

16/06/09

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2. July

K-theory of the C^* -algebras of free groups

F_n : free group on n generators a_1, \dots, a_n .

- Universal property: for any G , the data of a group morphism $F_n \rightarrow G$ is equivalent to the data of elements g_1, \dots, g_n in G .
- For any C^* -algebra A , a C^* -morphism $C^*F_n \rightarrow A$ is equivalent to the datum of a family of unitaries u_1, \dots, u_n in $\mathcal{U}(A)$. (No such thing for $C_n^*F_n$)
- Left regular representation: $\lambda: C^*F_n \rightarrow C_n^*F_n \subset \mathcal{B}(\ell^2(F_n))$

Special case: $n=1$ $C^*\mathbb{Z} \xrightarrow{\sim} C_n^*\mathbb{Z} \cong C(S^1)$

For $n \geq 2$, $C_0^*F_n \xrightarrow{\lambda} C_n^*F_n$ is not an isomorphism since F_n is not amenable.
 (u_1, \dots, u_n) ($\lambda(a_1), \dots, \lambda(a_n)$)

Th 1 [Cuntz] $\left[\begin{array}{l} K_0(C^*F_n) \cong \mathbb{Z} \text{ , generated by the class of } 1 \\ K_1(C^*F_n) \cong \mathbb{Z}^n \text{ , generated by classes of } u_1, \dots, u_n \end{array} \right.$

Th 2 [Pimsner-Viculescu] $\left[\begin{array}{l} K_0(C_n^*F_n) \cong \mathbb{Z} \text{ , generated by } 1 \\ K_1(C_n^*F_n) \cong \mathbb{Z}^n \text{ , generated by } \lambda(a_1), \dots, \lambda(a_n) \end{array} \right.$

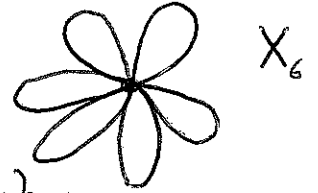
Th 3 [Cuntz] $\left[\begin{array}{l} F_n \text{ is } \mathcal{U}\text{-amenable, which implies that} \\ K_* (C^*F_n) \xrightarrow{\lambda_*} K_* (C_n^*F_n) \\ \text{is an isomorphism.} \end{array} \right.$

Remark:

Th 1 + Th 2 $\Rightarrow K_* (C^*F_n) \xrightarrow{\lambda_*} K_* (C_n^*F_n)$

Topology: F_n can be seen as the fundamental group

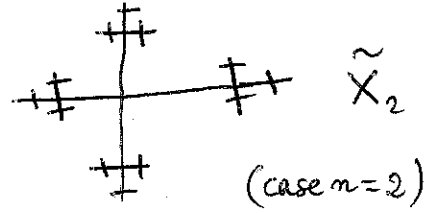
$\pi_1(X_n)$, $X_n = n$ circles with one common point.



Continuous functions on X_n :

$$C(X_n) = \{ (f_1, \dots, f_n) \mid \forall i \in \{1, \dots, n\}, f_i \in C(S^1), f_i(1) = \dots = f_n(1) \}$$

Universal cover of $X_n \longrightarrow \tilde{X}_n$ is a tree



Exercise: what is the K -theory of X_n ? (of $C(X_n)$)

$$0 \longrightarrow C_0(\mathbb{R}) \oplus \dots \oplus C_0(\mathbb{R}) \longrightarrow C(X_n) \longrightarrow \mathbb{C} \longrightarrow 0$$

\leadsto six-term exact sequence $\Rightarrow K_0 = \mathbb{Z}$
 $K_1 = \mathbb{Z}^n$



