PRO–EXCISIVE FUNCTORS

Michael Weiss and Bruce Williams

Abstract. We classify homotopy invariant and *pro–excisive* functors $\boldsymbol{F}$ from Euclidean neighbourhood retracts to spectra. The pro–excision axiom ensures that $\pi_*\boldsymbol{F}$ is a generalized "locally finite" homology theory.

0. Introduction

Index theorems usually involve some form of homology, as a receptacle for the *symbol*. Proofs of index theorems often involve some form of *locally finite homology* as well. Locally finite homology is so useful in this connection because it has some contravariant features in addition to the usual covariant ones. Thus, a proper map $f : X \to Y$ (details below) between locally compact spaces induces a map f_* in locally finite homology, going in the same direction ; but an inclusion j of an open subset $V \subset X$ induces a *wrong way* map j^* from the locally finite homology of X to that of V. In the case of singular locally finite homology, this is clear from the definition

$$H_*^{\ell f}(X) := \lim_U H_*(X, U)$$

where the inverse limit is taken over all $U \subset X$ with compact complement. (Note the excision property $H_*(X, U) \cong H_*(V, V \cap U)$ which applies when $X \smallsetminus U$ is compact and contained in V.)

In practice, when locally finite homology makes its appearance in the proof of an index theorem, or elsewhere, it may not be immediately recognizable as such. If the contravariant features are not needed, then theorem 1.2 below, essentially quoted from [WWA], should solve the problem. A recognition problem of this sort with a similar solution appears in the work on the Novikov conjecture of [CaPe]. (To see the similarity, note for example that a reduced Steenrod homology theory applied to one-point compactifications of locally compact subsets of some $\mathbb{R}^n$ makes a perfectly good locally finite homology theory.) If the contravariant features *are* relevant, then theorem 1.2 is not good enough and instead theorem 2.1 below should be used. We have used it in [DWW], in the proof of a parametrized index theorem for the algebraic K-theory *Euler class*. The proof is outlined below in §3. We emphasize that this proof is not analytic. It is part of a topological reply to the paper by Bismut and Lott [BiLo] and to the "Lott challenge" which asked for an explanation in topological language of a Riemann–Roch theorem for flat vector bundles [BiLo, Thm. 0.1].

1991 *Mathematics Subject Classification.* Primary 55N20; Secondary 55N07, 55P42, 19D10, 57Q10.

While this may not be directly related to the Novikov conjecture, the reader should realize that index theorems relating Euler class and Euler characteristic can often be strengthened by keeping track of Poincaré duality. For example, the Euler characteristic of a closed smooth oriented manifold M^{4k} may be regarded for simplicity as an element in the topological K-group $K_0^{\mathrm{top}}(*) \cong \mathbb{Z}$. Keeping track of the Poincaré duality, one obtains much more: an element in

$$K_0^{\mathrm{top}}(B\mathbb{Z}/2) \cong \mathbb{Z} \oplus \widehat{\mathbb{Z}}_2$$

[AtSe] whose first component is the Euler characteristic of M, and whose second component is the difference of Euler characteristic and signature divided by two. The relationship between Euler class and $\mathcal{L}$–class is similar. A strengthened index theorem along these lines, parametrized and with plenty of algebraic K-theory instead of topological K-theory, is already implicit in [WW3] and may be more explicit in the next revision.

1. Excision and Proper Maps

Recall that a map $f : X \to Y$ between locally compact spaces is *proper* if it extends to a continuous pointed map $f^\bullet : X^\bullet \to Y^\bullet$ between their one-point compactifications. Note also that, in general, not every pointed continuous map $X^\bullet \to Y^\bullet$ is of the form $f^\bullet$ for a proper $f : X \to Y$. Let $\mathcal{E}$ be the category of ENR's (euclidean neighborhood retracts), with *proper* maps as morphisms. Let $\mathcal{E}^\bullet$ be the larger category whose objects are the ENR's, and where a morphism from $X \to Y$ is a continuous pointed map $X^\bullet \to Y^\bullet$. The goal is to characterize functors of the form

$$X \mapsto X^\bullet \wedge \boldsymbol{Y}\,, \tag{*}$$

where $\boldsymbol{Y}$ is a CW–spectrum and an Ω–spectrum, by their homotopy invariance and excision properties. (We call $\boldsymbol{Y}$ an *Ω–spectrum* if the adjoints of the structure maps $\Sigma Y_n \to Y_{n+1}$ are homotopy equivalences $Y_n \to \Omega Y_{n+1}$.) Reason for setting this goal: $\pi_*(X^\bullet \wedge \boldsymbol{Y})$, as a functor in the variable X, has all the properties one expects from a locally finite homology theory—details below, just before Thm. 1.3. We may view (*) as a functor from $\mathcal{E}$ to spectra, or as a functor from $\mathcal{E}^\bullet$ to spectra, so the task is twofold.

