15/06/09

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THE DIRAC - DUAL DIRAC METHOD & BAUM-CONNES CONJECTURE

→ G. Kasparov (87) Introduction of the method to show BC for large classes of groups.

1. $C_0(X)$-algebras (cf. Dixmier, Douady, Kasparov)

A C^*-algebra with a non-degenerate $*$-hom $\phi : C_0(X) \longrightarrow ZM(A)$.

For $x \in X$, define a fiber $A_x = A/I_x$, $I_x = \phi(C_0(X \setminus \{x\}))A$

$a \in A$ can be viewed as $x \longmapsto a_x := a + I_x \in A_x$.

Prop: A is a $C_0(X)$-algebra $\Leftrightarrow$ $\exists$ continuous map $\varphi : \mathrm{Prim}(A) \longrightarrow X$ (same topology as $\hat{A}$)

$\leadsto \phi : C_0(X) \to C^b(\mathrm{Prim}(A)) \simeq ZM(A)$, $\phi(f) := f \circ \varphi$

Thm: [Dauns - Hofmann] $ZM(A) \simeq C^b(\hat{A})$

$z \in ZM(A)$, $\pi \in \hat{A}$, $\pi(z)\pi(a) = \pi(za) = \pi(az) = \pi(a)\pi(z)$

$\Rightarrow \pi(z) := f_z(\pi) \cdot 1$ and the isomorphism is $z \longmapsto f_z$

Examples:

1. $C_0(X) \otimes A \simeq C_0(X, A)$ trivial bundle

2. X Riemannian manifold. $\leadsto C_\tau(X)$: Clifford bundle.

$C_\tau(X)|_x = \mathcal{C}l(T_xX)$ with respect to $\langle\ ,\ \rangle_x$.

If $X = \mathbb{R}^n$, the bundle is trivial, $C_\tau(\mathbb{R}^n) = C_0(\mathbb{R}^n) \otimes \mathcal{C}l(n)$

Balanced tensor product : [maximal, otherwise more complicated.]

A, B $C_0(X)$-algebras

$$A \otimes_X B = A \underset{max}{\otimes} B \Big/ \langle af \otimes b - a \otimes fb \rangle$$

$C_0(X)$-algebra with fibers $A_x \underset{max}{\otimes} B_x$. (Kind of a pointwise tensor product).

Pull-backs :

A $C_0(X)$-algebra , $\varphi : X \longrightarrow Y$

$$\varphi^* A := C_0(Y) \underset{C_0(X)}{\otimes} A \quad : \quad C_0(Y)\text{-algebra}$$

Fibers: $(\varphi^* A)_x = A_{\varphi(y)}$

$\rightsquigarrow$ Restriction to subspaces: $Y \subseteq X$, inclusion $i : Y \hookrightarrow X$

$$A|_Y := i^* A \qquad (\text{same fibers as } A)$$

If $\mathcal{U}$ is open, get an ideal in $C_0(X)$, $A|_{\mathcal{U}} = C_0(\mathcal{U})A$.

Proper G-algebras

Def: X: G-space , A: G-algebra.

Then A is an $X \rtimes G$-algebra if $\exists$ a G-equivariant $\phi : C_0(X) \longrightarrow ZM(A)$.

A is a proper $X \rtimes G$-algebra if X is a proper G-space.

Recall: a proper G-space is always locally induced.

If X is a proper G-space, then $\forall x \in X$, there exists a G-invariant open neighbourhood $\mathcal{U}_x$ of x such that

$$\mathcal{U}_x = G \times_{K_x} \underbrace{Y_x}_{\text{diagonal action}} \rightsquigarrow C_0(G \times_{K_x} Y_x) \cong \mathrm{Ind}_{K_x}^G (C_0(Y_x))$$

Assume A is an $X \rtimes G$-algebra, X proper G-space.

Then $A|_{\mathcal{U}_x} \cong \mathrm{Ind}_{K_x}^G A|_{Y_x}$.

there is a map $\mathrm{Prim}(A|_{\mathcal{U}_x}) \longrightarrow \mathcal{U}_x = G \times_{K_x} Y_x \longrightarrow G/K_x$

$$[g,y] \longmapsto gK_x \qquad (G\text{-map})$$

Prop: Let A be a proper G-algebra. Then $A \rtimes G \underset{\cong}{\longrightarrow} A \rtimes_r G$

<u>Proof:</u> both algebras are $C_0({}_G\backslash X)$-algebras. (G-invariance of functions)

It is enough to find an open cover of ${}_G\backslash X$: $(V_i)_i$ such that

$$C_0(V_i)(A\rtimes G) \cong C_0(V_i)(A\rtimes_r G)$$

Choose a cover $(U_i)_i$ of X such that $U_i \cong G\times_{K_i} Y_i$.

Let $V_i = {}_G\backslash U_i$. Then we can assume $X = G\times_K Y$.