Construct $\pi: C^*F_n \xrightarrow{(\pi_1, \dots, \pi_n)} C(X_n)$

For $j \in \{1, \dots, n\}$, $\pi_j: F_n \longrightarrow \mathbb{Z}$, $\pi_j(a_i) = 0$ if $i \neq j$
 $\pi_j(a_j) = 1$

$$C^*F_n \xrightarrow{\pi} C(X_n)$$

$$a \longmapsto (\pi_1(a), \dots, \pi_n(a)) \in C(X_n)$$

$$\begin{matrix} C^*\mathbb{Z} \\ \cong \\ C(S^1) \end{matrix}$$

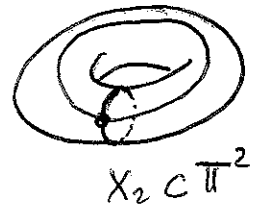
$f_i = \pi_i(a)$ function on S^1 : $f_i(1) = 1_{\mathbb{Z}} \cdot \pi_i(a) = 1_{F_n}(a)$.

Re: π doesn't factorize through $C_n^*F_n$ (unless $n=1$)

Alternative: $F_n \longrightarrow F_n^{ab} = F_n / [F_n, F_n] = \mathbb{Z}^n$

$$C^*F_n \longrightarrow C^*\mathbb{Z}^n \cong C(\mathbb{T}^n) \longrightarrow C(X_n)$$

$$X_n \subset \mathbb{T}^n$$



$$j \circ \pi : C^*F_2 \longrightarrow M_2(C^*F_2)$$

$$a \longmapsto \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$

$$b \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} \sim \begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix}$$

by a rotation trick

$$\left. \begin{array}{l} C^*F_2 \longrightarrow M_2(C^*F_2) \\ g \longmapsto \begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix} \end{array} \right\} = \begin{bmatrix} \text{id} & 0 \\ 0 & \tau_{C^*} \end{bmatrix}. \quad \text{This gives } j_* \circ \pi_*.$$

Maps of morphisms don't make sense at the level of algebras, but do at the level of K -theory. (The category of abelian groups is an abelian category)

$$\hat{\tau} : C(X_2) \longrightarrow C^*F_2$$

$$\begin{array}{ccc} & \searrow & \nearrow \\ & C & \end{array}$$

$$(j_* - \hat{\tau}_*) \circ \pi_* = \text{id}$$

$$\pi_* \circ (j_* - \hat{\tau}_*) = \text{id}$$

This proves that π induces an isomorphism in K -theory.

$$\begin{array}{c} \text{Th} \\ \left| \begin{array}{l} K(C^*F_n) \xrightarrow[\pi_*]{\sim} K(C(X_n)) \\ \downarrow \lambda_* \\ K(C^*F_n) \end{array} \right. \end{array}$$

n=2 $\pi : C^*F_2 \xrightarrow{(a,b)} C(X_2)$, $\pi_* : K(C^*F_2) \longrightarrow K(C(X_2))$

Construct $j : C(X_2) \longrightarrow M_2(C^*F_2)$

"Dual" maps to π_1, π_2 : $f_1 : C(S^1) \xrightarrow{\sim} C\mathbb{Z} \xrightarrow{\alpha} C^*F_2$
 $\qquad\qquad\qquad 1 \longmapsto a$
 $f = \sum c_n e^{in\theta} \longmapsto \sum c_n a^n$

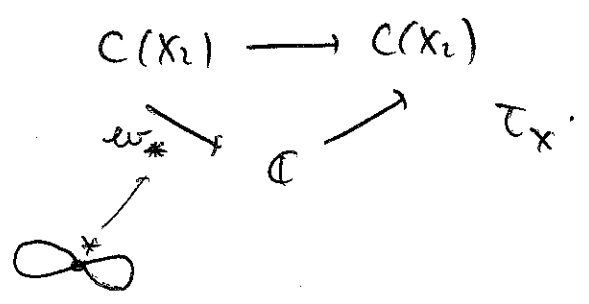
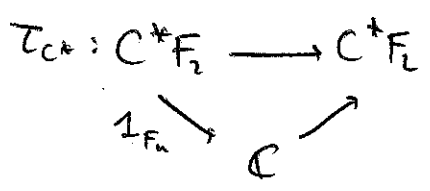
$f_2 : C(S^1) \longrightarrow C^*F_2$
 $f = \sum c_n e^{in\theta} \longmapsto \sum c_n b^n$

Define $j : (f_1, f_2) \longmapsto \begin{bmatrix} j_1(f_1) & 0 \\ 0 & j_2(f_2) \end{bmatrix}$

$j_* : K(C(X_2)) \longrightarrow K(M_2(C^*F_2)) = K(C^*F_2)$

Is j_* an inverse for π_* ?