1.1. Terminology. A commutative square of locally compact spaces and proper maps

$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ \downarrow & & \downarrow \\ X_3 & \longrightarrow & X_4 \end{array}$$

is a *proper homotopy pushout square* if the resulting proper map from the homotopy pushout of $X_3 \leftarrow X_1 \to X_2$ to X_4 is a proper homotopy equivalence. A covariant functor $\boldsymbol{F}$ from $\mathcal{E}$ to CW–spectra is *homotopy invariant* if it takes proper homotopy equivalences to homotopy equivalences. A homotopy invariant $\boldsymbol{F}$ is *excisive* if it takes homotopy pushout squares in $\mathcal{E}$ to homotopy pushout squares of spectra, and $\boldsymbol{F}(\emptyset)$ is contractible.

Suppose that $\boldsymbol{F}$ is homotopy invariant and excisive, and suppose that X in $\mathcal{E}$ is *discrete.* For any $y \in X$, we have a homotopy equivalence

$$\boldsymbol{F}(X \smallsetminus y) \vee \boldsymbol{F}(y) \longrightarrow \boldsymbol{F}(X)$$

by excision, and hence a projection $\boldsymbol{F}(X) \to \boldsymbol{F}(y)$, well defined up to homotopy. We call $\boldsymbol{F}$ *pro–excisive* if these projection maps induce an isomorphism

$$\pi_n \boldsymbol{F}(X) \longrightarrow \prod_{y \in X} \pi_n \boldsymbol{F}(y) \qquad\qquad (n \in \mathbb{Z}) .$$

Example. Let $\boldsymbol{Y}$ be a CW–spectrum. Assume also that $\boldsymbol{Y}$ is an Ω–spectrum, or that $\boldsymbol{Y}$ is the suspension spectrum of a CW–space. Then the functor taking X to the standard CW–approximation of $X^\bullet \wedge \boldsymbol{Y}$ is a homotopy invariant and pro–excisive functor from ENR's to CW–spectra. See [WWA, 4.3] for the proof. *Warning:* Be sure to use the correct topology on $X^\bullet$. Note that X can be a countably infinite discrete set, or the universal cover of a wedge of two circles, or worse. *Illustration:* Suppose $\boldsymbol{Y}$ is the sphere spectrum. Then $\pi_n(X^\bullet \wedge \boldsymbol{Y}) \cong \pi_n^s(X^\bullet)$ and this can be interpreted via transversality as the bordism group of framed manifolds equipped with a *proper map* to X. The excision properties follow easily from this interpretation.

In the following theorem we invoke a functorial construction which associates to each CW–spectrum $\boldsymbol{Y}$ a CW–Ω–spectrum $\boldsymbol{Y}_\Omega$ and a homotopy equivalence $\boldsymbol{Y} \to \boldsymbol{Y}_\Omega$.

1.2. Theorem. *If $\boldsymbol{F}$ from $\mathcal{E}$ to CW–spectra is homotopy invariant and pro–excisive, then there exists a chain of natural weak homotopy equivalences*

$$\boldsymbol{F}(X) \;\simeq\; \dots \;\simeq\; X^\bullet \wedge \boldsymbol{F}(*)_\Omega \,.$$

Proof. This is contained in Cor. 4.8 of [WWA], which is about homotopy invariant and pro–excisive functors from *control spaces* to spectra. A control space is a pair $(\bar{X}, X)$ where $\bar{X}$ is compact, X is open dense in $\bar{X}$, and X is an ENR. A morphism of control spaces, $f : (\bar{X}_1, X_1) \to (\bar{X}_2, X_2)$, is a map of pairs such that $f^{-1}(X_2) = X_1$. Writing $\mathcal{C}$ for the category of control spaces, we see that $\mathcal{E}$ is a *retract* of $\mathcal{C}$ via

$$X \mapsto (X^\bullet, X) \qquad\qquad (\bar{X}, X) \mapsto X \,.$$

Hence any homotopy invariant and strongly excisive $\boldsymbol{F}$ on $\mathcal{E}$ determines one on $\mathcal{C}$, and this is covered by Cor. 4.8 of [WWA]. □