Then $A \simeq \mathrm{Ind}_K^G A|_Y$.

$A\rtimes G = \mathrm{Ind}_K^G A_Y\rtimes G \underset{\text{Morita}}{\sim} A_Y\rtimes K$ by Green's imprimitivity theorem

But K is compact (hence amenable) so $\mathrm{Ind}_K^G A_Y\rtimes_r G \underset{\text{Morita}}{\sim} A_Y\rtimes_r K$ ■

Since $\mathrm{Ind}_K^G A_Y\rtimes G = \mathrm{Ind}_K^G A_Y\rtimes_r G$

and $A_Y\rtimes K = A_Y\rtimes_r K$ ■

$$\begin{tikzcd}
\lim\limits_{Z\subset EG} KK^G_*(C_0(Z),B) \arrow[r, "J^G_{(r)}"] \arrow[d, "\sigma_B(\eta)"] & KK_*(C_0(Z)\rtimes G, B\rtimes_{(r)}G) \arrow[r, "\lambda_Z\otimes\cdot"] \arrow[d, "J^G(\sigma_B(\eta))"] & K_*(B\rtimes_{(r)}G) \arrow[d, "J^G(\sigma_B(\eta))"] \\
\lim\limits_{Z\subset EG} KK^G_*(C_0(Z),B\otimes\mathcal{A}) \arrow[r] \arrow[d, "\sigma_B(D)"] & KK_*(C_0(Z)\rtimes G, B\otimes\mathcal{A}\rtimes_{(r)}G) \arrow[r] \arrow[d, "J^G(\sigma_B(D))"] & K_*((B\otimes\mathcal{A})\rtimes_{(r)}G) \arrow[d, "J^G(\sigma_B(D))"] \\
\lim\limits_{Z\subset EG} KK_*(C_0(Z),B) \arrow[r] & KK_*(C_0(Z)\rtimes G, B\rtimes_{(r)}G) \arrow[r] & K_*(B\rtimes_{(r)}G)\,\gamma
\end{tikzcd}$$

$$\lambda_Z \in K_*(C_0(Z)\rtimes G)$$

Second line is an isom since $\mathcal{A}$ is a proper G-algebra.

First column is $\sigma_B(\gamma)$. $KK^G_*(C_0(Z),B) \xrightarrow[\gamma\otimes_{\mathbb{C}}\cdot]{\cdot\otimes_{\mathbb{C}}\gamma} KK^G_*(C_0(Z),B)$ $\gamma\in KK^G(\mathbb{C},\mathbb{C})$

Last column is prod with γ. since Kasparov prod over $\mathbb{C}$ is commutative.

Aim: want to show BC is an isomorphism for all proper G-algebras.

Using some Mayer-Vietoris and induction arguments, we may assume $A \simeq \mathrm{Ind}_K^G B$, K compact subgroup of G. Use Green-Julg theorem.

If $H < G$ subgroup.

$$\begin{tikzcd} KK^H(C_0(Y), B) \arrow[r] \arrow[d, "\iota_H^G"] & K_*^{top}(H;B) \arrow[r, "\mu_B"] \arrow[d, "\simeq"] \arrow[dr, phantom, "\circlearrowleft"] & K_*(B \rtimes_{(r)} H) \arrow[d, "\simeq \text{ (Green's imprimit.)}"] \\ KK^G(C_0(G \times_H Y), \mathrm{Ind}_H^G B) \arrow[r] & K_*^{top}(G, \mathrm{Ind}_H^G B) \arrow[r, "\mu_{\mathrm{Ind} B}"'] & K_*(\mathrm{Ind} B \rtimes G) \end{tikzcd}$$

We are reduced to induced algebras from subgroups, and we know BC for compact groups.

Idea: assume $\mathcal{A}$ is a proper G-algebra and there exist $D \in KK^G(\mathcal{A}, \mathbb{C})$ (Dirac) and $\eta \in KK^G(\mathbb{C}, \mathcal{A})$ (Dual Dirac) such that

$$D \otimes_{\mathbb{C}} \eta = 1_{\mathcal{A}} \in KK^G(\mathcal{A}, \mathcal{A}), \quad \eta \otimes_{\mathcal{A}} D = 1_{\mathbb{C}} \in KK^G(\mathbb{C}, \mathbb{C}) = R(G)$$

representation ring of G when G is compact.

