A HAEFLIGER STYLE DESCRIPTION OF THE EMBEDDING CALCULUS TOWER

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Abstract. Let M and N be smooth manifolds. The calculus of embeddings produces, for every $k \geq 1$, a *best degree* $\leq k$ *polynomial approximation* to the cofunctor taking an open $V \subset M$ to the space of embeddings from V to N. In this paper a description of these polynomial approximations in terms of equivariant mapping spaces is given, for $k \geq 2$. The description is new only for $k \geq 3$. In the case $k = 2$ we recover Haefliger's approximation and the known result that it is the best degree ≤ 2 approximation.

0. Introduction

Let M and N be smooth manifolds, without boundary for simplicity, $\dim(M) = m$ and $\dim(N) = n$ where $n > 3$. The calculus of embeddings [10], [11], [3], [2] produces certain 'Taylor' approximations $\mathcal{T}_k\mathrm{emb}(M,N)$ to the space $\mathrm{emb}(M,N)$ of smooth embeddings from M to N. In more detail, there are maps

$$\eta_k\colon\ \mathrm{emb}(M,N) \to \mathcal{T}_k\mathrm{emb}(M,N),$$

one for each $k \geq 1$, and there are maps $r_k\colon\ \mathcal{T}_k\mathrm{emb}(M,N) \to \mathcal{T}_{k-1}\mathrm{emb}(M,N)$ such that $r_k\eta_k = \eta_{k-1}$. The map η_k is $(1-m+k(n-m-2))$–connected; therefore if $n > m+2$ one has

$$\mathrm{emb}(M,N) \simeq \operatorname*{holim}_k\ \mathcal{T}_k\mathrm{emb}(M,N).$$

(*Remark on notation:* In this paper we use a calligraphic $\mathcal{T}$ for Taylor approximations and reserve the roman T for tangent spaces and the like.)

The method of embedding calculus is to relate $\mathrm{emb}(M,N)$ to spaces of embeddings $\mathrm{emb}(V,N)$ where V runs through the open subsets of M which are disjoint unions of finitely many open balls. In particular, $\mathcal{T}_k\mathrm{emb}(M,N)$ is defined as

$$\operatorname*{holim}_{V\in\mathcal{O}k}\ \mathrm{emb}(V,N)$$

where $\mathcal{O}k$ is the poset (ordered by inclusion) of open subsets of M which are diffeomorphic to $\{1,2,\dots,j\} \times \mathbb{R}^m$ for some $j \leq k$. The map η_k from $\mathrm{emb}(M,N)$ to $\mathcal{T}_k\mathrm{emb}(M,N)$ is determined by the restriction maps $\mathrm{emb}(M,N) \to \mathrm{emb}(V,N)$ for $V \in \mathcal{O}k$.

This definition of $\mathcal{T}_k\mathrm{emb}(M,N)$ is convenient in many respects, but from a geometric point of view it is awkward; for example, there is no obvious action of the (topological) group of diffeomorphisms $M \to M$ on $\mathcal{T}_k\mathrm{emb}(M,N)$. Our goal here

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is to define by elementary geometric methods spaces $\Theta_k(M,N)$ for $k \geq 1$, depending nicely on M and N, and to show that $\Theta_k(M,N)$ is homotopy equivalent to $\mathcal{T}_k\mathrm{emb}(M,N)$ for $k \geq 2$. The construction $\Theta_2(M,N)$ is already known, cf.§4 of [3]. It is Haefliger's approximation [4] to $\mathrm{emb}(M,N)$, the homotopy pullback of

$$\begin{array}{ccc} & & \mathrm{ivmap}^{\mathbb{Z}/2}(M\times M, N\times N) \\ & & \big\downarrow \text{incl.} \\ \mathrm{map}(M,N) & \xrightarrow{f\mapsto f\times f} & \mathrm{map}^{\mathbb{Z}/2}(M\times M, N\times N). \end{array}$$

Here $\mathrm{ivmap}^{\mathbb{Z}/2}(M\times M, N\times N)$ is the space of *strictly isovariant* smooth $\mathbb{Z}/2$–maps from $M\times M$ to $N\times N$. (*Definition:* Let X,Y be smooth manifolds on which a finite group G acts; a smooth G–map $f\colon X\to Y$ is *strictly isovariant* if $(Tf)^{-1}(TY^H) = TX^H$ for every subgroup H of G, where $Tf\colon TX\to TY$ is the differential of f.)

There are projection maps $\Theta_{k+1}(M,N)\to\Theta_k(M,N)$ which model the canonical projections $\mathcal{T}_{k+1}\mathrm{emb}(M,N)\to\mathcal{T}_k\mathrm{emb}(M,N)$, for $k\geq 2$. These will be clear from the definition of $\Theta_k(M,N)$ given below. In the case $k=1$ one can proceed as follows. The recommended geometric substitute for $\mathcal{T}_2\mathrm{emb}(M,N)$ is still $\Theta_2(M,N)$, as defined above and below. The recommended geometric substitute for $\mathcal{T}_1\mathrm{emb}(M,N)$ is the space of pairs (g,e) where $g\colon M\to N$ is continuous and $e: TM\to f^*(TN)$ is a vector bundle monomorphism; we denote it here by $\bar\Theta_1(M,N)$ since it is a refinement of $\Theta_1(M,N)$. The recommended geometric substitute for the canonical projection $\mathcal{T}_2\mathrm{emb}(M,N)\to\mathcal{T}_1\mathrm{emb}(M,N)$ is the composition

$$\Theta_2(M,N) \xrightarrow{\text{proj.}} \mathrm{ivmap}^{\mathbb{Z}/2}(M\times M,N\times N) \xrightarrow{v} \bar\Theta_1(M,N);$$

here v is obtained by restricting the strictly isovariant maps $M\times M\to N\times N$ to the diagonals, and keeping track of the induced map of normal bundles (of the diagonals), which one identifies with the tangent bundles of the diagonals.

