Part II

The Regulator Map for Cyclotomic Fields

§1. The Main Theorem

Let $F = \mathbb{Q}(\mu_N)$, μ_N the group of N-th roots of unity and $N > 1$, and let $X = \mathrm{Spec}(F)$. For $\zeta \in \mu_N$ we have set

$$L_n(\zeta) = (\ldots, L_n(\alpha\zeta), \ldots)_{\alpha: F\to\mathbb{C}} \in (\mathbb{C}/\mathbb{R}(n))^{X(\mathbb{C})} ,$$

where

$$L_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$$

is the polylogarithm function (see Part I, §1). The purpose of this part is to present Beilinson's proof of the following theorem on the regulator map

$$r_{\mathcal{D}}: H^1_A(X,\mathbb{Q}(n)) \to H^1_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(n)).$$

<u>(1.1) Theorem:</u> For every $n \geq 1$ we have a map of $\mathrm{Gal}(F/\mathbb{Q})$-sets

$$\varepsilon_{n+1}: \mu_N \smallsetminus \{1\} \to H^1_A(X,\mathbb{Q}(n+1))$$

such that, for $\zeta \in \mu_N \smallsetminus \{1\}$,

$$r_{\mathcal{D}}(\varepsilon_{n+1}(\zeta)) = L_{n+1}(\zeta) .$$

This theorem may be seen as a complete and explicit description of the regulator map for $X = \mathrm{Spec}(\mathbb{Q}(\mu_N))$ in the following sense. Let $\mu_N^\times$ be the set of primitive N-th roots of unity and consider the linear map

$$L_{n+1}: \mathbb{Q}^{\mu_N^\times} = \bigoplus_{\zeta\in\mu_N^\times} \mathbb{Q}\zeta_* \to (\mathbb{C}/\mathbb{R}(n+1))^{X(\mathbb{C})}$$

given by $\zeta_* \mapsto L_{n+1}(\zeta)$. The Galois group $G = \mathrm{Gal}(F/\mathbb{Q})$ acts on $\mu_N^\times$ and $X(\mathbb{C})$, and thus on both vector spaces, and L_{n+1} is a G-homomorphism. Moreover, if c denotes the involution $\zeta_* \mapsto (-1)^n(\zeta^{-1})_*$ on the left hand side and on the right hand side the involution given by complex conjugation on $\mathbb{C}/\mathbb{R}(n+1)$

and on $X(\mathbb{C})$, then

$$\mathcal{L}_{n+1}(cv) = c\mathcal{L}_{n+1}(v) \ .$$

We indicate by $[\ \ ]^+$ the fixed module of c and obtain:

<u>(1.2) Corollary:</u> For $n \geq 1$ we have a canonical G-isomorphism

$$\varepsilon_{n+1}: [\mathbb{Q}^{\mu_N^\times}]^+ \xrightarrow{\cong} H^1_A(X,\mathbb{Q}(n+1))$$

such that the diagram

$$\begin{tikzcd}
H^1_A(X,\mathbb{Q}(n+1)) \arrow[r, "r_{\mathcal{D}}"] & H^1_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(n+1)) \arrow[d, equal] \\
{[\mathbb{Q}^{\mu_N}]^+} \arrow[u, "\varepsilon_{n+1}", "\cong"'] \arrow[r, "\mathcal{L}_{n+1}"] & {[\mathbb{C}/\mathbb{R}(n+1)^{X(\mathbb{C})}]^+}
\end{tikzcd}$$

is commutative.