2. Excision and One-Point Compactification

Here we are interested in functors from $\mathcal{E}^\bullet$, the "enlarged" category of ENR's, to spectra. Such a functor will be called *homotopy invariant* and *pro–excisive* if its restriction to $\mathcal{E}$ has these properties. Before proving any theorems about pro–excisive functors on $\mathcal{E}^\bullet$, we elucidate the structure of $\mathcal{E}^\bullet$. Note that every diagram of ENR's of the form

(*) $$X \supset V \xrightarrow{g} Y$$

where V is open in X and g is proper, gives rise to a continuous pointed map $X^\bullet \to Y^\bullet$ which agrees with g on V and maps the complement of V to the base point in $Y^\bullet$. Clearly every continuous pointed map $X^\bullet \to Y^\bullet$ arises in this way, for unique V and g, so that (*) may be regarded as the description of a typical morphism in $\mathcal{E}^\bullet$. If it happens that $V = X$ in (*), then the morphism under consideration is in $\mathcal{E}$. If it happens that $g = \mathrm{id}$, then we must think of the morphism as some kind of *reversed inclusion.* We see from (*) that every morphism in $\mathcal{E}^\bullet$ can be written in the form gh, where g is in $\mathcal{E}$ and h is a reversed inclusion of an open subset. This decomposition is unique.

A functor $\boldsymbol{F}$ from $\mathcal{E}^\bullet$ to CW–spectra is therefore a gadget which to every X in $\mathcal{E}^\bullet$ associates a CW–spectrum $\boldsymbol{F}(X)$, to every proper map $g : X \to Y$ a map $\boldsymbol{F}(X) \to \boldsymbol{F}(Y)$, and to every inclusion $V \subset X$ of an open subset, a wrong way map $\boldsymbol{F}(X) \to \boldsymbol{F}(V)$ which we think of as a *restriction map.* Certain associativity relations and identity relations must hold—after all, a functor is a functor.

Similarly, a natural transformation $\tau : \boldsymbol{F}_1 \to \boldsymbol{F}_2$ between functors from $\mathcal{E}^\bullet$ to spectra is a gadget which to every X in $\mathcal{E}^\bullet$ associates a map $\boldsymbol{F}_1(X) \to \boldsymbol{F}_2(X)$ such that the diagrams

$$\begin{array}{ccc} \boldsymbol{F}_1(X) & \xrightarrow{\tau} & \boldsymbol{F}_2(X) \\ \Big\downarrow{\scriptstyle g_*} & & \Big\downarrow{\scriptstyle g_*} \\ \boldsymbol{F}_1(Y) & \xrightarrow{\tau} & \boldsymbol{F}_2(Y) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \boldsymbol{F}_1(X) & \xrightarrow{\tau} & \boldsymbol{F}_2(X) \\ \Big\downarrow{\scriptstyle \text{restriction}} & & \Big\downarrow{\scriptstyle \text{restriction}} \\ \boldsymbol{F}_1(V) & \xrightarrow{\tau} & \boldsymbol{F}_2(V) \end{array}$$

commute, for every proper $g : X \to Y$ and every inclusion of an open subset $V \subset X$. In particular, the natural weak homotopy equivalences in the following theorem 2.1. are gadgets of this type. Thus 2.1 is *not* a formal consequence of 1.2.

2.1. Theorem. *If $\boldsymbol{F}$ from $\mathcal{E}^\bullet$ to CW–spectra is homotopy invariant and pro–excisive, then there exists a chain of natural weak homotopy equivalences*

$$\boldsymbol{F}(X) \;\simeq\; \dots \;\simeq\; X^\bullet \wedge \boldsymbol{F}(*)_\Omega\,.$$

