

THE GREEN-JULG THEOREM

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ABSTRACT. We give a short KK -theoretic proof of the Green-Julg Theorem, i.e., we show that for any compact group G and any G - C^* -algebra B the group $KK_*^G(\mathbb{C}, B)$ is canonically isomorphic to $K_*(B \rtimes G)$.

Let G be a locally compact group and let A and B be two G - C^* -algebras. Then the equivariant KK -groups $KK^G(A, B) =: KK_0^G(A, B)$ are defined as the set of all homotopy classes of triples (\mathcal{E}, Φ, T) , where

- (1) $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$ is a \mathbb{Z}_2 -graded Hilbert B -module endowed with a grading preserving action $\gamma : G \rightarrow \text{Aut}(\mathcal{E})$;
- (2) $\Phi = \begin{pmatrix} \Phi_0 & 0 \\ 0 & \Phi_1 \end{pmatrix}$ is a G -equivariant $*$ -homomorphism;
- (3) $T = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}$ is an operator in $\mathcal{L}(\mathcal{E})$ such that

$$[\Phi(a), T], (T^* - T)\Phi(a), (T^2 - 1)\Phi(a), (\text{Ad } \gamma_s(T) - T)\Phi(a) \in \mathcal{K}(\mathcal{E})$$

for all $a \in A$ and $s \in G$.

If G is compact, then it was shown by Kasparov that we may assume without loss of generality that the operator T is G -equivariant (by replacing T by $T^G = \int_G \text{Ad } \gamma_s(T) ds$ if necessary) and that Φ is nondegenerate. In particular, if $A = \mathbb{C}$, then $KK^G(\mathbb{C}, B)$ can be described as the set of homotopy classes of pairs (\mathcal{E}, T) such that T is G -invariant and

$$T^* - T, T^2 - 1 \in \mathcal{K}(\mathcal{E}).$$

Similarly, we can describe $K_0(B \rtimes G) = KK(\mathbb{C}, B \rtimes G)$. We start with the following easy lemma:

Lemma 0.1. *Let G be a compact group and let B be a G - C^* -algebra. Suppose that \mathcal{E}_B is a G -equivariant Hilbert B -module. Then \mathcal{E}_B becomes a pre-Hilbert $B \rtimes G$ -module if we define the right action of $B \rtimes G$ on \mathcal{E}_B and the $B \rtimes G$ -valued inner products by the formulas*

$$e \cdot f := \int_G \gamma_s(e \cdot f(s^{-1})) ds \quad \text{and} \quad \langle e_1, e_2 \rangle_{B \rtimes G}(s) := \langle e_1, \gamma_s(e_2) \rangle_B$$

for $e, e_1, e_2 \in \mathcal{E}$ and $f \in C(G, B) \subseteq B \rtimes G$. Denote by $\mathcal{E}_{B \rtimes G}$ its completion. Moreover, if (\mathcal{E}_B, T) represents an element of $KK^G(\mathbb{C}, B)$ with T being G -invariant, then T extends to an operator on $\mathcal{E}_{B \rtimes G}$ such that $(\mathcal{E}_{B \rtimes G}, T)$ represents an element of $KK(\mathbb{C}, B \rtimes G)$.¹

Proof. First note that the above defined right action of $C(G, B)$ on \mathcal{E}_B extends to an action of $B \rtimes G$. For this we observe that the pair (Ψ, γ) , with $\Psi : B \rightarrow \mathcal{L}_{\mathcal{K}(\mathcal{E})}(\mathcal{E}_B)$ given by the formula $\Psi(b)(e) = e \cdot b^*$, is a covariant homomorphism of (B, G, β) on the left Hilbert

¹We are grateful to Walther Paravicini for pointing out a mistake in a previous version of this lemma!

$\mathcal{K}(\mathcal{E}_B)$ -module \mathcal{E}_B . Then $e \cdot f = (\Psi \times \gamma(f))(e)$ for $f \in C(G, B)$, and the right hand side clearly extends to all of $B \rtimes G$.

It is easily seen that $\langle \cdot, \cdot \rangle_{B \rtimes G}$ is a well defined $B \rtimes G$ -valued inner product which is compatible with the right action of $B \rtimes G$ on \mathcal{E}_B . So we can define $\mathcal{E}_{B \rtimes G}$ as the completion of \mathcal{E}_B with respect to this inner product.

If $T \in \mathcal{L}_B(\mathcal{E}_B)$, then T determines an operator $T^G \in \mathcal{L}_{B \rtimes G}(\mathcal{E}_{B \rtimes G})$ by the formula

$$T^G(e) = \int_G \gamma_t(T(\gamma_{t^{-1}}(e))) dt, \quad \text{for } e \in \mathcal{E}_B \subseteq \mathcal{E}_{B \rtimes G},$$

and one checks that $T \rightarrow T^G$ is a $*$ -homomorphism from $\mathcal{L}_B(\mathcal{E}_B)$ to $\mathcal{L}_{B \rtimes G}(\mathcal{E}_{B \rtimes G})$. In particular, it follows that every G -invariant operator on \mathcal{E}_B extends to an operator on $\mathcal{E}_{B \rtimes G}$. If $e_1, e_2 \in \mathcal{E}_B \subseteq \mathcal{E}_{B \rtimes G}$, then a short computation shows that the corresponding finite rank operator $\Theta_{e_1, e_2} \in \mathcal{K}(\mathcal{E}_{B \rtimes G})$ is given by the formula

$$\Theta_{e_1, e_2} = \tilde{\Theta}_{e_1, e_2}^G$$

if $\tilde{\Theta}_{e_1, e_2} \in \mathcal{K}(\mathcal{E}_B)$ denotes the corresponding finite rank operator on \mathcal{E}_B . This easily implies that the remaining part of the lemma. \square

Theorem 0.2 (Green-Julg Theorem). *Let G be a compact group and let B be a G - C^* -algebra. Then the map*

$$\mu : KK^G(\mathbb{C}, B) \rightarrow KK(\mathbb{C}, B \rtimes G); \mu([(E_B, T)]) = [(E_{B \rtimes G}, T)]$$

is an isomorphism.