Terminology. Unless otherwise stated, *smooth map* from M to N will mean: a C^k–map $M\to N$ for some fixed $k\gg 0$. Hence the *space of smooth maps* from M to N is really the space of C^k–maps from M to N, with the compact–open C^k–topology alias weak topology. See §2 of [5]. Similarly a *smooth embedding* from M to N is to be understood as a C^k–embedding (= C^k–immersion which maps M homeomorphically onto its image). The *space of smooth embeddings* from M to N is defined as a subspace of the space of smooth maps from M to N. (In [11] and [3] the preferred models for all kinds of mapping spaces and embedding spaces were simplicial sets or geometric realizations of such. These models don't go very well with group actions, so we decided not to use them here.)

1. The geometric model

Fix M and N, as above. Let R and S be finite sets, $R\subset S$. Denote by $\mathrm{map}(M^S,N^R)$ the space of smooth maps $M^S\to N^R$. Call a smooth map f from M^S to N^R *admissible* if, for every equivalence relation ρ on R, we have

$$(Tf)^{-1}(TN^{R/\rho}) = TM^{S/\rho}$$

where Tf is the differential of f. Here S/ρ is short for the quotient of S obtained by identifying elements $x,y\in S$ whenever $x,y\in R$ and $x\rho y$; we are using inclusions

$N^{R/\rho} \subset N^R$ and $M^{S/\rho} \subset M^S$. — Let $\mathrm{amap}(M^S, N^R) \subset \mathrm{map}(M^S, N^R)$ be the subspace consisting of the admissible maps.

Definition 1.1. $$\Theta_k(M,N) := \left(\operatorname*{holim}_{\substack{R,S\subset\{1,\dots,k\}\\ R\subset S}} \mathrm{amap}(M^S, N^R) \right)^{\Sigma_k}.$$

Remark 1.2. The space $\mathrm{amap}(M^S, N^R)$ depends contravariantly on the variable $R \subset \{1,\dots,k\}$ and covariantly on $S \subset \{1,\dots,k\}$. So we may regard

$$(S,R) \mapsto \mathrm{amap}(M^S, N^R)$$

as a functor on the poset whose elements are pairs (S,R) with $R \subset S \subset \{1,\dots,k\}$, the ordering being defined by

$$(S_1,R_1) \le (S_2,R_2) \quad \Leftrightarrow \quad S_1 \subset S_2 \text{ and } R_2 \subset R_1.$$

The homotopy limit of this functor (which appears in 1.1) has a standard description which we recall in section 2. There we also note that the standard description simplifies to the following: the space of natural transformations

$$[0,1]^{S\setminus R} \longrightarrow \mathrm{amap}(M^S, N^R).$$

(Both domain and codomain are to be viewed as functors in the variable (S,R) and we still assume $R \subset S \subset \{1,\dots,k\}$. Specifically we identify $[0,1]^{S\setminus R}$ with the space of all maps $f\colon \{1,\dots,k\} \to [0,1]$ which satisfy $f(x) = 0$ for all $x \in R$ and $f(x) = 1$ for all $x \notin S$; this gives the functorial dependence on S and R.)

Theorem 1.3. $\Theta_k(M,N) \simeq \mathcal{T}_k\mathrm{emb}(M,N)$ *for* $k \ge 2$.

Remark 1.4. This can be formulated with more precision, as follows. Let $\mathcal{O} = \mathcal{O}(M)$ be the poset of open subsets of M. For $V \in \mathcal{O}$ we have a map $\bar\eta_k$ from $\mathrm{emb}(V,N)$ to $\Theta_k(V,N)$ given by

$$g \mapsto \left(V^S \xrightarrow{\text{proj.}} V^R \xrightarrow{g^R} N^R \right)_{R\subset S\subset\{1,\dots,k\}}.$$

This amounts to a natural transformation between cofunctors in the variable $V \in \mathcal{O}$. We will check (in section 3) that the cofunctor $V \mapsto \Theta_k(V,N)$ is polynomial of degree $\le k$, cf. [11], and that $\bar\eta_k\colon \mathrm{emb}(V,N) \to \Theta_k(V,N)$ specializes to a weak homotopy equivalence for $V \in \mathcal{O}k$. This means that $\bar\eta_k$ has the properties which characterize the k–th Taylor approximation; so there exists a chain of weak homotopy equivalences *under* $\mathrm{emb}(V,N)$ relating $\mathcal{T}_k\mathrm{emb}(V,N)$ to $\Theta_k(V,N)$, natural in $V \in \mathcal{O}$. Specializing this to $V = M$, we obtain 1.3.

Illustration. Here we show that $\bar\eta_2\colon \mathrm{emb}(M,N) \to \Theta_2(M,N)$ agrees with Haefliger's approximation to $\mathrm{emb}(M,N)$. The homotopy limit which appears in 1.1 is the homotopy pullback of a diagram

$$\begin{array}{ccc} & & \mathrm{amap}(M\times M, N\times N) \\ & & \downarrow \\ \mathrm{map}(M,N)\times\mathrm{map}(M,N) & \longrightarrow & \mathrm{map}(M\times M,N)\times\mathrm{map}(M\times M,N) \end{array}$$

where the horizontal arrow is $(f_1, f_2) \mapsto (f_1 p_1, f_2 p_2)$ and the vertical arrow is $g \mapsto (q_1 g, q_2 g)$, the p_i and q_i being appropriate projections. Taking fixed points under the action of Σ_2 now, we obtain the homotopy pullback of

$$\begin{array}{ccc} & & \mathrm{ivmap}^{\mathbb{Z}/2}(M \times M, N \times N) \\ & & \downarrow \\ \mathrm{map}(M, N) & \longrightarrow & \mathrm{map}(M \times M, N) \end{array}$$

where the horizontal arrow is $f \mapsto f p_1$ and the vertical one is $g \mapsto q_1 g$. It only remains to observe

$$\mathrm{map}(M \times M, N) \cong \mathrm{map}^{\mathbb{Z}/2}(M \times M, N \times N)\,.$$

2. Homotopy limits, homotopy ends and edgewise subdivision

Let $\mathcal{C}$ be a small category. Recall that the *limit* of a functor F from $\mathcal{C}$ to spaces is the space of all natural transformations from the constant functor $c \mapsto *$ to F; it is topologized as a subspace of $\prod_c F(c)$. The *homotopy limit* of F, denoted $\mathrm{holim}\, F$, is the corealization (alias Tot) of the cosimplicial space

$$[i] \mapsto \prod_{c_0 \to c_1 \to \cdots \to c_i} F(c_i)$$

where $c_0 \to c_1 \to \cdots \to c_i$ runs through the diagrams in $\mathcal{C}$ having that shape. See [1] for motivation. There is a canonical inclusion $\lim F \to \mathrm{holim}\, F$.