Lemma : $j_* \circ \pi_* = \text{id}_{K(C^*F_2)} + (\tau_{C^*})_*$
 $\pi_* \circ j_* = \text{id}_{K(C(X_2))} + (\tau_X)_*$
 where



Aim: prove $K(C^*F_2) \xrightarrow{\lambda_*} K(C_n^*F_2)$ is an isomorphism.

Tools (Cuntz): $KK^G(A, B)$ (objects are not really equivariant, but up to compacts).

The key is the construction of an element in the commutative ring $KK^G(\mathbb{C}, \mathbb{C})$, with unit $1 = [\mathbb{C} \xrightarrow{\circ} 0]$ with trivial rep on \mathbb{C} .
 (If G is compact, $KK^G(\mathbb{C}, \mathbb{C}) = \mathcal{R}(G)$)

Lemma: $G = F_2$. There exist two unitary representations (π_0, H_0) and (π_1, H_1) of G and $T: H_0 \rightarrow H_1$ such that:

- 1/ $\left(\begin{array}{ccc} H_0 & \xrightarrow{T} & H_1 \\ \pi_0 & & \pi_1 \end{array} \right)$ defines a class in $KK^G(\mathbb{C}, \mathbb{C})$, that is
 $1 - T^*T, 1 - TT^*$ and $T\pi_0(g) - \pi_1(g)T$ are compact
- 2/ π_0 and π_1 are weakly contained in λ . $(\pi_i: C^*(G) \xrightarrow{G} \mathcal{L}(H_i))$
 $\searrow C_n^*(G)$
- 3/ In $KK^G(\mathbb{C}, \mathbb{C})$, $[\begin{array}{ccc} H_0 & \xrightarrow{T} & H_1 \\ \pi_0 & & \pi_1 \end{array}] \simeq \underset{[\mathbb{C} \rightarrow 0]}{1_G}$ (homotopy).

This lemma implies that λ_* is an isomorphism:

$$j^G: KK^G(\mathbb{C}, \mathbb{C}) \rightarrow KK^G(C^*G, C^*G)$$

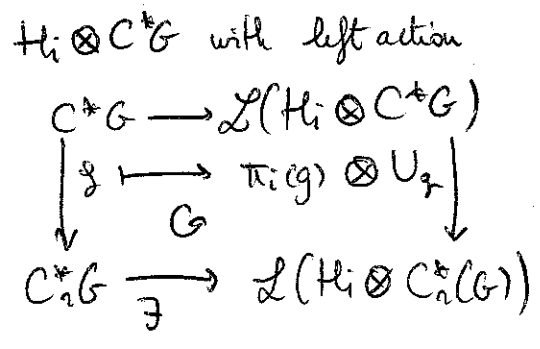
$$j_n^G: KK^G(\mathbb{C}, \mathbb{C}) \rightarrow KK^G(C_n^*G, C_n^*G)$$

$$\lambda: C^*G \rightarrow C_n^*G, \lambda_* \in KK(C^*G, C_n^*G).$$

Let $\gamma \in KK^G(\mathbb{C}, \mathbb{C})$ be the class $[\begin{array}{ccc} H_0 & \xrightarrow{T} & H_1 \end{array}]$.

