$ and $\mathcal{Z}$ as above, let X be a space and let $\tau : \mathcal{Z} \longrightarrow X$ be a natural transformation (where we think of X as a constant functor on $\mathcal{A}$). This induces a map

$$\tau_* : |\mathcal{Z}| \longrightarrow X \quad ; \quad (a, b) \mapsto \tau(a)\,.$$

Example 1.2. Let X be a $CW - space$, and let $\{X_\gamma \mid \gamma \in \Gamma\}$ be a collection of CW-subspaces of X. Let $\mathcal{A}$ be the *nerve* of the collection of subspaces: i.e., $\mathcal{A}$ has one k–simplex for each finite subset $\Gamma' \subset \Gamma$ such that

$$\bigcap_{\gamma \in \Gamma'} X_\gamma \quad \neq \quad \emptyset.$$

Of course, we let $\mathcal{Z}(s) = \cap X_\gamma$ where the intersection is taken over all γ which are vertices of s. The inclusions $\mathcal{Z}(s) \subset X$ define a natural transformation

$$\tau : \mathcal{Z} \longrightarrow X\ .$$

If X is the union of the X_γ, then $\tau_* : |\mathcal{Z}| \longrightarrow X$ is a homotopy equivalence.

The proof consists essentially in showing that the fibers of τ_* are contractible spaces. For $x \in X$, the fiber of τ_* over x is homeomorphic to the full simplex spanned by vertices γ such that $x \in X_\gamma$.

Example 1.3. Let X be a smooth n–manifold, and let $\{X_\gamma \mid \gamma \in \Gamma\}$ be a collection of open subsets of X. Define $\mathcal{A}$, $\mathcal{Z}$ and τ as before. *If X is the union of the X_γ, then $\tau_* : |\mathcal{Z}| \longrightarrow X$ is a homotopy equivalence.*

The proof is by reduction to the previous example (use triangulations of X). Details are left to the reader. The smoothness assumption is unnecessary, but it makes the proof easier.

In the situation of 1.2 or 1.3, assuming that X is the union of the X_γ, we have the canonical filtration of $|\mathcal{Z}|$ by vertical skeletons $|\mathcal{Z}|(k)$. Now a filtration of a space *always* gives rise to a filtration of its singular chain complex, and then to a spectral sequence converging to the homology of the space. Here we obtain a spectral sequence converging to the homology of $|\mathcal{Z}|$, which is the homology of X. This is the *Leray* spectral sequence. We are not going to use it. We will use the filtration of $|\mathcal{Z}|$ by vertical skeletons.

2. Cell content

The reader should now have [Gro1] before his/her eyes, more specifically, section 2.3 of [Gro1]. First a remark on terminology: As I understand it, Gromov means by a *ball* in the Riemannian manifold V a certain open subset $B = B(x, R)$ of V, equipped with the (sometimes additional) structure of a center x and radius R. For example, if V is closed and D is the diameter of V, then $B(x, 2D)$ and $B(x, 10D)$ must be regarded as different balls in V, although the underlying subsets of V are both equal to V. If $B = B(x, R)$ is a ball in V, and λ is a positive real number, then λB denotes the ball $B = B(x, \lambda R)$.

In section 2.3, Gromov defines the *content* of a ball B in V as the rank of the inclusion homomorphism

$$H_*(\tfrac{1}{5}B; F) \longrightarrow H_*(B; F)\ .$$

Further, he writes:

Quotation 2.1. "Take a ball B and cover the concentric ball $\frac{1}{5}B$ by some open balls B_i, where $i = 1, \dots, N$, all of the same radius. Consider also the concentric coverings $\{\lambda_j B_i\}$, where $j = 0, 1, \dots, n+1$ and $\lambda_j = 10^j$. Suppose that all balls $5\lambda_j B_i$ (where $j = 0, \dots, n+1$ and $i = 1, \dots, N$) are contained in B, and let the contents of these balls be bounded by a constant p, that is

$$\mathrm{Cont}(5\lambda_j B_i) \leq p\ .$$

Denote by J the index [Gro1,2.2] of the system $\{5\lambda_{n+1} B_i\}$, where $i = 1, \dots, N$.

The content of B satisfies the following inequality:

$$\mathrm{Cont}(B) \leq (n+1)pJ\ ."$$

(End of quotation.)

Definition 2.2. The *cell content* of a ball B in V is $\leq q$ if there exists a CW–space Y with at most q cells, and maps

$$\tfrac{1}{5}B \xrightarrow{f} Y \xrightarrow{g} B$$

such that gf is homotopic to the inclusion.