Most of this chapter is a digression on *ends* and *homotopy ends*, which are special cases of limits and homotopy limits, respectively. The digression is useful because the homotopy limit which appears in 1.1 is almost a homotopy end.

Starting with the category $\mathcal{C}$, make another category $\mathcal{C}'$ whose objects are the arrows $f\colon c_1 \to c_2$ in $\mathcal{C}$; a morphism in $\mathcal{C}'$ from $f\colon c_1 \to c_2$ to $g\colon d_1 \to d_2$ is a commutative diagram

$$\begin{array}{ccc} c_1 & \xrightarrow{f} & c_2 \\ \uparrow & & \downarrow \\ d_1 & \xrightarrow{g} & d_2. \end{array}$$

There is a forgetful functor $J\colon \mathcal{C}' \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$, given on objects by the assignment $(f\colon c_1 \to c_2) \mapsto (c_1, c_2)$.

Definition 2.1. The *end* of a functor E from $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ to spaces (for example) is defined by $\mathrm{end}\, E := \lim EJ$. The *homotopy end* of E is defined by

$$\mathrm{hoend}\, E := \mathrm{holim}\, EJ\,.$$

See [7] for more about ends. Our definition of *end* is somewhat different in spirit from MacLane's, but certainly equivalent.

Proposition 2.2. *The homotopy end of E is homeomorphic to the corealization alias* Tot *of the cosimplicial space*

$$[i] \quad \mapsto \prod_{c_0 \to c_1 \to \cdots \to c_i} E(c_0, c_i)$$

where $c_0 \to c_1 \to \cdots \to c_i$ runs through the diagrams in $\mathcal{C}$ having that shape.

Example. Let F_a and F_b be functors from $\mathcal{C}$ to spaces; for simplicity assume that $F_a(c)$ is compact for all objects c in $\mathcal{C}$. Let $E(c_0, c_1) = \mathrm{map}(F_a(c_0), F_b(c_1))$ for objects c_0, c_1 in $\mathcal{C}$. Then E is a functor on $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ and $\mathrm{end}\, E$ can be identified with the space of natural transformations from F_a to F_b. What is $\mathrm{hoend}\, E$? Using the alternative definition of homotopy ends given in 2.2, we find that a point ω in $\mathrm{hoend}\, E$ gives us, for each c in $\mathcal{C}$, a map $\omega(c)\colon F_a(c) \to F_b(c)$; for each morphism $g\colon c_0 \to c_1$ in $\mathcal{C}$, a map $\omega(g)\colon \Delta^1 \times F_a(c_0) \to F_b(c_1)$ which is a homotopy from $\omega(c_1)F_a(g)$ to $F_b(g)\omega(c_0)$; for each diagram

$$c_0 \xrightarrow{g_0} c_1 \xrightarrow{g_1} c_2$$

in $\mathcal{C}$, a map $\omega(g_0, g_1)\colon \Delta^2 \times F_a(c_0) \to F_b(c_2)$ which restricts to $\omega(g_1)F_a(g_0)$, $\omega(g_1 g_0)$ and $F_b(g_1)\omega(g_0)$ on $d_i\Delta^2 \times F_a(c_0)$ for $i = 0, 1, 2$ respectively; and so on. Thus, ω is a transformation $F_a \to F_b$ which is *natural up to all higher homotopies.*

Clearly proposition 2.2 is a special case of the following:

Proposition 2.3. *For any functor F from $\mathcal{C}'$ to spaces,* $\mathrm{holim}\, F$ *is homeomorphic to the corealization of*

$$[i] \quad \mapsto \quad \prod_{c_0 \to c_1 \to \cdots \to c_i} F(c_0 \to c_i).$$

Here $c_0 \to c_1 \to \cdots \to c_i$ runs through the diagrams of the indicated shape in $\mathcal{C}$, and $c_0 \to c_i$ is the composite morphism in $\mathcal{C}$, alias object in $\mathcal{C}'$, determined by such a diagram.

The proof will be given after lemma 2.4 and corollary 2.5, below.

The construction $\mathcal{C} \mapsto \mathcal{C}'$ corresponds, via nerves, to a construction on simplicial sets, the *edgewise subdivision* of Quillen and/or Segal, cf. [8]. Let U be the endofunctor of the category of nonempty finite totally ordered sets given by $U(S) = S^{\mathrm{op}} \amalg S$. (Here $\amalg$ indicates a disjoint union with the lexicographic ordering, so that all elements of the left hand summand are $<$ than all elements of the right summand summand.) Let $\mathfrak{X}$ be a simplicial set, to be viewed as a contravariant functor from nonempty finite totally ordered sets to sets. The edgewise subdivision of $\mathfrak{X}$ is $\mathfrak{X} \circ U$. (Admittedly this is the opposite of Segal's edgewise subdivision of $\mathfrak{X}$, which is $\mathfrak{X} \circ U^{\mathrm{op}}$, where $U^{\mathrm{op}}(S) = (U(S))^{\mathrm{op}}$.)