Let B be any C^*-algebra.

$$\sigma_B(D) = 1_B \otimes D \in KK^G(B \otimes \mathcal{A}, B)$$
$$\sigma_B(\eta) = 1_B \otimes \eta \in KK^G(B, B \otimes \mathcal{A})$$

Then we can get a KK-equivalence to a proper algebra $\rightsquigarrow$ $\blacksquare$

$$\begin{tikzcd} K_*(B \rtimes G) \arrow[r, "\Lambda_{B,*}", "\Uparrow \cong"'] \arrow[d, "\text{KK-equiv.}"'] \arrow[dr, phantom, "\circlearrowleft"] & K_*(B \rtimes_r G) \arrow[d, "\text{KK-equiv.}"] \\ K_*((B \otimes \mathcal{A}) \rtimes G) \arrow[r, "\Lambda_{B\otimes\mathcal{A}\, *}", "\underset{\text{by properness}}{\simeq}"'] & K_*((B \otimes \mathcal{A}) \rtimes_r G) \end{tikzcd}$$

Sometimes, we have $\mathcal{D} \otimes_{\mathbb{C}} \eta = 1_A$ but $\eta \otimes_A \mathcal{D} \neq 1_{\mathbb{C}}$.

Call $\gamma := \eta \otimes_{A} \mathcal{D} \in KK^G(\mathbb{C},\mathbb{C})$.

Pull it back to $\mathcal{E}G$: $\quad p : \mathcal{E}G \longrightarrow \{pt\}$

$$p^*(\gamma) \in \mathcal{R}KK^G(\mathcal{E}G, C_0(\mathcal{E}G), C_0(\mathcal{E}G)) \qquad \left(= RKK^G(\mathcal{E}G, \mathbb{C}, \mathbb{C}) \right)$$

$p^*(\gamma) = 1 \in RKK^G(\mathcal{E}G, \mathbb{C}, \mathbb{C})$.

In such a situation, we get injectivity for BC, γ has to act as 1, not to be $= 1$.

<u>About RKK</u>:

A, B : $X \rtimes G$ - algebras. $\quad \mathbb{E}^G(X; A, B) = \{ (E,T) \in \mathbb{E}^G(A,B)$, such that $f\xi = \xi f$ for $\xi \in E$ and $f \in C_0(X) \}$

(more precisely, $f a \xi = a \xi f$.
assume left action of A non-degenerate

T automatically commutes to X.

Forgetful map $RKK^G(X; A, B) \xrightarrow{F} KK^G(A,B)$

Construction of σ_B can be done in a balanced way.

$f : Y \longrightarrow X$, $\quad f^* : RKK(X, -) \longrightarrow RKK(Y, -)$ $\quad$ (pull-back)

$p^*(\gamma) = 1 \in RKK^G(\mathcal{E}G; C_0(\mathcal{E}G), C_0(\mathcal{E}G))$

If $Z \subseteq \mathcal{E}G$ is G-compact, get a restriction

$$\begin{array}{ccc} p_Z^*(\gamma) & \in & RKK^G(Z; C_0(Z), C_0(Z)) \\ \| & & \downarrow \text{forgetful map} \\ 1 & & \\ \text{I} & & \\ 1 & \in & KK^G(C_0(Z), C_0(Z)) \end{array}$$

<u>Kasparov</u>: assume that G acts properly and isometrically on a complete Riemannian manifold X.

$$\mathcal{D} \in KK^G(C_\tau(X), \mathbb{C}) \quad , \quad \Theta \in RKK^G(X, C_0(X), C_0(X) \otimes C_\tau(X))$$

$$\Theta \otimes_{C_\tau(X)} \mathcal{D} = \Theta \otimes_{C_0(X) \otimes C_\tau(X)} \sigma_{C_0(X)}(\mathcal{D}) = 1_X \in RKK^G(X, C_0(X), C_0(X))$$

Def : X is <u>special</u> if there exists $\eta \in KK^G(\mathbb{C}, C_\tau(X))$ such that

a) $p_X^*(\eta) = \Theta$

b) $D \otimes_{\mathbb{C}} \eta = 1_{C_\tau(X)} \in KK^G(C_\tau(X), C_\tau(X))$

$\gamma = \eta \otimes_{C_\tau(X)} D$, $p_{\mathcal{E}G}^*(\gamma) \overset{?}{=} 1$

If X is special, $p_X^*(\gamma) = 1$: $p_X^*(\gamma) = \sigma_{C_0(X)}(\eta \otimes_{C_\tau(X)} D) = \sigma_{C_0(X)}(\eta) \otimes \sigma_{C_0(X)}(D)$

$= \Theta \otimes \sigma_{C_0(X)}(D) = 1$

Cor : if $X = \mathcal{E}G$ is a special G-manifold, there exists abstract Dirac - Dual Dirac.

$\rightsquigarrow$ $\gamma = 1$: BC
$\rightsquigarrow$ $\gamma \neq 1$, get injectivity

- G almost connected, K maximal compact subgroup.

  Then $X = G/K \simeq \mathcal{E}G$. Such X is a special manifold.

  G/K is still universal for $H < G$.

  $\Rightarrow$ Novikov conjecture passes to discrete subgroups (injectivity of μ).

- Higson. Kasparov : amenable groups have $\gamma = 1$.
- $SO(n,1)$, $SU(n,1)$: $\gamma = 1$