Proof: We define the map

$$\varepsilon_{n+1}: \mathbb{Q}^{\mu_N^\times} = \bigoplus_{\zeta\in\mu_N^\times} \mathbb{Q}\zeta_* \to H^1_A(X,\mathbb{Q}(n+1))$$

by $\zeta_* \mapsto \varepsilon_{n+1}(\zeta)$ and have to show that its restriction to $[\mathbb{Q}^{\mu_N^\times}]^+$ is an isomorphism. Tensoring the above diagram with $\mathbb{R}$, $r_{\mathcal{D}}$ becomes an isomorphism. It therefore suffices to show that the map $\mathcal{L}_{n+1}$ becomes also an isomorphism. Now

$$\mathbb{R}^{\mu_N^\times} \xrightarrow{\mathcal{L}_{n+1}} (\mathbb{C}/\mathbb{R}(n+1))^{X(\mathbb{C})}$$

is a homomorphism of $\mathbb{R}[G]$-modules of rank 1. We tensor this map with $\mathbb{C}$ and then decompose it into a direct sum according to the different characters χ of G. Setting

$$e_\chi = \sum_{\tau\in G} \chi^{-1}(\tau)\tau \in \mathbb{C}[G]$$

we have $\mathbb{C}[G] = \bigoplus_\chi \mathbb{C}e_\chi$ and the components

$$\mathcal{L}_{n+1,\chi}: e_\chi(\mathbb{C}^{\mu_N^\times}) \to e_\chi((\mathbb{C}\otimes\mathbb{C}/\mathbb{R}(n+1))^{X(\mathbb{C})})$$

of $\mathcal{L}_{n+1}$ are homomorphism of 1-dimensional $\mathbb{C}$-vector spaces and are given by

$$\mathcal{L}_{n+1,\chi}(e_\chi(\zeta_*)) = e_\chi\mathcal{L}_{n+1}(\zeta) = \sum_{\tau\in G} \chi^{-1}(\tau) \otimes \mathcal{L}_{n+1}(\tau\zeta) \ .$$

It was shown in the proof of (5.1) in Part I that the right hand side is nonzero if $\chi(-1) = (-1)^n$ so that $L_{n+1,\chi}$ is an isomorphism in that case. But only those characters contribute to the spaces in the diagram.

§2. Universal Symbols

In [17] Loday has constructed symbols in K-theory which have a larger domain of definition than the Steinberg symbols. Since we need in fact symbols only in absolute cohomology ([21] §3) we shall content ourselves with stating the results in this language, especially since then the analogous construction in Deligne cohomology becomes transparent. To stress this fact we shall actually first state the results in Deligne cohomology and then indicate how to translate them into the other language.

Let $\mathbb{A}^{n+1}$ be affine n+1-space over $\mathrm{Spec}(\mathbb{R})$, with coordinate functions $X_o,\dots,X_n$. Let $Y \subseteq \mathbb{A}^{n+1}$ be the divisor $X_o = 0$ and let $U := \mathbb{A}^{n+1} \setminus \bigcup_{i=1}^{n} (X_i=0)$ be the complement of the union of the remaining n coordinate hyperplanes. Let $\phi := 1 - X_o\cdot\ldots\cdot X_n$ and let D be the smooth divisor $\phi = 0$. We denote by a subscript U, resp. (ϕ), the intersection with U, resp. with the complement of D. Note that $Y_{(\phi)} = Y$. We consider the open immersion of closed pairs of schemes

$$(U_{(\phi)}, Y_U) \to (\mathbb{A}^{n+1}_{(\phi)}, Y)\ .$$

For the definition of the (algebraic) Deligne cohomology of a closed pair (which also can be considered as a simplicial scheme; [7] §6.3) we refer to [3] §1 or [10] §5.

<u>(2.1) Proposition</u>: The restriction homomorphism

$$H^p_{\mathcal{D}}(\mathbb{A}^{n+1}_{(\phi)}, Y; \mathbb{Z}(q)) \to H^p_{\mathcal{D}}(U_{(\phi)}, Y_U; \mathbb{Z}(q))$$

is an isomorphism for all p and q.