Lemma 2.3. Keeping the hypotheses of 2.1 in all other respects, suppose that the *cell contents* of the balls $5\lambda_j B_i$ (where $j = 0, \dots, n$ and $i = 1, \dots, N$) are bounded by a constant p.

Then the cell content of B is not greater than pJ.

(This will be proved in the next section.) Now return to the assumptions and notation of Theorem B above; in particular, let D be the diameter of V. Lemma 2.3 implies, by arguments identical with Gromov's, that for any $x \in V$ the cell content of $B(x, 10D)$ is bounded by $\mathcal{C}^{1+\kappa D}$ for suitable $\mathcal{C}$ independent of V (but depending on the dimension n). Since

$$B(x, 10D) = \tfrac{1}{5}B(x, 10D) = V \qquad \text{"as sets"},$$

this means that V can be dominated by a cell complex with at most $\mathcal{C}^{1+\kappa D}$ cells.

3. Proof of the lemma

Using example 1.3, we can deduce lemma 2.3 from the following statement.

Proposition 3.1. *Let $\mathcal{A}$ be a compact simplicial complex with J simplices. Let*

$$\mathcal{Z}_0 \xrightarrow{T_0} \mathcal{Z}_1 \xrightarrow{T_1} \mathcal{Z}_2 \to \cdots \to \mathcal{Z}_n \xrightarrow{T_n} \mathcal{Z}_{n+1}$$

be a diagram of functors (contravariant, from $\mathcal{A}$ to spaces) and natural transformations. Assume that, for each simplex $s \subset \mathcal{A}$ and each $j \in \{0, 1, \dots, n\}$, the map from $\mathcal{Z}_j(s)$ to $\mathcal{Z}_{j+1}(s)$ given by T_j has a (strict) factorization

$$(*) \qquad \mathcal{Z}_j(s) \xrightarrow{\alpha_{j,s}} Y_{j,s} \xrightarrow{\beta_{j,s}} \mathcal{Z}_{j+1}(s)$$

where $Y_{j,s}$ is homotopy equivalent to a CW–space with not more than p cells. Then the map from $|\mathcal{Z}_0|(n)$ to $|\mathcal{Z}_{n+1}|$ induced by $T_n T_{n-1} \dots T_1 T_0$ has a factorization

$$(**) \qquad |\mathcal{Z}_0|(n) \longrightarrow W \longrightarrow |\mathcal{Z}_{n+1}|$$

where W is homotopy equivalent to a CW–space with not more than pJ cells.

Interpretation 3.2. Let $\mathcal{A}$ be the nerve of the collection of open sets $\{\lambda_{n+1}B_i\}$ (notation of 2.1 and 2.3 above). Define $\mathcal{Z}_j$ by

$$\mathcal{Z}_j(s) = \bigcap_{i \text{ vertex of } s} \lambda_j B_i \qquad \text{for } s \subset \mathcal{A} \text{ and } 0 \le j \le n+1.$$

The natural transformations T_j are given by inclusion for $0 \le j \le n$. Gromov's interpolation argument (in [Gro1, 2.3]) and the assumptions in 2.2 above imply that the factorizations (*) exist. (They can be made strict by converting certain maps into fibrations.) Therefore the factorization (**) exists. Now

$$|\mathcal{Z}_0| \simeq \bigcup_{1\le i\le N} B_i$$

by 1.3, and the right–hand side contains $\frac{1}{5}B$. Similarly

$$|\mathcal{Z}_{n+1}| \simeq \bigcup_{1\le i\le N} \lambda_{n+1} B_i$$

and the right–hand side is contained in B. We conclude that the inclusion

$$\tfrac{1}{5}B \hookrightarrow B$$

has a factorization

$$\tfrac{1}{5}B \longrightarrow W \longrightarrow B$$

where W is homotopy equivalent to a CW–space with not more than pJ cells. (Never mind the difference between $|\mathcal{Z}_0|$ and $|\mathcal{Z}_0|(n)$: the inclusion of $|\mathcal{Z}_0|(n)$ in $|\mathcal{Z}_0|$ is n–connected, so any map from an n–manifold such as $\frac{1}{5}B$ to $|\mathcal{Z}_0|$ can be deformed into $|\mathcal{Z}_0|(n)$.) This shows that the cell content of B is at most pJ.