$$j^G(\gamma) \in KK(C^*G, C^*G)$$

$$j_n^G(\gamma) \in KK(C_n^*G, C_n^*G)$$



Condition 2) gives that $\pi_i \propto \lambda$. Then

$$\begin{array}{ccc} C^*G & \longrightarrow & \mathcal{L}(H \otimes C^*G) \\ \downarrow & \nearrow & \downarrow \\ C_n^*G & \longrightarrow & \mathcal{L}(H \otimes C_n^*G) \end{array}$$


$$\Rightarrow \exists \mu \in KK(C_n^*G, C^*G)$$

Recall $\lambda_* \in KK(C^*G, C_n^*G)$

$$\text{One has } \left\{ \begin{array}{l} \lambda_* \otimes \mu = j^G(\gamma) \in KK(C^*G, C^*G) \\ \mu \otimes \lambda_* = j_n^G(\gamma) \in KK(C_n^*G, C_n^*G) \end{array} \right.$$

Condition 3) gives that $\gamma = 1$, so $j^G(\gamma)$ and $j_n^G(\gamma) = 1$.

$$\left. \begin{array}{l} \lambda_* \otimes \mu = 1_{KK(C^*G, C^*G)} \\ \mu \otimes \lambda_* = 1_{KK(C_n^*G, C_n^*G)} \end{array} \right\} \Rightarrow \lambda^* \text{ isomorphism.}$$

Proof of the lemma: G is a group acting on a tree  here $\#$ action is free

Tree: $\Delta^0 = \{\text{vertices}\}$, $\Delta^1 = \{\text{edges}\}$

$$\begin{array}{ccc} \Delta^1 & \xrightarrow{s} & \Delta^0 \text{ source map} \\ & \searrow t & \Delta^0 \text{ target map} \end{array}$$

$$\Delta^1 \xrightarrow{op} \Delta^1 \quad \nearrow \longmapsto \searrow$$

+ no loops.

$$H_0 := \ell^2(\Delta^0) = \left\{ f: \Delta^0 \rightarrow \mathbb{C} \mid \sum_{x \in \Delta^0} |f(x)|^2 < \infty \right\}, \pi_0 \text{ unit rep on } H_0.$$

Action of G defined on vertices. Then on edges: $g_0 \xrightarrow{e} g_1 \iff g_0x \xrightarrow{ge_1} g_1y$

Proof of condition 1/

$$H_1 := \ell^2(\Delta^1) = \left\{ f: \Delta^1 \rightarrow \mathbb{C} \mid \frac{1}{2} \sum_{e \in \Delta^1} |f(e)|^2 < \infty \right\}$$

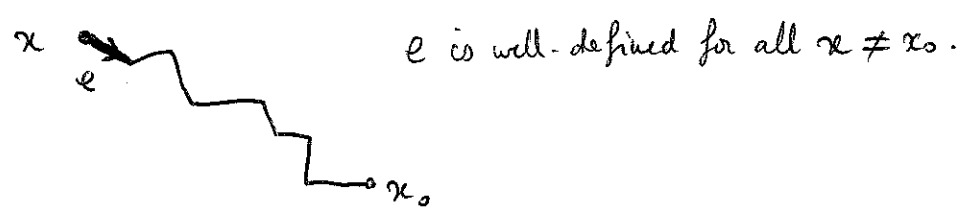
$$f(e^{\sigma}) = -f(e)$$

π_1 unitary rep of G on H_1 .

" Think of H_0 as functions and H_1 as 1-forms - "

$T: \ell^2 \Delta^0 \rightarrow \ell^2 \Delta^1$ analogue of Bott element.

Chose $x_0 \in \Delta^0$. For all $x \in \Delta^0$, $\exists!$ geodesic $x \rightarrow x_0$

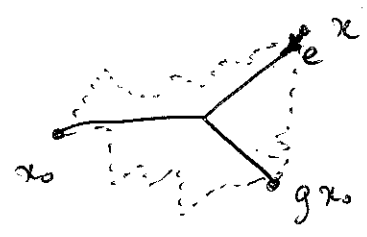


Define T_{x_0} by $\left\{ \begin{array}{l} T_{x_0} \sigma_x = \sigma_e \text{ for } x \neq x_0, \text{ where } \sigma_e(e) = 1 \\ T_{x_0} \sigma_{x_0} = 0 \end{array} \right. \left\{ \begin{array}{l} \sigma_e(e) = 1 \\ \sigma_e(e^{\sigma}) = -1 \\ \sigma_e(e') = 0 \text{ if } e' \notin \{e, e^{\sigma}\} \end{array} \right.$