Lemma 2.4. *The nerve of $\mathcal{C}'$ is isomorphic to the edgewise subdivision of the nerve of $\mathcal{C}$.*

Proof. An i–simplex in the nerve of $\mathcal{C}'$ is the same thing as a commutative diagram

$$\begin{array}{ccccccccc} c_0 & \longleftarrow & c_1 & \longleftarrow & \cdots\cdots & \longleftarrow & c_{i-1} & \longleftarrow & c_i \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & & & \downarrow{\scriptstyle f_{i-1}} & & \downarrow{\scriptstyle f_i} \\ d_0 & \longrightarrow & d_1 & \longrightarrow & \cdots\cdots & \longrightarrow & d_{i-1} & \longrightarrow & d_i \end{array}$$

in $\mathcal{C}$. Deleting the redundant arrows labelled $f_1, \dots, f_i$ gives a subdiagram which is an i–simplex in the edgewise subdivision of the nerve of $\mathcal{C}$. □

Corollary 2.5. $|\mathcal{C}'| \cong |\mathcal{C}|$.

Proof. Segal [8] gives a natural homeomorphism $h\colon |\mathfrak{X}\circ U| \to |\mathfrak{X}|$ for any simplicial set $\mathfrak{X}$. We describe it briefly. By naturality, it suffices to look at the cases where $\mathfrak{X}$ is the nerve of the totally ordered set $\{0,\dots,i\}$ for some $i \geq 0$, so that $|\mathfrak{X}| = \Delta^i$. In such a case h can be described or characterized as follows:

- It is linear on the (realizations of) the nondegenerate simplices of $\mathfrak{X}\circ U$.
- The value of h on (the realization of) a 0–simplex of $\mathfrak{X}\circ U$ alias 1–simplex of $\mathfrak{X}$ is the barycenter of the corresponding edge or vertex of $|\mathfrak{X}| = \Delta^i$. □

Proof of 2.3. Let Y_1 be the disjoint union of

$$F(c'_i) \times V(c'_0 \to \cdots \to c'_i)$$

where $c'_0 \to \cdots \to c'_i$ runs through the nondegenerate simplices in the nerve of $\mathcal{C}'$, and where $V(c'_0 \to \cdots \to c'_i)$ is the corresponding (open) cell of $|\mathcal{C}'|$. Let Y_2 be the disjoint union of $F(c_0 \to c_i) \times V(c_0 \to \cdots \to c_i)$ where $c_0 \to \cdots \to c_i$ runs through the nondegenerate simplices in the nerve of $\mathcal{C}$, and again $V(c_0 \to \cdots \to c_i)$ is the corresponding cell of $|\mathcal{C}|$. Let $p_1\colon\ Y_1 \to |\mathcal{C}'|$ and $p_2\colon\ Y_2 \to |\mathcal{C}|$ be the projections. (We do not put any topologies on Y_1 or Y_2.) Now the two spaces in 2.3 which we have to compare can be identified, as sets, with subsets of the section sets of p_1 and p_2, respectively. Using this to label elements, we can write down the desired homeomorphism as $s \mapsto (h(x) \mapsto s(x))$, where h comes from the proof of 2.5. □

We return to the homotopy limit in 1.1. Let $\mathcal{C}$ be the poset of subsets of $\{1,2,\dots,k\}$, ordered by inclusion. Then $\mathcal{C}'$ is the poset of pairs (R,S) with $R \subset S \subset \{1,2,\dots,k\}$, with the ordering described in 1.2. The homotopy limit in 1.1 is the homotopy limit of the functor on $\mathcal{C}'$ given by $(R,S) \mapsto \operatorname{amap}(M^S,N^R)$. By 2.3, we can also describe it as the corealization of the cosimplicial space

$$[i] \quad \mapsto \prod_{S_0\subset S_1\subset\cdots\subset S_i} \operatorname{amap}(M^{S_i},N^{S_0})$$

where $S_0 \subset S_1 \subset \cdots \subset S_i$ runs through diagrams in $\mathcal{C}$ of the indicated shape. Since $\mathcal{C}$ is a poset, this simplifies as follows:

Proposition 2.6. *The homotopy limit in 1.1 is homeomorphic to the corealization of the incomplete cosimplicial space (i. e. , cosimplicial space without degeneracy operators)*

$$[i] \quad \mapsto \prod_{S_0\subsetneq S_1\subsetneq\cdots\subsetneq S_i\subset\{1,\dots,k\}} \operatorname{amap}(M^{S_i},N^{S_0}).$$

Here the strings $S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_i$ with fixed $S_0 = R$ and $S_i = S$ can be regarded as the i–simplices of an incomplete simplicial set whose geometric realization happens to be a cube $[0,1]^{S\setminus R}$. Hence we obtain the statement made in 1.2: The homotopy limit in 1.1 is homeomorphic to the space of natural transformations from $(S,R) \mapsto [0,1]^{S\setminus R}$ to $(S,R) \mapsto \operatorname{amap}(M^S,N^R)$, assuming $R \subset S \subset \{1,\dots,k\}$.

Returning to the expression in 2.6, we proceed to take fixed points of the action of Σ_k. Note that Σ_k also acts on the set of strings

$$S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_i$$

(as in 2.6) for each i, and in each orbit of that action there is exactly one string of the form $\{1,2,\dots,k_0\} \subset \{1,2,\dots,k_1\} \subset \cdots \subset \{1,2,\dots,k_i\}$ with $k_0 < k_1 < \cdots < k_i$.