Proof: We shall use the connecting homomorphisms

$$H^p_{\mathcal{D}}(\mathbb{A}^{n+1}_{(\phi)}, \mathbb{Z}(q)) \to H^{p-1}_{\mathcal{D}}(D, \mathbb{Z}(q-1)) \quad \text{and}$$

$$H^p_{\mathcal{D}}(U_{(\phi)}, \mathbb{Z}(q)) \to H^{p-1}_{\mathcal{D}}(D, \mathbb{Z}(q-1))$$

in the Gysin sequence ([14] 1.19). This is justified since D is a smooth divisor in $\mathbb{A}^{n+1}$ and in U. These homomorphisms induce the oblique arrows in the following commutative diagram

$$\begin{tikzcd}
H^p_{\mathcal{D}}(\mathbb{A}^{n+1}_{(\phi)}, Y; \mathbb{Z}(q)) \arrow[d] \arrow[dr] & \\
& H^{p-1}_{\mathcal{D}}(D, \mathbb{Z}(q-1)) \\
H^p_{\mathcal{D}}(U_{(\phi)}, Y_U; \mathbb{Z}(q)) \arrow[ur] &
\end{tikzcd}$$

We show that the vertical arrow is an isomorphism by proving the oblique arrows to be isomorphisms. The proofs for both arrows being virtually identical we limit ourselves to treating the lower arrow. We use the following commutative diagram with exact rows and columns. The rows are part of the relative cohomology sequence and the columns come from the Gysin sequence; the oblique arrows are defined by the commutativity requirement. For typographical simplicity we leave out the subscript $\mathcal{D}$ and the coefficients.

$$\begin{tikzcd}
H^{p-1}(U) \arrow[d] \arrow[dr] & & & H^p(U) \arrow[d] \arrow[dr] & \\
H^{p-1}(U_{(\phi)}) \arrow[r] \arrow[d] & H^{p-1}(Y_U) \arrow[r, "(*)"] & H^p(U_{(\phi)}, Y_U) \arrow[r] \arrow[dr, dashed] & H^p(U_{(\phi)}) \arrow[r] \arrow[d] & H^p(Y_U) \\
\vdots & & & H^{p-1}(D) \arrow[d, "(*)"] & \\
& & & H^{p+1}(U) \arrow[d] \arrow[dr] & \\
& & & H^{p+1}(U_{(\phi)}) \arrow[r] & H^{p+1}(Y_U)
\end{tikzcd}$$

We note that the closed immersion $Y_U \to U$ is isomorphic to $\mathbb{G}_m^n \to \mathbb{A}^1 \times \mathbb{G}_m^n$. By homotopy invariance all solid oblique arrows are isomorphisms. Therefore all starred arrows are zero. It follows that the dotted oblique arrow is an isomorphism as asserted. q.e.d.

We now are in a position to define the Loday symbol in Deligne cohomology. We consider the cohomology classes

$$\{\phi\} \in H^1_{\mathcal{D}}(U_{(\phi)}, Y_U; \mathbb{Z}(1)) \quad \text{and}$$

$$\{X_1\}, \ldots, \{X_n\} \in H^1_{\mathcal{D}}(U_{(\phi)}, \mathbb{Z}(1)) ;$$

here we have used the identifications ([10] 2.12)

$$H^1_{\mathcal{D}}(U_{(\phi)}, \mathbb{Z}(1)) = \mathcal{O}(U_{(\phi)})^{\times}_{\mathrm{alg}} \quad \text{and}$$

$$H^1_{\mathcal{D}}(U_{(\phi)}, Y_U; \mathbb{Z}(1)) = \ker(\mathcal{O}(U_{(\phi)})^{\times}_{\mathrm{alg}} \to \mathcal{O}(Y_U)^{\times}_{\mathrm{alg}}).$$

We form the cup-product

$$\{\phi\} \cup \{X_1\} \cup \ldots \cup \{X_n\} \in H^{n+1}_{\mathcal{D}}(U_{(\phi)}, Y_U; \mathbb{Z}(n+1)).$$

The inverse image of this element under the restriction isomorphism of (2.1) is the <u>Loday symbol in Deligne cohomology</u>

$$\{\phi, X_1, \ldots, X_n\}_{\mathcal{D}} \in H^{n+1}_{\mathcal{D}}(\mathbb{A}^{n+1}_{(\phi)}, Y; \mathbb{Z}(n+1)) .$$

We denote by the same symbol also its image in $H^{n+1}_{\mathcal{D}}(\mathbb{A}^{n+1}_{(\phi)}, Y; R(n+1))$ for any subring R in $\mathbb{R}$.