Proof. Let $\mathcal{E}^\bullet_1$ be the full subcategory of $\mathcal{E}^\bullet$ consisting of those objects which are geometric realizations of simplicial sets. Let $\mathcal{E}^\bullet_2$ be the full subcategory of $\mathcal{E}^\bullet$ consisting of the objects X such that $X^\bullet$ is homeomorphic to the geometric realization of a pointed finitely generated simplicial set. Note that $\mathcal{E}^\bullet_2$ is equivalent to the category of finitely generated pointed simplicial sets, where the morphisms are the pointed continuous maps between geometric realizations. Let $\iota_1 : \mathcal{E}^\bullet_1 \to \mathcal{E}^\bullet$ and $\iota_2 : \mathcal{E}^\bullet_2 \to \mathcal{E}^\bullet$ be the inclusion functors. Our strategy is to show that $\boldsymbol{F}\iota_2$ determines $\boldsymbol{F}\iota_1$, up to a natural chain of weak homotopy equivalences. It is comparatively easy to analyze $\boldsymbol{F}\iota_2$ and to recover $\boldsymbol{F}$ from $\boldsymbol{F}\iota_1$, up to a natural chain of weak homotopy equivalences.

We shall pretend that $\mathcal{E}^\bullet$, $\mathcal{E}^\bullet_1$ and $\mathcal{E}^\bullet_2$ are *small* categories ; the truth is of course that they are *equivalent* to small categories. We can also throw in a few technical assumptions about $\boldsymbol{F}$, as follows. Each spectrum $\boldsymbol{F}(X)$ is made up of pointed simplicial sets $F_n(X)$ and simplicial maps from $\Sigma F_n(X)$ to $F_{n+1}(X)$, for $n \in \mathbb{Z}$. For each morphism $X \to X'$ in $\mathcal{E}^\bullet$, the induced map $\boldsymbol{F}(X) \to \boldsymbol{F}(X')$ is in fact a *function* [Ad, III§2], given by compatible simplicial maps $F_n(X) \to F_n(X')$ for all $n \in \mathbb{Z}$. Finally, each $\boldsymbol{F}(X)$ is an Ω–spectrum. These assumptions facilitate the definition of homotopy limits.

Step 1. For Z in $\mathcal{E}_1^\bullet$ we have the canonical map

$$\boldsymbol{F}\iota_1(Z) \longrightarrow \operatorname*{holim}_K \boldsymbol{F}\iota_2(Z \smallsetminus K)\,. \tag{**}$$

where K runs over all closed subsets of Z such that $Z \smallsetminus K$ is in $\mathcal{E}_2^\bullet$ and $Z \smallsetminus K$ has *compact* closure in Z. We want to think of (**) as a natural transformation between *functors* on $\mathcal{E}_1^\bullet$, in the variable Z. Thus if Z_1 and Z_2 are in $\mathcal{E}_1^\bullet$, and $f : Z_1^\bullet \to Z_2^\bullet$ is a continuous pointed map, and $L \subset Z_2$ is closed, and the closure of $Z_2 \smallsetminus L$ in Z_2 is compact, we let $f^*(L)$ be the inverse image of L under f, minus the base point. Then we have

$$\operatorname*{holim}_K \boldsymbol{F}\iota_2(Z_1 \smallsetminus K) \longrightarrow \operatorname*{holim}_L \boldsymbol{F}\iota_2(Z_1 \smallsetminus f^*L) \longrightarrow \operatorname*{holim}_L \boldsymbol{F}\iota_2(Z_2 \smallsetminus L)\,.$$

Step 2. We shall verify that the codomain of (**) has certain excision properties. From a strict pushout square of (realized) simplicial sets, all in $\mathcal{E}_1^\bullet$, and simplicial maps

$$\begin{array}{ccc} Z_1 & \xrightarrow{\subset} & Z_2 \\ \downarrow & & \downarrow \\ Z_3 & \xrightarrow{\subset} & Z_4 \end{array} \tag{***}$$

and a cofinite simplicial subset $K \subset Z_4$, we obtain another pushout square

$$\begin{array}{ccc} K_1 & \longrightarrow & K_2 \\ \downarrow & & \downarrow \\ K_3 & \longrightarrow & K_4 \end{array}$$

where K_i is the inverse image of K in Z_i. Then the spaces $Z_i \smallsetminus K_i$ form a proper homotopy pushout square, so that the spectra $\boldsymbol{F}\iota_2(Z_i \smallsetminus K_i)$ form a homotopy pushout square, alias homotopy pullback square. Passing to homotopy limits and noting that a homotopy limit of homotopy pullback squares is a homotopy pullback square, we see that the codomain of (**) does indeed have certain excision properties: it takes (***) to a homotopy pullback square alias homotopy pushout square. (We have taken the liberty to "prune" the indexing categories for the homotopy limits involved. This is justified by [DwKa, 9.3].)