We will denote its stabilizer (alias isotropy) group by $\mathrm{st}(k_0,\dots,k_i)$. It is isomorphic to

$$\Sigma_{k_0}\times\Sigma_{k_1-k_0}\times\cdots\times\Sigma_{k_i-k_{i-1}};$$

it is contained in Σ_{k_i} and projects to Σ_{k_0}, and so acts on $\mathrm{amap}(M^{k_i},N^{k_0})$. Bearing all this in mind, we have the following rather explicit description of $\Theta_k(M,N)$:

Proposition 2.7. *The space $\Theta_k(M,N)$ defined in 1.1 is naturally homeomorphic to the corealization of the incomplete cosimplicial space*

$$[i]\quad\mapsto\prod_{0\le k_0<k_1<\cdots<k_i\le k}\big(\mathrm{amap}(M^{k_i},N^{k_0})\big)^{\mathrm{st}(k_0,\dots,k_i)}.$$

3. Polynomial behavior

Here we show that the cofunctor $V\mapsto\Theta_k(V,N)$ on $\mathcal{O}(M)$ is polynomial of degree $\le k$. See 1.4. The argument is standard; compare example 2.4 of [11]. Most of it can be seen in the proof of the following easier statement:

Proposition 3.1. *Let X be any space. For any $k\ge 0$, the cofunctor $V\mapsto \mathrm{map}(V^k,X)$ on $\mathcal{O}(M)$ is polynomial of degree $\le k$.*

Proof. Suppose given $V\in\mathcal{O}(M)$ and pairwise disjoint subsets $A_0,A_1,\dots A_k$ of V which are closed in V. For $i\in\{0,1,\dots,k\}$ let $V_i\colon=V\setminus A_i$ and for $S\subset\{0,1,\dots,k\}$ let $V_S\colon=\bigcap_{i\in S}V_i$. By the pigeonhole principle we have

$$V^k=\bigcup_{i\in\{0,\dots,k\}}(V_i)^k.$$

This implies by lemma 3.2 below that the canonical projection from the homotopy colimit of the $(V_S)^k$ for *nonempty* $S\subset\{0,\dots,k\}$ to V^k is a homotopy equivalence. Hence the map which it induces, from $\mathrm{map}(V^k,X)$ to

$$\mathrm{map}\left(\operatorname*{hocolim}_{\substack{S\subset\{0,\dots,k\}\\ S\ne\emptyset}}(V_S)^k,\quad X\right)\quad\cong\quad\operatorname*{holim}_{\substack{S\subset\{0,\dots,k\}\\ S\ne\emptyset}}\mathrm{map}((V_S)^k,X),$$

is a homotopy equivalence. Therefore $V\mapsto\mathrm{map}(V^k,X)$ is polynomial of degree $\le k$. $\square$

Lemma 3.2. *For a paracompact space Z with open cover $\{W_\alpha\mid\alpha\in\Lambda\}$, the canonical projection*

$$p\colon\ \operatorname*{hocolim}_{\substack{S\subset\Lambda\\ 0<|S|<\infty}}\bigcap_{\alpha\in S}W_\alpha\quad\longrightarrow\quad Z$$

is a homotopy equivalence.

Proof. Choose a partition of unity $\{\psi_\alpha\colon W_\alpha\to I\}$ subordinate to the open cover $\{W_\alpha\}$. Think of the domain of p as a quotient of

$$\coprod_{\substack{S\subset\Lambda\\ 0<|S|<\infty}}\Delta(S)\times\bigcap_{\alpha\in S}W_\alpha$$

where $\Delta(S)$ denotes the simplex spanned by S, of dimension $|S|-1$. We will describe points in $\Delta(S)$ by their barycentric coordinates. — For $z \in Z$ let $S(z) = \{\alpha \in \Lambda \mid z \in W_\alpha\}$. The formula

$$z \quad \mapsto \quad ((\psi_\alpha(z))_{\alpha \in S(z)}, z) \in \Delta(S(z)) \times \bigcap_{\alpha \in S(z)} W_\alpha$$

defines a section σ of p. A homotopy $\{h_t\}$ from σp to the identity is defined by

$$h_t((y,z)) = (ty + (1-t)\sigma(z), z)$$

for $y \in \Delta(S)$ and $z \in \bigcap_{\alpha \in S} W_\alpha$. □

Proposition 3.3. *Let K, L be finite sets, where $K \subset L$. Let G be any subgroup of $\Sigma_K \times \Sigma_{L \setminus K}$. The cofunctor on $\mathcal{O}(M)$ given by*

$$V \quad \mapsto \big(\mathrm{amap}(V^L, N^K)\big)^G$$

is polynomial of degree $\leq |L|$.

Proof. Let $\ell := |L|$. Suppose given $V \in \mathcal{O}(M)$ and pairwise disjoint subsets $A_0, A_1, \dots A_\ell$ of V which are closed in V. For $i \in \{0, 1, \dots, \ell\}$ let $V_i \colon = V \setminus A_i$ and for $S \subset \{0, 1, \dots, \ell\}$ let $V_S \colon = \bigcap_{i \in S} V_i$. By the pigeonhole principle we have

$$V^\ell = \bigcup_{i \in \{0, \dots, \ell\}} (V_i)^\ell .$$

Thus the $(V_i)^\ell$ constitute an open cover of V^ℓ; and moreover the cover is invariant under the action of G on V^ℓ. Choose a subordinate partition of unity which is also invariant under G. From the proof of 3.2, this choice of partition of unity gives us a homotopy inverse σ^* for the canonical map

$$\mathrm{map}(V^L, N^K) \longrightarrow \operatorname*{holim}_{S \neq \emptyset} \ \mathrm{map}((V_S)^L, N^K)$$

where S runs through the nonempty subsets of L; more precisely, a strict left inverse σ^* and a homotopy $\{h_t^*\}$ showing that the left inverse is also a homotopy right inverse. By inspection, σ^* restricts to a G–map

$$\mathrm{amap}(V^L, N^K) \longrightarrow \operatorname*{holim}_{S \neq \emptyset} \ \mathrm{amap}((V_S)^L, N^K)$$

and each h_t^* restricts to a G–map

$$\operatorname*{holim}_{S \neq \emptyset} \ \mathrm{amap}((V_S)^L, N^K) \longrightarrow \operatorname*{holim}_{S \neq \emptyset} \ \mathrm{amap}((V_S)^L, N^K).$$

Hence σ^* restricts to a homotopy inverse for the canonical map

$$\big(\mathrm{amap}(V^L, N^K)\big)^G \longrightarrow \operatorname*{holim}_{S \neq \emptyset} \ \big(\mathrm{amap}((V_S)^L, N^K)\big)^G . \quad \square$$

Corollary 3.4. *The cofunctor on $\mathcal{O}(M)$ given by $V \mapsto \Theta_k(V, N)$ is polynomial of degree $\leq k$.*

Proof. In addition to 3.3 use 2.7 and observe that corealization commutes with homotopy (inverse) limits. □

Remark. The relevant homotopy limits in this proof are taken over the poset of nonempty subsets of $\{0, 1, \dots, k\}$. Do not confuse $\{0, 1, \dots, k\}$ with $\{1, \dots, k\}$; the numbers $0, 1, \ldots, k$ serve as indices for $k+1$ pairwise disjoint closed subsets A_i of some open subset V of M, while $\{1, \ldots, k\}$ appears in the definition of Θ_k. Note that a cofunctor on $\mathcal{O}(M)$ which is polynomial of degree $\leq \ell$ with $\ell \leq k$ is also polynomial of degree $\leq k$.