<u>(2.2) Corollary</u> (of the proof of (2.1)): The image of the Loday symbol $\{\phi, X_1, \ldots, X_n\}_{\mathcal{D}}$ under the isomorphism

$$H^{n+1}_{\mathcal{D}}(\mathbb{A}^{n+1}_{(\phi)}, Y; \mathbb{Z}(n+1)) \xrightarrow{\cong} H^n_{\mathcal{D}}(D, \mathbb{Z}(n))$$

is equal to the cup-product $\{X_1\} \cup \ldots \cup \{X_n\}$ of the elements $\{X_i\} \in H^1_{\mathcal{D}}(D, \mathbb{Z}(1))$.

<u>Proof:</u> We have denoted here by the same symbol $\{X_i\}$ the cohomology class of the algebraic function X_i <u>on</u> D (which is invertible). The result follows from the fact that the Gysin sequence for $D \to U$ is a sequence of $H^*_{\mathcal{D}}(U, \mathbb{Z}(*))$-modules under the cup-product (reference lacking) and the fact that the connecting homomorphism in the Gysin sequence

$$\begin{array}{ccc} H^1_{\mathcal{D}}(U_{(\phi)}, \mathbb{Z}(1) & \longrightarrow & H^0_{\mathcal{D}}(D, \mathbb{Z}) \\ \| & & \| \\ \mathcal{O}(U_{(\phi)})^{\times}_{\mathrm{alg}} & \xrightarrow{\mathrm{ord}_D} & \mathbb{Z} \end{array}$$

maps $\{\phi\}|U_{(\phi)}$ to 1 ([14] 3.1.1).

We now wish to define the <u>Loday symbol in absolute cohomology</u>

$$\{\phi, X_1, \ldots, X_n\}_A \in H_A^{n+1}(\mathbb{A}^{n+1}_{(\phi)}, Y; \mathbb{Q}(n+1)) .$$

Here $\mathbb{A}^{n+1}$ is the affine space over $\operatorname{Spec}(\mathbb{Q})$ and the other schemes, too, are taken over $\operatorname{Spec}(\mathbb{Q})$. In fact, both (2.1) and (2.2) transpose to absolute cohomology since the only facts about $\mathcal{D}$-cohomology that we used were 1. functoriality, 2. the extension of the cohomology theory together with the cup-product to affine simplicial schemes, 3. the existence of the Gysin sequence, 4. the homotopy invariance, and 5. the properties of the Gysin sequence used in the proof of (2.2). As for the corresponding facts about absolute cohomology, 1. is obvious, 2. is achieved using the homotopy limit ([3] 2.2.1; compare also [26]), 3. is proved in [23] Thm. 9 or [24]5.2, 4. is a well-known theorem of Quillen ([19] p. 114), and for 5. we refer to [11] 7.14 and [19] § 7.5.16.

(2.3) Corollary: The regulator map ([21] §4)

$$r_{\mathcal{D}}: H_A^{n+1}(\mathbb{A}^{n+1}_{(\phi)}, Y; \mathbb{Q}(n+1)) \to H_{\mathcal{D}}^{n+1}(\mathbb{A}^{n+1}_{(\phi)\mathbb{R}}, Y_{\mathbb{R}}; \mathbb{Q}(n+1))$$

maps $\{\phi, X_1, \ldots, X_n\}_A$ to $\{\phi, X_1, \ldots, X_n\}_{\mathcal{D}}$.