In particular, the filtration of an arbitrary Z in $\mathcal{E}_1^\bullet$ by skeletons leads to a filtration of domain and codomain of (**), and by excision and inspection the induced maps of filtration quotients are weak homotopy equivalences. Hence (**) is a weak homotopy equivalence, by induction on the dimension of Z.

Step 3. The functor $\boldsymbol{F}\iota_2$ on $\mathcal{E}_2^\bullet$ is homotopy invariant and excisive in the following sense. For any Z in $\mathcal{E}_2^\bullet$, the map

$$\boldsymbol{F}\iota_2(Z \times [0,1]) \to \boldsymbol{F}\iota_2(Z)$$

induced by projection is a homotopy equivalence. The square of spectra

$$\begin{array}{ccc} \boldsymbol{F}\iota_2(Z_1 \cap Z_2) & \longrightarrow & \boldsymbol{F}\iota_2(Z_1) \\ \downarrow & & \downarrow \\ \boldsymbol{F}\iota_2(Z_2) & \longrightarrow & \boldsymbol{F}\iota(Z_1 \cup Z_2) \end{array}$$

is a weak homotopy pushout square provided $Z_1^\bullet, Z_2^\bullet$ are geometric realizations of pointed simplicial subsets of a finitely generated pointed simplicial set. Further, $\boldsymbol{F}\iota_2(\emptyset)$ is weakly contractible. This follows directly from our hypotheses on $\boldsymbol{F}$.

Step 4. The functor $\boldsymbol{F}\iota_2$ is related to $Z \mapsto Z^\bullet \wedge \boldsymbol{F}(*)$ by a chain of natural weak homotopy equivalences. The argument follows the lines of [WWA, §1]. For Z in $\mathcal{E}_2^\bullet$ let $\mathrm{simp}(Z^\bullet)$ be the category whose objects are maps $\Delta^n \to Z^\bullet$ and whose morphisms are linear maps $f_* : \Delta^m \to \Delta^n$, over $Z^\bullet$, induced by some monotone f from $\{0, 1, \dots, m\}$ to $\{0, 1, \dots, n\}$. Let $\boldsymbol{F}_{Z^\bullet}$ be the functor from $\mathrm{simp}(Z^\bullet)$ to spectra taking $g : \Delta^n \to Z^\bullet$ to $\boldsymbol{F}(\Delta^n)$, and let

$$\boldsymbol{F}^{\%}(Z^\bullet) := \operatorname{hocolim} \boldsymbol{F}_{Z^\bullet}\,.$$

Each $g : \Delta^n \to Z^\bullet$ in $\mathrm{simp}(Z^\bullet)$ is a morphism $\Delta^n \to Z$ in $\mathcal{E}_2^\bullet$ which induces g_* from $\boldsymbol{F}\iota_2(\Delta^n) = \boldsymbol{F}(\Delta^n)$ to $\boldsymbol{F}\iota_2(Z)$. Collecting all these, we have the *assembly*

$$\alpha : \boldsymbol{F}^{\%}(Z^\bullet) \longrightarrow \boldsymbol{F}\iota_2(Z)\,.$$

The domain of α is homotopy invariant and excisive like $\boldsymbol{F}\iota_2$, but $\boldsymbol{F}^{\%}(\emptyset^\bullet)$ need not be contractible. However, the map of vertical (homotopy) cofibers in

$$\begin{array}{ccc} \boldsymbol{F}^{\%}(\emptyset^\bullet) & \xrightarrow{\ \alpha\ } & \boldsymbol{F}\iota_2(\emptyset) \\ \downarrow & & \downarrow \\ \boldsymbol{F}^{\%}(Z^\bullet) & \xrightarrow{\ \alpha\ } & \boldsymbol{F}\iota_2(Z) \end{array}$$

is a natural transformation between homotopy invariant and excisive functors in the variable Z, and it is clearly a homotopy equivalence when Z is a point and when Z is empty. By Eilenberg–Steenrod arguments, it is always a homotopy equivalence. Further, $\boldsymbol{F}^{\%}(Z^\bullet)$ can be related to $Z_+^\bullet \wedge \boldsymbol{F}(*)$ as in [WWA, §1].

