13-07-09 (Münster) : KK^{ban} and the Baum–Connes Conjecture

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* BC with coefficient (?) :

$$\begin{tikzcd} K_*^{top}(\underline{E}G; B) \arrow[r] \arrow[dr, "\cong ?"'] & K_*(B \rtimes_r G) \\ & K_*(\mathcal{A}(G,B)) \arrow[u, "\cong ?"'] \end{tikzcd}$$

* $\mathcal{E}$: class of groups, contains: • G locally compact acting geometrically (continuously, properly, isometrically) on :
  * complete, simply connected Riemann mflds with s.c. ≤ 0
  * affine buildings
  * uniformly locally finite, weakly geodesic, weakly bolic metric spaces.

  • G discrete, acting geometrically on weakly

  G a-T-menable

* for $\mathcal{C}$: γ-element, BC + coefficients is injective
  Lafforgue : Bost with coeff. is surjective.

$\mathcal{C}' = \{ G$ loc. compact $\curvearrowright$ geometrically on

+ compl. simply connected Riem. mflds, s.c ≤ 0

s.c is bdd below + derivative of the curvature tensor is bounded

+ uniformly locally finite, weakly geodesics, weakly bolic metric spaces + bolicity condition $\}$

* $\mathcal{C}' \subset \mathcal{C}$: For $\mathcal{C}'$: Bost is surjective.

Definition: An alg. norm $\|\cdot\|_{\mathcal{A}}$ on $C_c(G)$ is called "unconditional" if:

$$\forall f_1, f_2 \in C_c(G) : (\forall g \in G, |f_1(g)| \leq |f_2(g)|) \Rightarrow (\|f_1\|_{\mathcal{A}} \leq \|f_2\|_{\mathcal{A}}).$$

The completion of $C_c(G)$ with respect to $\|\cdot\|_{\mathcal{A}}$ is denoted by $\mathcal{A}(G)$.

Now, on $C_c(G, B)$ with B a Banach algebra, define

$$\|f\|_{\mathcal{A}(G,B)} := \| g \mapsto \|f(g)\|_B \|_{\mathcal{A}}$$

Question: What is $KK^{ban}(A, B)$ for Banach algebras A, B?

1. What are B-Hilbert modules?

Definition 1.1: A right Banach B-module E is a right B-module which is, at the same time a Banach space such that $\forall e \in E$ and $\forall b \in B$ one has

$$\|eb\| \leq \|e\| \|b\|.$$

Definition 1.2: A right Banach B-pair is a pair $E=(E^<, E^>)$ together with a map $\langle \cdot, \cdot\rangle : E^< \times E^> \longrightarrow B$ such that

(i) $E^<$ is a left Banach B-module;

(ii) $E^>$ is a right " "

(iii) $\langle \cdot, \cdot \rangle$ is $\mathbb{C}$-bilinear and satisfies:

(a) $\langle be^<, e^>\rangle = b\langle e^<, e^>\rangle$,

(b) $\langle e^<, e^> b\rangle = \langle e^<, e^>\rangle b$

(c) $\| \langle e^<, e^>\rangle \|_B \leq \|e^<\|_{E^<} \cdot \|e^>\|_{E^>}$,

for all $b \in B$, $e^< \in E^<$ and $e^> \in E^>$.

If B is a C^*-algebra and $\mathcal{H}$ is a Hilbert B-module then $(\mathcal{H}^*, \mathcal{H})$ is a Banach B-pair (under $be := eb^*$).

2. What are adjointable operators ?

Definition 2.1: Let E_B and F_B be Banach B-pairs. An operator from E to F is a pair $T = (T^<, T^>)$ such that

(1) $T^< \in {}_B\mathcal{L}(F^<, E^<)$; i.e. $E^< \xleftarrow{T^<} F^<$

(2) $T^> \in \mathcal{L}_B(E^>, F^>)$; i.e. $E^> \xrightarrow{T^>} F^>$

(3) $\forall\, e^> \in E^>, f^< \in F^<$: $\langle T^< f^<, e^>\rangle_B = \langle f^<, T^> e^>\rangle_B$

The space of all operators from E to F is called $\mathcal{L}_B(E, F)$ (Banach space with norm $\|T\| = \sup\{\|T^<\|, \|T^>\|\}$).

3. What are compact operators ?

Definition 3.1: Let E and F be B-pairs. Let $e^< \in E^<$, $f^> \in E^>$. Define:

$|f^>\rangle\langle e^<|^> : E^> \longrightarrow F^>,\ e^> \longmapsto f^>\langle e^<, e^>\rangle$.

$|f^>\rangle\langle e^<|^< : F^< \longrightarrow E^<,\ f^< \longmapsto \langle f^<, f^>\rangle e^<$.

This defines an element $|f^>\rangle\langle e^<| \in \mathcal{L}_B(E,F)$.
Call an element in the closed linear span $\mathcal{K}_B(E,F)$ of these operators "compact".

Examples 3.2. 1) Let X be a Banach space. Then (X', X) is a Banach $\mathbb{C}$-pair, and $\mathcal{K}_{\mathbb{C}}(X', X)$ consists of approximate operators on X.

2) $\mathcal{K}_B(B) = cl\{ \underbrace{|b\rangle\langle c|}, b \in B \} = cl\{ b \cdot \mid b \in B \} \subseteq \mathcal{L}_B(B)$

↳ multiplication with bc from the other side

⚠ $b \mapsto b\cdot$, $B \longrightarrow \mathcal{K}_B(B)$ is neither surjective, nor injective in general.

4. What is $\mathbb{E}^{ban}(A,B)$?

It is the set of triples (E, ϕ, T), where E is a graded B-pair, $\phi: A \longrightarrow \mathcal{L}_B(E)$ is a Banach algebra homomorphism (i.e.: $\langle e^< a, e^> \rangle = \langle e^<, a e^> \rangle$
and T is an odd element in $\mathcal{L}_B(E)$ such that:

a) $\forall a \in A: [\phi(a), T] \in \mathcal{K}_B(E)$,

b) $\forall a \in A: \phi(a)(T^2 - 1) \in \mathcal{K}_B(E)$.

5. What is $KK^{ban}(A,B)$?

Define $KK^{ban}(A,B) := \mathbb{E}^{ban}(A,B) / \mathbb{E}^{ban}(A, B[0,1])$.
One shows that $KK^{ban}(A,B)$ is an abelian group which acts on K-theory: $KK^{ban}(A,B) \longrightarrow Hom(K_0(A), K_0(B))$.

- We have to point out that there is no Kasparov product known for KK^{ban}.

However, there is an isomorphism

$$KK^{ban}(\mathbb{C}, B) \cong K_0(B).$$