4. Behavior on finite sets

To complete the proof of 1.3 we must show that for every open $V \subset M$ which is diffeomorphic to $\mathbb{R}^m \times L$ for a finite set L of cardinality $\leq k$, the canonical map

$$\mathrm{emb}(V, N) \longrightarrow \Theta_k(V, N)$$

is a weak homotopy equivalence. It is convenient to separate the task into a non–tangential and a tangential part. The goal here is to establish the non–tangential part:

Proposition 4.1. *The canonical map* $\mathrm{emb}(L, N) \longrightarrow \Theta_k(L, N)$ *is a homotopy equivalence if* L *is a finite set of cardinality* $\leq k$.

For the moment suppose that L is any finite set, not necessarily of cardinality $\leq k$.

Lemma 4.2.
$$\operatorname*{holim}_{\substack{R,S \subset \{1,\ldots,k\} \\ R \subset S}} \mathrm{amap}(L^S, N^R) \cong \operatorname*{holim}_{\substack{g\colon S \to L \\ S \subset \{1,\ldots,k\}}} \mathrm{emb}(g(S), N).$$

Explanation. The homotopy limit is taken over the poset $L^{\leq k}$ whose objects are pairs (S, g) with $S \subset \{1, \ldots, k\}$ and $g\colon S \to L$. The ordering is by inclusion over L; that is, $(S_1, g_1) \leq (S_2, g_2)$ means $S_1 \subset S_2$ and $g_1 = g_2 \mid S_1$.

Proof. By 2.6, the left hand term in 4.2 is homeomorphic to the corealization of the incomplete cosimplicial space

$$[i] \quad \mapsto \prod_{S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_i \subset \{1,\ldots,k\}} \mathrm{amap}(L^{S_i}, N^{S_0}).$$

Fixing i and the string $S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_i$ for the moment, we have an easy identification

$$\mathrm{amap}(L^{S_i}, N^{S_0}) \cong \prod_{g\colon S_i \to L} \mathrm{emb}(g(S_0), N) \cong \prod_{g_0, g_1, \ldots, g_i} \mathrm{emb}(g_0(S_0), N)$$

where the g_r for $0 \leq r \leq i$ are maps $S_r \to L$ such that

$$(S_0, g_0) < (S_1, g_1) < \cdots < (S_i, g_i)$$

in $L^{\leq k}$. Hence for fixed i we have an identification

$$\prod_{S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_i} \mathrm{amap}(L^{S_i}, N^{S_0}) \quad \cong \prod_{(S_0,g_0) < (S_1,g_1) < \cdots < (S_i,g_i)} \mathrm{emb}(g_0(S_0), N).$$

Using these identifications for all i, one finds that the face operators are exactly the ones that appear in the definition of the right hand term of 4.2. (For the present purposes this can be and should be defined as the corealization of an appropriate *incomplete* cosimplicial space, because the indexing category $L^{\leq k}$ is a poset). □

Let $\mathcal{D}_k(L)$ be the set of functions $f\colon L \to \mathbb{N} = \{0, 1, 2, \ldots\}$ which satisfy

$$\sum_{x \in L} f(x) \leq k\,.$$

The *support* of $f \in \mathcal{D}_k(L)$ is

$$\operatorname{supp}(f) := f^{-1}(\{1,2,3,\dots\}).$$

We view $f \mapsto \operatorname{supp}(f)$ as a functor from the poset $\mathcal{D}_k(L)$, with the usual ordering, to the poset of subsets of L. We are still assuming that L is a finite set.

Corollary 4.3. $\Theta_k(L,N) \cong \operatorname{holim}_{f\in\mathcal{D}_k(L)} \operatorname{emb}(\operatorname{supp}(f),N).$

Proof. There is a map from the right hand side in 4.3 to the right hand side in 4.2, induced by the functor $L^{\leq k} \to \mathcal{D}_k(L)$ which takes $(S,g) \in L^{\leq k}$ to $f_{S,g}\colon L \to \mathbb{N}$ with $f_{S,g}(x) = |g^{-1}(x)|$. Note that $\operatorname{supp}(f_{S,g}) = g(S)$. By inspection, the map is a homeomorphism of the right hand side in 4.3 with the Σ_k–fixed points of the right hand side in 4.2. □

Completion of the proof of 4.1 . The canonical map mentioned in 4.1 has now been identified with the composition of the equally canonical maps

$$\operatorname{emb}(L,N) \longrightarrow \operatorname{holim}_{J\subset L} \operatorname{emb}(J,N) \longrightarrow \operatorname{holim}_{f\in\mathcal{D}_k(L)} \operatorname{emb}(\operatorname{supp}(f),N).$$

The first of these is a homotopy equivalence because L is a terminal element in the poset of subsets of L, alias initial object in the opposite poset. If $|L| \leq k$ the second one is also a homotopy equivalence because it is induced by the functor $f \mapsto \operatorname{supp}(f)$ from $\mathcal{D}_k(L)$ to the poset of subsets of L; and that functor has a left adjoint. The left adjoint takes a subset J of L to $f_J \in \mathcal{D}_k(L)$ with $f_J(x) = 1$ if $x \in J$ and $f_J(x) = 0$ if $x \notin J$. □

5. Behavior on tubular neighbourhoods of finite sets

In this section we suppose that V is a tubular neighborhood of a finite set $L \subset M$, with $|L| \leq k$. The goal is to show:

Proposition 5.1. *In this situation, the canonical map* $\operatorname{emb}(V,N) \longrightarrow \Theta_k(V,N)$ *is a homotopy equivalence.*

The proof of 5.1 will take up the entire section. It uses the description of $\Theta_k(V,N)$ given in 2.7. Therefore we begin with an investigation of the spaces $\operatorname{amap}(V^S,N^R)$ and their symmetries, for $R \subset S \subset \{1,2,\dots,k\}$. Denote by $\operatorname{ajet}(V^S,N^R;L^S)$ the space of 1–jets at L^S of admissible maps $V^S \to N^R$. (An element of that space is an equivalence class of admissible maps $V^S \to N^R$, two such maps being equivalent if they agree to first order at all points of $L^S \subset V^S$.)