Step 5. We must verify that $\boldsymbol{F}$ can be recovered from $\boldsymbol{F}\iota_1$. For every X in $\mathcal{E}^\bullet$, the map

$$\boldsymbol{F}(X) \longrightarrow \operatorname*{hocolim}_{Z \to X} \boldsymbol{F}(Z)$$

is a homotopy equivalence. The hocolim is taken over the category with objects $Z \to X$, where Z is in $\mathcal{E}_1^\bullet$. The claim is obvious for X in $\mathcal{E}_1^\bullet$. The general case follows because every X in $\mathcal{E}^\bullet$ is a retract of some object in $\mathcal{E}_1^\bullet$. $\square$

Remark. Theorems 1.3 and 2.1 are reminiscent of uniqueness and existence theorems for generalized *Steenrod homology theories*, [KKS], [EH], [M]. For variations and applications see [CaPe].

3. An Example

For the purposes of this section, a *control space* is a pair of spaces $(\bar{Y}, Y)$ where $\bar{Y}$ is *locally* compact Hausdorff, Y is open dense in $\bar{Y}$, and Y is an ENR. Informally, the set $\bar{Y} \smallsetminus Y$ is the *singular set*, whereas Y is the *nonsingular set*. A *morphism* of control spaces is a continuous *proper* map of pairs $f : (\bar{Y}, Y) \to (\bar{Z}, Z)$ such that $f^{-1}(Z) = Y$. Note that we are less restrictive here than in [WWA, §4] because we allow $\bar{Y}$ to be noncompact. In any case these ideas come from [ACFP].

Fix a control space $(\bar{Y}, Y)$. By a *geometric module* on Y we mean a free abelian group B with an (internal) direct sum decomposition

$$B = \bigoplus_{x \in Y} B_x$$

where each B_x is finitely generated, and the set $\{x \in Y \mid B_x \neq 0\}$ is closed and discrete in Y. Given two geometric modules B and B' on Y, a *controlled homomorphism* $f : B \to B'$ is a group homomorphism, with components $f_y^x : B_x \to B_y$ say, subject to the following condition. For any $z \in \bar{Y} \smallsetminus Y$ and any neighborhood V of z in $\bar{Y}$, there exists a smaller neighborhood W of z in $\bar{Y}$ such that $f_y^x = 0$ and $f_x^y = 0$ whenever $x \in W$ and $y \notin V$. Clearly the composition of two geometric homomorphisms $B \to B'$, $B' \to B''$ is a geometric homomorphism $B \to B''$.

For a geometric module B on Y and a neighborhood U of $\bar{Y} \smallsetminus Y$ in Y, we let B^U be the geometric submodule of B which is the direct sum of the B_x for $x \in U$. Given B and B', as before, a *germ* of controlled homomorphisms from B to B' is an equivalence class of pairs $(U, f : B^U \to B')$. Here $(U, f : B^U \to B')$ and $(W, g : B^W \to B')$ are *equivalent* if f and g agree on $B^{U \cap W}$.

A controlled homomorphism germ as above is *invertible* if it is an isomorphism in the germ category. Geometric modules on Y and invertible germs of controlled homomorphisms between them form a symmetric monoidal category. With this we can associate a K-theory spectrum, using the construction of [Se], say. Since it depends ultimately on $(\bar{Y}, Y)$, we denote it by $\boldsymbol{E}(\bar{Y}, Y)$. It is clear that $\boldsymbol{E}(\bar{Y}, Y)$ is covariantly functorial in $(\bar{Y}, Y)$. Finally we put

$$\boldsymbol{F}(X) := \boldsymbol{E}\big(X \times [0,1], X \times [0,1)\big)$$

so that $\boldsymbol{F}$ is a functor from ENR's and proper maps to spectra. It is well known [ACFP] that $\boldsymbol{F}$ is homotopy invariant and excisive on the category of *compact* polyhedra and piecewise linear maps. It follows immediately that $\boldsymbol{F}$ is also homotopy invariant and excisive on the category of compact ENR's, since every compact ENR is a retract of a compact polyhedron. Using the result of [Ca], in addition to arguments proving excision as in [ACFP] or [Vog], one can verify that $\boldsymbol{F}$ is in fact homotopy invariant and pro–excisive on the category $\mathcal{E}$ of all ENR's and their proper maps. We hope to give more details elsewhere.