$T^* T = 1 - p$, with $p =$ orthogonal proj. on $\mathbb{C} \sigma_{x_0}$
 $T T^* = 1$

What is $T_{x_0} - \pi_1(g) T_{x_0} \pi_0(g)^{-1}$?

$T_{x_0} - \pi_1(g) T_{x_0} \pi_0(g)^{-1} = T_{x_0} - T_{gx_0}$



for there are no loops.

$\Rightarrow T_{x_0} - T_{gx_0}$ is of finite rank, hence compact.

Prove condition 2/.

$G = F_2, \Delta^0 = G \pi_0 = \lambda$ on $\ell^2 G$
 $\Delta^1 = \underbrace{G \cup G \cup G \cup G}_{\text{or}} \pi_1 = \lambda \oplus \lambda$

Proof of condition 3/

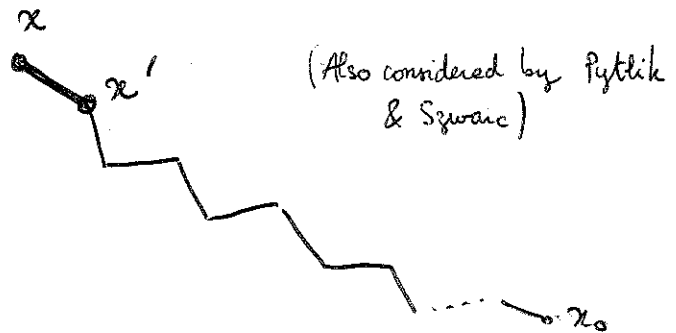
$$\begin{matrix} \ell^2 \Delta^0 & \xrightarrow{T} & \ell^2 \Delta^1 \\ \pi_0 & & \pi_1 \end{matrix}$$

Family of representations $\rho_t(g) = V_t^{-1} \pi_0(g) V_t$ $t \in [0, 1]$ \mathcal{A} :

V_t : bounded operator $\ell^2 \Delta^0 \rightarrow \ell^2 \Delta^0$, $V_0 = id$ $f_0 = \pi_0$

V_t^{-1} : $\left\{ \begin{array}{l} \text{bounded if } t \in [0, t_{\text{critical}}[\\ \text{unbounded (densely defined) if } t \in [t_{\text{critical}}, 1] \end{array} \right.$

Notations: \cdot $\mathbb{P}: \ell^2 \Delta^0 \rightarrow \ell^2 \Delta^0$
 $\delta_x \mapsto \delta_{x'}$
 $\delta_{x_0} \mapsto 0$



\cdot p : projection on $\mathbb{C} \delta_{x_0}$.

Consider: $V_t = 1 - t\mathbb{P} + (\sqrt{1-t^2} - 1)p$

Lemma: $V_t^* V_t$ commutes with $\pi_0(g)$

"Proof": $\mathbb{P} + \mathbb{P}^*$ turns out to be the operator sending δ_x to the sum of its neighbours:
 $\mathbb{P} + \mathbb{P}^* = S: \delta_x \mapsto \sum_{y \text{ neigh. of } x} \delta_y$,

whereas $\mathbb{P}^* \mathbb{P} = Q + p$, where $Q \delta_x = \#(\text{neighbours of } x) \delta_x$.

Then $V_t^* V_t = 1 - tS + t^2 Q$. S, Q are G -invariants.

Consequence: $V_t^{-1} \pi_0(g) V_t$ extends to a unitary.

$$\delta\text{-}\lim_{t \rightarrow 1} V_t^{-1} \pi_0(g) V_t = T^* \pi_0(g) T + p \quad (T^* T = 1)$$