Lemma 5.2. *The projection* $\operatorname{amap}(V^S,N^R) \to \operatorname{ajet}(V^S,N^R;L^S)$ *is an equivariant homotopy equivalence, with respect to the action of* $\Sigma_{S\setminus R} \times \Sigma_R$. *Furthermore there is an equivariant and natural homotopy equivalence*

$$\operatorname{ajet}(V^S,N^R;L^S) \longrightarrow \operatorname{map}(L^{S\setminus R},\operatorname{ajet}(V^R,N^R;L^R)).$$

Proof. For the first part, choose a complete riemannian metric on N. Also, choose a riemannian metric on V such that each component becomes isomorphic as a riemannian manifold to $\mathbb{R}^m$ with the standard metric. Next, let X be the space of maps $g\colon V^S \to N^R$ which are admissible in a neighbourhood of L^S. The map which we are investigating is a composition

$$\operatorname{amap}(V^S,N^R) \hookrightarrow X \longrightarrow \operatorname{ajet}(V^S,N^R;L^S).$$

Using the exponential maps for V^S and N^R determined by the riemannian metrics on V and N one finds that $X \to \mathrm{ajet}(V^S, N^R; L^S)$ is an equivariant homotopy equivalence. It remains to show that the inclusion $\mathrm{amap}(V^S, N^R) \hookrightarrow X$ is also an equivariant homotopy equivalence. This can be done by a shrinking argument. That is, there is an equivariant homotopy inverse of the form $g \mapsto g \circ h_{g,1}$ for $g \in X$. Here

$$\{h_{g,t} \mid 0 \le t \le 1\}$$

is a suitable equivariant smooth isotopy of embeddings $V^S \to V^S$, relative to a neighbourhood of L^S and depending continuously on $g \in X$. It is assumed that $h_{g,0}$ is the identity and $h_{g,1}$ has sufficiently small image, so that $g \circ h_{g,1}$ is indeed admissible on all of V^S. To construct $\{h_{g,t}\}$ simultaneously for all $g \in X$, use partitions of unity, noting that X is metrizable.

For the second part, let $p\colon V^S \to V^R$ be the projection. An element of the space $\mathrm{ajet}(V^S, N^R; L^S)$ can be thought of as a map $a\colon L^S \to N^R$ together with linear maps

$$b_x\colon T_x(V^S) \to T_{a(x)}(N^R),$$

one for each $x \in L^S$, subject to some conditions. An element of

$$\mathrm{map}(L^{S \setminus R}, \mathrm{ajet}(V^R, N^R; L^R))$$

can be thought of as a map $a\colon L^S \to N^R$ together with linear maps

$$c_x\colon T_{p(x)}(V^R) \to T_{a(x)}(N^R),$$

subject to some conditions. The equivariant homotopy equivalence that we need is induced by the inclusions

$$T_{p(x)}(V^R) = \prod_{s \in R} T_{s(x)}V \quad \longrightarrow \quad T_x(V^S) = \prod_{s \in S} T_{s(x)}V. \quad \square$$

The maps given in 5.2 should be viewed as natural transformations of functors on the poset with elements (S, R), compare 1.1:

$$\mathrm{amap}(V^S, N^R) \longrightarrow \mathrm{ajet}(V^S, N^R; L^S) \longrightarrow \mathrm{map}(L^{S \setminus R}, \mathrm{ajet}(V^R, N^R; L^R)).$$

The equivariance statement in 5.2 shows (with some inspection) that these natural transformations respect the Σ_k–symmetries. This leads us to the next lemma:

Lemma 5.3. *For $S \subset \{1, 2, \ldots, k\}$ and $g \in L^S \subset V^S$ let $\mathrm{ajet}(V^S, N^S; g)$ be the space of 1–jets of admissible maps $V^S \to N^S$ at g. There is a Σ_k–equivariant homeomorphism*

$$\operatorname*{holim}_{\substack{R,S \subset \{1,\ldots,k\} \\ R \subset S}} \mathrm{map}(L^{S \setminus R}, \mathrm{ajet}(V^R, N^R; L^R)) \quad \cong \quad \operatorname*{holim}_{\substack{g\colon S \to L \\ S \subset \{1,\ldots,k\}}} \mathrm{ajet}(V^S, N^S; g).$$

The proof resembles that of 4.2 and will be left to the reader. — For the next lemma we resurrect the poset $\mathcal{D}_k(L)$ of section 4. For $f \in \mathcal{D}_k(L)$ let

$$S(f) := \coprod_{x \in L} \{1, \ldots, f(x)\}, \qquad \Sigma(f) = \prod_{x \in L} \Sigma_{f(x)}$$

so that $\Sigma(f)$ acts canonically on $S(f)$. Let $f^\natural\colon S(f) \to L$ be the evident projection; then $f^\natural \in L^{S(f)} \subset V^{S(f)}$.