Proof. Note first that we can apply the same formula to a homotopy, so the map is well defined. We now define a map $\nu : KK(\mathbb{C}, B \rtimes G) \rightarrow KK^G(\mathbb{C}, B)$ and show that it is inverse to μ .

For this let $L^2(G, B)$ denote the Hilbert B -module defined as the completion of $C(G, B)$ with respect to the B -valued inner product

$$\langle f, g \rangle_B = \int_G \beta_s(f(s^{-1})^* g(s^{-1})) ds$$

and the right action of B on $L^2(G, B)$ given by $(f \cdot b)(t) = f(t)\beta_t(b)$ for $f \in C(G, B), b \in B$. There is a well defined left action of $B \rtimes G$ on $L^2(G, B)$ given by convolution when restricted to $C(G, B) \subseteq B \rtimes G$ (and $C(G, B) \subseteq L^2(G, B)$). We even have $B \rtimes G \subseteq \mathcal{K}(L^2(G, B))$. To see this we simply note that $\mathcal{K}(L^2(G, B)) = C(G, B) \rtimes G$ by Green's imprimitivity theorem (where G acts on $C(G)$ by left translation), and $B \rtimes G$ can be viewed as a subalgebra of $C(G, B) \rtimes G$ in a canonical way.

Let $\sigma : G \rightarrow \text{Aut}(L^2(G, B))$ be defined by

$$\sigma_s(f)(t) = f(ts); \quad f \in C(G, B).$$

Then σ is compatible with the action β of G on B . Moreover, a short computation shows that the homomorphism of $B \rtimes G$ into $\mathcal{L}(L^2(G, B))$ given by convolution is equivariant with respect to the trivial G -action on $B \rtimes G$ and the action $\text{Ad } \sigma$ on $L^2(G, B)$. Assume now that $(\tilde{\mathcal{E}}, T)$ represents an element of $KK(\mathbb{C}, B \rtimes G)$. Then $\tilde{\mathcal{E}} \otimes_{B \rtimes G} L^2(G, B)$ equipped with the action $\text{id} \otimes \sigma$ is a G -equivariant Hilbert B -module and $(\tilde{\mathcal{E}} \otimes L^2(G, B), T \otimes 1)$

represents an element of $KK^G(\mathbb{C}, B)$ (here we use the fact that $B \rtimes G \subseteq \mathcal{K}(L^2(G, B))$). Thus we define

$$\nu : KK(\mathbb{C}, B \rtimes G) \rightarrow KK^G(\mathbb{C}, B); \quad \nu([\tilde{\mathcal{E}}, T]) = [(\tilde{\mathcal{E}} \otimes_{B \rtimes G} L^2(G, B), T \otimes 1)].$$

Again, applying the same formula to homotopies implies that ν is well defined.

To see that ν is an inverse to μ one checks:

- (a) Let \mathcal{E}_B be a Hilbert B -module and let $\mathcal{E}_{B \rtimes G}$ be the corresponding Hilbert $B \rtimes G$ -module as described in Lemma 0.1. Then

$$\mathcal{E}_B \odot C(G, B) \rightarrow \mathcal{E}_B; e \otimes f \mapsto e \cdot f = \int_G \gamma_s(e \cdot f(s^{-1})) ds$$

extends to a G -equivariant isometric isomorphism between $\mathcal{E}_{B \rtimes G} \otimes_{B \rtimes G} L^2(G, B)$ and \mathcal{E}_B .

- (b) Let $\tilde{\mathcal{E}}$ be a Hilbert $B \rtimes G$ -module. Then

$$\tilde{\mathcal{E}} \odot C(G, B) \rightarrow \tilde{\mathcal{E}}; e \otimes f \mapsto e \cdot f$$

determines an isometric isomorphism

$$(\tilde{\mathcal{E}} \otimes_{B \rtimes G} L^2(G, B))_{B \rtimes G} \cong \tilde{\mathcal{E}}$$

as Hilbert $B \rtimes G$ -modules.

Both results follow from some straightforward computations. Note that for the proof of (b) one should use the fact that for all $x \in B \rtimes G$ and $f, g \in C(G, B)$ the element of $B \rtimes G$ given by the continuous function $s \mapsto \langle x^* \cdot f, \sigma_s(g) \rangle_B$ coincides with $f^* * x * g$, where we view f, g as elements of $B \rtimes G$. This follows by direct computations for $x \in C(G, B)$, and since both expressions are continuous in x , it follows for all $x \in B \rtimes G$. Finally, it is trivial to see that the operators match up in both directions. \square

Remark 0.3. We should remark, that the above Theorem 0.2 is a special case of a more general result for crossed products by proper actions due to Kasparov and Skandalis (see [2, Theorem 5.4]). There also exist important generalizations to proper groupoids by Tu [3, Proposition 6.25] and Paravicini [4], where the latter provides a version within Lafforgue's Banach KK-theory. For the original proof of the Green-Julg theorem for compact groups we refer to [1].

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