The functor $\boldsymbol{F}$ has an extension to $\mathcal{E}^\bullet$, the “enlarged” category of ENR's. To understand this extension, recall the canonical decomposition of morphisms $X \to Y$ in $\mathcal{E}^\bullet$ as reversed inclusion $X \supset V$ of an open subset, followed by a proper map $g : V \to Y$. We know already how the proper map $g : V \to Y$ induces $g_* : \boldsymbol{F}(V) \to \boldsymbol{F}(Y)$, so that our task now is to produce a *restriction map* $\boldsymbol{F}(X) \to \boldsymbol{F}(V)$ which we may (pre–)compose with g_*. Now the word *restriction* almost gives it away. Recall that $\boldsymbol{F}(X)$ was constructed as the K-theory of a certain symmetric monoidal category whose objects are the geometric modules on $X \times [0,1)$. We can indeed *restrict* a geometric module B on $X \times [0,1)$ to $V \times [0,1)$ by discarding all the B_z for $z \notin V \times [0,1)$. Similarly, if $f : B \to B'$ is a morphism of geometric modules on $X \times [0,1)$, we restrict by discarding all the f_z^y where $y \notin V \times [0,1)$ or $z \notin V \times [0,1)$.

Now it is important to realize that restriction, as we have defined it, is *not* a functor from the category of geometric modules on $X \times [0,1)$ to the category of geometric modules on $V\times[0,1)$. It does not respect composition of controlled homomorphisms. However, restriction is compatible with passage to controlled homomorphism germs, and after passage to germs restriction does respect composition. This is easily verified. Hence we have enough of a restriction functor to get an induced restriction map $\boldsymbol{F}(X)\to\boldsymbol{F}(V)$.

It is known [ACFP] that $\boldsymbol{F}(*)=\boldsymbol{E}\big([0,1],[0,1)\big)\simeq\mathbb{S}^1\wedge\boldsymbol{K}(\mathbb{Z})$ where $\boldsymbol{K}(\mathbb{Z})$ is the algebraic K-theory spectrum of $\mathbb{Z}$. Therefore, by theorem 2.1, there exists a chain of natural weak homotopy equivalences

$$(**)\qquad \boldsymbol{F}(X)\quad\simeq\quad\ldots\quad\simeq\quad X^\bullet\wedge\mathbb{S}^1\wedge\boldsymbol{K}(\mathbb{Z})$$

for X in $\mathcal{E}^\bullet$. We stress once again that each of the natural homotopy equivalences in the chain is natural for arbitrary morphisms in $\mathcal{E}^\bullet$, not just those in the subcategory $\mathcal{E}$.

The functor $\boldsymbol{F}$ has a nonlinear version which is used in [DWW] to state and prove an index theorem. We proceed to explain how, keeping the linear $\boldsymbol{F}$ for simplicity. A key fact is that $\boldsymbol{F}$ comes equipped with a rule which selects for each X in $\mathcal{E}^\bullet$ a point $\langle\!\langle X\rangle\!\rangle$ in the infinite loop space $\Omega^{\infty+1}\boldsymbol{F}(X)$, the *microcharacteristic* of X. It is a refined sort of Euler characteristic. For example, when X is compact and connected, then the component of $\langle\!\langle X\rangle\!\rangle$ in

$$\pi_0(\Omega^{\infty+1}\boldsymbol{F}(X))\cong\mathbb{Z}$$

(see (**) above) is the "usual" Euler characteristic of X. Note however that the microcharacteristic is a point, not a connected component. Microcharacteristics enjoy some naturality. Namely, bending the truth just a little, we may say that for any open subset $V\subset X$, the restriction $\Omega^{\infty+1}\boldsymbol{F}(X)\to\Omega^{\infty+1}\boldsymbol{F}(V)$ (explained earlier) takes $\langle\!\langle X\rangle\!\rangle$ to $\langle\!\langle V\rangle\!\rangle$.

Let $\gamma: E\to X$ be a fiber bundle whose fibers E_x are homeomorphic to $\mathbb{R}^n$. We can make another fibration on X, the *Euler fibration* of γ, with infinite loop space fiber $\Omega^{\infty+1}\boldsymbol{F}(E_x)$ over $x\in X$. Then $x\mapsto\langle\!\langle E_x\rangle\!\rangle$ determines a section of the new fibration, which we call the *Euler section.* It is a refined sort of Euler class. Note that we have omitted a number of serious technical points (what is the topology on the total space of the Euler fibration ; why is the Euler section continuous).