It is clear that, having handled the universal case, we can extend the domain of definition of the Loday symbol. To fix ideas we consider the case of $\mathcal{D}$-cohomology. Let $X = \operatorname{Spec}(A)$ be an affine scheme over $\operatorname{Spec}(\mathbb{R})$ and $S \subseteq X$ be a closed subscheme defined by the ideal I in A . Let $f, a_1, \ldots, a_n \in A$ be elements such that f is a unit, that the a_i are not zero divisors, and finally that

$$a_o := \frac{1-f}{a_1 \cdot \ldots \cdot a_n}$$

lies in $I \subseteq A$. We obtain a morphism of pairs of schemes over $\operatorname{Spec}(\mathbb{R})$

$$h: (X,S) \to (\mathbb{A}^{n+1}_{(\phi)}, Y)$$

defined by $h^*(X_i) = a_i$ for $i = 0, \ldots, n$. We put

$$\{f, a_1, \ldots, a_n\}_{\mathcal{D}} := h^*(\{\phi, X_1, \ldots, X_n\}_{\mathcal{D}}) \in H_{\mathcal{D}}^{n+1}(X, S; \mathbb{Z}(n+1)).$$

This notation is justified since when in addition the $a_1, \ldots, a_n$ are all units in A the symbol $\{f, a_1, \ldots, a_n\}_{\mathcal{D}}$

reduces to the Steinberg symbol (in relative cohomology), i.e., to the cup-product $\{f\} \cup \{a_1\} \cup \ldots \cup \{a_n\}$.

We shall need to "calculate" the Loday symbol in $\mathcal{D}$-cohomology explicitly. To this end Beilinson formulates a lemma (7.0.2 in [3]) whose proof he leaves as an exercise. We were unable to even understand the assertion in the generality it is stated; and even in the special case where the assertion makes perfect sense and which would be sufficient for our purposes here we were unable to prove it. This situation is highly unsatisfactory since this lemma is absolutely crucial to the proof of theorem (1.1). We shall first state the lemma in the special case and then comment on it.

(2.4) Crucial lemma: We use the notation introduced after (2.3) and assume in addition that X is smooth of dimension $\leq n$. Let Z be a relative singular C^∞-homology n-cycle on $X(\mathbb{C})$ modulo $S(\mathbb{C})$. We make the assumption that there is a branch log f of the logarithm of f which is single valued in a neighbourhood of the support $|Z|$ of Z and vanishes at every point $z \in |Z|$ such that $z \in S(\mathbb{C})$ or $a_1(z) \cdot \ldots \cdot a_n(z) = 0$ (because $a_0 \in I$ these are points where $f(z) = 1$). Then the following equality of numbers in $\mathbb{C}/\mathbb{Q}(n+1)$ holds:

$$\langle Z, \{f, a_1, \ldots, a_n\}_{\mathcal{D}} \rangle = \int_Z \log f \, d\log a_1 \wedge \ldots \wedge d\log a_n .$$

Here on the left side we have used the isomorphism

$$H^*_{\mathcal{D}}(X_{/\mathbb{C}}, S_{/\mathbb{C}}; \mathbb{Z}(n+1)) \xrightarrow{\cong} H^{*-1}(X(\mathbb{C}), S(\mathbb{C}); \mathbb{C}/\mathbb{Z}(n+1))$$

([3] 1.1), valid since $\dim X \leq n$, to evaluate the $\mathcal{D}$-cohomology class on a relative homology n-cycle. On the right side there is the integral of a differential form which is regular in a neighbourhood of $|Z|$, as follows easily from the assumption on the branch $\log f$ and the fact that $a_0 \in I$.