Lemma 5.4.

$$\left(\underset{\substack{g\colon S\to L\\ S\subset\{1,\dots,k\}}}{\operatorname{holim}} \ \operatorname{ajet}\left(V^S, N^S; g\right)\right)^{\Sigma_k} \cong \underset{f\in\mathcal{D}_k(L)}{\operatorname{holim}} \left(\operatorname{ajet}\left(V^{S(f)}, N^{S(f)}; f^\natural\right)\right)^{\Sigma(f)}.$$

Proof. There is a straightforward map from right hand side to left hand side; by inspection it is a homeomorphism. □

Now choose an embedding $e\colon L\to N$, in other words, a base point in $\operatorname{emb}(L,N)$. This of course makes each space $\operatorname{emb}(\operatorname{supp}(f),N)$ for $f\in\mathcal{D}_k(L)$ into a pointed space. For $x\in L$ we abbreviate $T_x := T_xM$ and $T_{e(x)} := T_{e(x)}N$. Evaluation at $f^\natural$ gives a map

$$\left(\operatorname{ajet}\left(V^{S(f)}, N^{S(f)}; f^\natural\right)\right)^{\Sigma(f)} \longrightarrow \operatorname{emb}(\operatorname{supp}(f),N).$$

Lemma 5.5. *This map is a fibration, and its fiber over the base point is*

$$\prod_{x\in L}\left(\operatorname{ahom}\left(T_x^{f(x)}, T_{e(x)}^{f(x)}\right)\right)^{\Sigma_{f(x)}}$$

where ahom(...) *denotes a space of linear and admissible maps.* □

Again, the identification in 5.5 should be seen as an isomorphism of contravariant functors, now in the variable $f\in\mathcal{D}_k(L)$. — Write $?\otimes\mathbb{R}^{f(x)}$ for $?^{f(x)}$ and split $\mathbb{R}^{f(x)}$ into irreducible representations of $\Sigma_{f(x)}$. The cases $f(x)=0$ and $f(x)=1$ are easy; when $f(x)\geq 2$ there are two irreducible summands, the trivial one–dimensional representation and the reduced permutation representation (of dimension $f(x)-1$), both with endomorphism field $\mathbb{R}$. See 5.7 below. This gives

$$\left(\operatorname{ahom}\left(T_x^{f(x)}, T_{e(x)}^{f(x)}\right)\right)^{\Sigma_{f(x)}} \cong \begin{cases} * & \text{if } f(x)=0\\ \operatorname{hom}(T_x, T_{e(x)}) & \text{if } f(x)=1\\ \operatorname{hom}(T_x, T_{e(x)})\times\operatorname{hom}^\sharp(T_x, T_{e(x)}) & \text{if } f(x)\geq 2\end{cases}$$

where $\operatorname{hom}^\sharp(\dots)$ denotes spaces of injective linear maps. For homotopy theoretic purposes the contractible terms $\operatorname{hom}(T_x, T_{e(x)})$ are not of interest. This brings us to the next lemma, which essentially completes the proof of 5.1. (A summary of the entire proof will be given, though.)

Lemma 5.6. $$\underset{f\in\mathcal{D}_k(L)}{\operatorname{holim}} \prod_{\substack{x\in L\\ f(x)\geq 2}} \operatorname{hom}^\sharp(T_x, T_{e(x)}) \simeq \prod_{x\in L}\operatorname{hom}^\sharp(T_x, T_{e(x)}).$$

Proof. Note first of all that the functor on $\mathcal{D}_k(L)$ whose homotopy limit we are interested in is contravariant; the induced maps are projection maps. — In the left hand side interchange homotopy limit and product to get

$$\prod_{x\in L} \underset{\substack{f\in\mathcal{D}_k(L)\\ f(x)\geq 2}}{\operatorname{holim}} \operatorname{hom}^\sharp(T_x, T_{e(x)}).$$

Now it suffices to show that, for each $x\in L$, the poset of all $f\in\mathcal{D}_k(L)$ with $f(x)\geq 2$ has contractible classifying space. But clearly it has a minimal element. (Here we are using the assumption $k\geq 2$.) □

Summary of proof of 5.1. Because of 4.1, it is enough to show that the following is homotopy cartesian:

$$\begin{array}{ccc} \mathrm{emb}(V,N) & \xrightarrow{\text{res.}} & \mathrm{emb}(L,N) \\ \big\downarrow \text{can.} & & \big\downarrow \text{can.} \\ \Theta_k(V,N) & \xrightarrow{\text{res.}} & \Theta_k(L,N). \end{array}$$

So let $e \in \mathrm{emb}(L,N)$. We need to understand the homotopy fiber of the lower horizontal map over the image of e. Using 5.2, 5.3, 5.4 and 4.3 we find that this is homotopy equivalent to the appropriate homotopy fiber, or fiber, of the map

$$\operatorname*{holim}_{f\in\mathcal{D}_k(L)} \left(\mathrm{ajet}\left(V^{S(f)}, N^{S(f)}; f^{\natural}\right)\right)^{\Sigma(f)} \quad \longrightarrow \quad \operatorname*{holim}_{f\in\mathcal{D}_k(L)} \mathrm{emb}(\mathrm{supp}(f), N)$$

given by evaluation at $f^{\natural}$. Therefore by 5.5 and 5.6, its homotopy type is that of the product

$$\prod_{x\in L} \mathrm{hom}^{\sharp}(T_x, T_{e(x)}).$$

But that is also the homotopy type of the fiber of $\mathrm{emb}(V,N) \to \mathrm{emb}(L,N)$ over e. Some inspection shows that the abstract homotopy equivalence between the two fibers so obtained agrees with the canonical map between them. □

Lemma 5.7. *Suppose $i \geq 2$. Let ρ be the reduced permutation representation of Σ_i on $\mathbb{R}^i/\mathbb{R}$. Then ρ is irreducible and has endomorphism field $\mathbb{R}$.*

Proof. Irreducibility is established in chapter 2, exercise 2.6 of Serre's book [9] on linear representations of finite groups. In fact this shows that the complexified representation $\rho \otimes_{\mathbb{R}} \mathbb{C}$ is still irreducible. We learn from Serre's book, chapter 13.2, paragraph about the three types of irreducible representations, that if the complexification of an irreducible real representation is still irreducible, then the original real representation has endomorphism field $\mathbb{R}$. □

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