Digression. The geometric significance of the Euler section is clearer in the *nonlinear* set-up: when $\dim(X)<(4n/3)-5$, say, the structure group of γ can be reduced from $\mathrm{TOP}(\mathbb{R}^n)$ to $\mathrm{TOP}(\mathbb{R}^{n-1})$ if and only if the (nonlinear) Euler section of γ is nullhomotopic. *End of digression.*

Let M^n be a closed topological manifold with tangent bundle $TM\to M$. The tangent bundle is a fiber bundle with distinguished "zero" section and with fibers homeomorphic to $\mathbb{R}^n$; we can sufficiently characterize it by assuming that it comes with an *exponential map* $\exp: TM\to M$ which is left inverse to the zero section and embeds each fiber. There is a map $\wp$ from $\Omega^{\infty+1}\boldsymbol{F}(M)$ to the space of sections of the Euler fibration of the tangent bundle which takes $z\in\Omega^{\infty+1}\boldsymbol{F}(M)$ to the section

$$x\mapsto\mathrm{res}(z)\in\Omega^{\infty+1}\boldsymbol{F}(\exp(T_xM))\cong\Omega^{\infty+1}\boldsymbol{F}(T_xM)$$

where res means restriction $\boldsymbol{F}(M)\to\boldsymbol{F}(\exp(T_xM))$. At this point it is necessary to know what the restriction maps do, not just that they exist. This is what we have theorem 2.1 for. It follows quite easily that $\wp$ is a version of *Poincaré duality*. In particular, $\wp$ is a homotopy equivalence. By the naturality property of microcharacteristics, $\wp$ takes the microcharacteristic $\langle\langle M\rangle\rangle$ to the Euler section of the tangent bundle. Hence we have proved a version of Heinz Hopf's index theorem: *the Poincaré dual of the Euler characteristic of M is the Euler class of M*. This works quite well for families, and then the advantage of working with *algebraic K-theory* as opposed to working with the group K_0 becomes apparent.

References

[ACFP]: D.R. Anderson, F.X. Connolly, S. Ferry and E.K. Pedersen, *Algebraic K-theory with continuous control at infinity*, J. Pure and Appl. Algebra **94** (1994), 25–47.

[Ad]: J.F.Adams, *Stable homotopy and generalised homology*, Chicago Lectures in Math., Chicago University Press, Chicago and London, 1974.

[AtSe]: M. Atiyah and G. Segal, *Equivariant K-theory and Completion*, J. Diff. Geometry **3** (1969), 1–18.

[BiLo]: J.-M. Bismut and J. Lott, *Flat vector bundles, direct images and higher real analytic torsion*, J. Amer. Math. Soc. (to appear).

[Ca]: G. Carlsson, *On the algebraic K-theory of infinite product categories*, K-Theory (to appear).

[CaPe]: G. Carlsson and E.K. Pedersen, *Controlled algebra and the Novikov conjectures for K- and L-theory*, Topology (to appear).

[DwKa]: W. Dwyer and D. Kan, *A classification theorem for diagrams of simplicial sets*, Topology **23** (1984), 139–155.

[DWW]: W. Dwyer, M. Weiss and B. Williams, *A parametrized index theorem for the algebraic K-theory Euler class*, Preprint, Feb. 1995, 63 pp.

[EH]: D.A. Edwards and H.M. Hastings, *Čech and Steenrod Homotopy Theories with Applications to Geometric Topology*, Lecture Notes in Math., vol. 542, Springer-Verlag, Berlin, New York, 1976.

[KKS]: D.S. Kahn, J. Kaminker and C. Schochet, *Generalized homology theories on compact metric spaces*, Michigan Math. J. **24** (1977), 203–224.

[M]: J. Milnor, *On the Steenrod Homology Theory*, these proceedings.

[Se]: G.B. Segal, *Categories and Cohomology Theories*, Topology **13** (1974), 293–312.

[Vog]: W. Vogell, *Algebraic K-theory of spaces, with bounded control*, Acta Math. **165** (1990), 161–187.

[WWA]: M. Weiss and B. Williams, *Assembly*, these proceedings.

[WW3]: M. Weiss and B. Williams, *Theory: Finale*, Preprint, 1994, 62pp.

Dept. of Mathematics, Univ. of Michigan, Ann Arbor, MI 48109-1003, USA
E-mail address: `msweiss@math.lsa.umich.edu`

Dept. of Mathematics, Univ. of Notre Dame, Notre Dame, IN 46556, USA
E-mail address: `bruce@bruce.math.nd.edu`