Proof of 3.1. Without loss of generality, $\dim(\mathcal{A}) \le n$, and then $|\mathcal{Z}_0|(n) = |\mathcal{Z}_0|$. Define a new contravariant functor $\mathcal{Y}$ from $\mathcal{A}$ to spaces by

$$\mathcal{Y}(s) = Y_{i,s} \qquad \text{where } i = n - \dim(s).$$

Induced maps are defined as follows. For simplices $s \subset t$ (proper inclusion) and $i = n - \dim(s)$ and $j = n - \dim(t)$ use the composition

$$Y_{j,t} \xrightarrow{\beta_{j,t}} \mathcal{Z}_{j+1}(t) \longrightarrow \mathcal{Z}_i(t) \xrightarrow{\mathcal{Z}_i(s\subset t)} \mathcal{Z}_i(s) \xrightarrow{\alpha_{i,s}} Y_{i,s}$$

where the unlabelled arrow is a specialization of $T_{i-1}\dots T_{j+1}$, or the identity if i equals $j+1$. (Check that this gives a functor.) There are obvious natural transformations

$$\mathcal{Z}_0 \longrightarrow \mathcal{Y} \longrightarrow \mathcal{Z}_{n+1}$$

with composition equal to $T_nT_{n-1}\dots T_0$. Hence we have a factorization

$$|\mathcal{Z}_0| \longrightarrow |\mathcal{Y}| \longrightarrow |\mathcal{Z}_{n+1}|$$

of the map induced by $T_n \dots T_0$. To complete the proof, apply the next lemma.

Lemma 3.3. *Let $\mathcal{A}$ be a simplicial complex with J simplices, and let $\mathcal{Y}$ be a contravariant functor from $\mathcal{A}$ to spaces. Assume that each $\mathcal{Y}(s)$ is homotopy equivalent to a CW–space with at most p cells. Then $|\mathcal{Y}|$ is homotopy equivalent to a CW–space with at most pJ cells.*

Proof. Use induction on J. For the induction step, let s be a simplex of maximal dimension in $\mathcal{A}$, and let $\mathcal{A}'$ be the complement of the interior of s in $\mathcal{A}$. Let $\mathcal{Y}'$ be the restriction of $\mathcal{Y}$ to $\mathcal{A}'$. Note that $|\mathcal{Y}|$ is the pushout of a diagram

$$\mathcal{Y}(s) \times s \hookleftarrow \mathcal{Y}(s) \times \partial s \to |\mathcal{Y}'|. \quad \square$$

4. Cell Rank

Now switch to §1 of [Abr2]. (I shall use somewhat different notation to be consistent, starting with V, U, U_0 where Abresch writes M, X, Y, respectively.) For open subsets $U_0 \subset U$ of the manifold V^n and $t > 0$, Abresch defines

$$\begin{aligned} \mathrm{rk}_j(U, U_0) &:= \mathrm{rank}(H_j(U_0; F) \to H_j(U; F)) \\ \mathrm{rk}^t_*(U, U_0) &:= \sum_{j \geq 0} \mathrm{rk}_j(U, U_0) \cdot t^j \,. \end{aligned}$$

Supposing that $B^0_i \subset B^1_i \subset \cdots \subset B^{n+1}_i$, for $1 \leq i \leq N$, are open subsets of V such that

$$U_0 \subset \bigcup_{i=1}^{N} B^0_j \quad \text{and} \quad U \supset \bigcup_{i=1}^{N} B^{n+1}_i \,,$$

he states the following lemma (which replaces 2.1).

Quotation 4.1. *Let $t > 0$, $t^{-1} \in \mathbb{N}$, and suppose that any B^n_i intersects at most t^{-1} distinct sets B^n_k, $i \neq k$; then there holds the following inequality:*

$$\begin{aligned} &\mathrm{rk}^t_*(U, U_0) \;\leq\; \mathrm{rk}^t_*\Big(\bigcup_{i=1}^{N} B^{n+1}_i \,, \bigcup_{i=1}^{N} B^0_i\Big) \\ &\leq\; (e-1)N \cdot \sup\Big\{\mathrm{rk}^t_*\Big(\bigcap_{i\in S} B^{j+1}_i \,, \bigcap_{i\in S} B^j_i\Big) \;\Big|\; 0 \leq j \leq n,\ \emptyset \neq S \subset \{1, \ldots, N\}\Big\} \,. \end{aligned}$$

Definitions 4.2. For a compact CW–space Y and $t > 0$, define

$$\sharp^t(Y) := \sum_{j \geq 0} (\text{number of } j\text{–cells in } Y) \cdot t^j \,.$$

For open subsets $U_0 \subset U$ in V and $t > 0$, let $\mathrm{crk}^t_*(U, U_0)$ be the minimum of all numbers $q \in \mathbb{N}$ such that there exist maps

$$U_0 \xrightarrow{f} Y \xrightarrow{g} U$$

where Y is a compact CW–space with $\sharp^t(Y) \leq q$, and gf is homotopic to the inclusion. If there is no such q let $\mathrm{crk}^t_*(U, U_0) = \infty$.

Lemma 4.3. *With rk^t_* replaced by crk^t_* throughout, the inequality in 4.1 remains correct.*

Proof. Let $\mathcal{A}$ be the nerve of the collection of open sets $\{B^{n+1}_i\}$, where $1 \leq i \leq N$. As in 3.2, define contravariant functors $\mathcal{Z}_j$ from $\mathcal{A}$ to spaces:

$$\mathcal{Z}_j(s) = \bigcap_{i \text{ vertex of } s} B^j_i$$

where $0 \leq j \leq n+1$. As in 3.2, there are natural transformations $\mathcal{Z}_j \to \mathcal{Z}_{j+1}$ for $0 \leq j \leq n$, given by inclusion. Let

$$p = \sup\Big\{\mathrm{crk}^t_*\Big(\bigcap_{i\in S} B^{j+1}_i \,, \bigcap_{i\in S} B^j_i\Big) \;\Big|\; 0 \leq j \leq n,\ \emptyset \neq S \subset \{1, \ldots, N\}\Big\} \,.$$

As in 3.1 (**) and proof of 3.1, we can construct a factorization

$$|\mathcal{Z}_0|(n) \longrightarrow W \longrightarrow |\mathcal{Z}_{n+1}|$$

where $W = |\mathcal{Y}|$ is the geometric realization of a contravariant functor $\mathcal{Y}$ from the n–skeleton $\mathcal{A}^n$ to spaces, and $\mathcal{Y}(s)$ is homotopy equivalent to a CW–space $X(s)$ such that

$$\sharp^t(X(s)) \leq p$$

for every face $s \subset \mathcal{A}^n$. We now have to show that

$$\mathrm{crk}^t_*(|\mathcal{Y}|) \leq (e-1)Np\,.$$

To this end we show first that $\sharp^t(\mathcal{A}^n) \leq (e-1)N$, using the hypotheses in 4.1 ; this is actually carried out in [Abr2, p.479]. (Beware that our t is Abresch's t^{-1}.) Then we finish with a variation on 3.3:

Lemma 4.4. *Let $\mathcal{B}$ be a compact simplicial complex with $\sharp^t(\mathcal{B}) = J$ and let $\mathcal{Y}$ be a contravariant functor from $\mathcal{B}$ to spaces. Assume that each $\mathcal{Y}(s)$ is homotopy equivalent to a CW–space $X(s)$ with $\sharp^t(X(s)) \leq p$. Then $|\mathcal{Y}|$ is homotopy equivalent to a CW–space X with $\sharp^t(X) \leq pJ$.*

The proof is by induction on the number of simplices in $\mathcal{B}$, like that of 3.3. □

5. Big spaces with small homology

Here is an example showing that Theorem B is stronger than Theorem A. Let M be a square matrix (size $k \times k$) with integer entries such that both M and $M - I_k$ have determinant ± 1; for instance, $k = 2$ and

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $\pi = \mathbb{Z}$, and let the generator of π act on $\mathbb{Z}^k$ by M . Let μ be the minimum number of generators of

$$E = \bigoplus_{i=1}^{s} \mathbb{Z}^k$$

as a π–module. Then $\mu k \geq s$, because $\hom_\pi(E, \mathbb{Z}^k)$ contains a free abelian group of rank s. Hence $\mu \geq s/k$. Let X be a wedge of sk spheres of dimension $d > 1$, and let $f : X \to X$ be a homotopy equivalence such that $H_d(X)$, with the action of π determined by f_*, is isomorphic to E as a π–module. Finally let Y be the mapping torus of f,

$$Y = X \times [0,1] \;/\; (x,1) \sim (f(x),0).$$

Then $\pi_1(Y) = \pi = \mathbb{Z}$, and $H_*(Y;F) \cong H_*(S^1;F)$ for any field F. But the number of cells in any CW–space dominating Y is $\geq \mu$, which is $\geq s/k$, which is as large as we please.

To obtain closed manifold examples of the same type, just make sure that Y embeds in a high–dimensional euclidean space. Then take a smooth regular neighbourhood and double along the boundary.

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