Note that the right side is independent of the choice of the branch of the logarithm. To see this we may assume the support $|Z|$ to be connected. If there is a $z \in |Z|$ such that $z \in S(\mathbb{C})$ or $a_1(z) \ldots a_n(z) = 0$ then the branch is unique. Otherwise the functions $a_1,\ldots,a_n$ are all non-vanishing on $|Z|$. Consider the morphism $a = (a_1,\ldots,a_n): X \setminus \bigcup_i \{a_i=0\} \to \mathbb{G}_m^n$. Denoting by $d\log T_1,\ldots,d\log T_n$ the standard generators of the de Rham cohomology of $\mathbb{G}_m^n$ we have

$$\int_Z d\log a_1 \wedge \ldots \wedge d\log a_n = \int_{a_*(Z)} d\log T_1 \wedge \ldots \wedge d\log T_n \in \mathbb{Z}(n) .$$

The assertion follows easily.

Using the fact that the differential form on the right side is regular even in a neighbourhood of $|Z| \cup S(\mathbb{C}) \cup \bigcup_i \{a_i=0\}$ and vanishes on $S(\mathbb{C})$ and using the assumption $\dim X \leq n$ together with Stokes' theorem one also can show that the integral only depends on the relative homology class of Z.

Let us consider the case $n = 1$ and where X is of dimension 1 and S is empty. Also assume that both f and a_1 are invertible functions on X. Then Beilinson states the following formula for the evaluation of the cup-product $\{f\} \cup \{a_1\} = \{f,a_1\}_{\mathcal{D}}$ on a 1-cycle $[\gamma]$ represented by a loop γ based on $s_o \in X$ (see [3] 1.3.1 resp. [2]):

$$<[\gamma],\{f,a_1\}_{\mathcal{D}}> = \int_\gamma \log f \, d\log a_1 - \log a_1(s_o) \cdot \int_\gamma d\log f .$$

Here $\log f$ and $\log a_1$ are branches of the logarithm which are continuous on γ outside s_o. If we assume, as in the statement of the lemma, that $\log f$ is continuous on γ then the second summand disappears which is the assertion of the lemma. As the next example keep all assumptions as before but take $S = \{s_o,\ldots,s_r\}$ non-empty. Then we may represent a relative 1-cycle as $[\gamma] = [\gamma_o] + \sum_{i=1}^{r} n_r[\gamma_i]$ where γ_i are curves joining s_o to s_i. The corresponding formula appears to be (put $n_o := 1$)

$$<[\gamma],\{f,a_1\}_{\mathcal{D}}> = \sum_i n_i \left(\int_{\gamma_i} \log f \, d\log a_1 - \log a_1(s_o) \cdot \int_{\gamma_i} d\log f \right) .$$

If $\log f$ is continuous on the γ_i and in addition satisfies the vanishing condition $\log f(s_i) = 0$ for $0 \leq i \leq r$ then the second sum vanishes and we obtain the formula of the lemma.

The only approach we can see to a proof of this lemma is to reduce to the universal situation. However, then the dimension hypothesis is not satisfied. Still, an evaluation as in the statement of the lemma, i.e., of a $\mathcal{D}$-cohomology class on a (relative) homology cycle (which would in general not vanish on a boundary) should exist even without this hypothesis. Working with coefficients $\mathbb{R}(n+1)$ which would suffice for our purposes this can be seen to be the case by using the "real version" of the $\mathcal{D}$-cohomology, representing a class by C^∞-differential forms of degree n (compare [3]1.2.5 or [10]2.16). The general case may possibly have something to do with Beilinson's evaluation map ([3]1.1.2).

§3. Special Symbols

We now apply the constructions of the preceding section. We wish to construct elements in some relative absolute cohomology group $H_A^{n+1}(\mathbb{A}_F^n, S.; \mathbb{Q}(n+1))$; this will be done in the following sections. Here we shall construct these elements on a big open subset. Let F be a number field and let $z \in F$, $z \neq 1$, have absolute value 1 under all embeddings of F in $\mathbb{C}$. We consider a rational function $f = f_{a,b}(z) \in F(t_1,\ldots,t_n)$ of the following form. Let $a = (a_{ij})$, $b = (b_{ij})$, $i=1,\ldots,n$, be matrices of positive integers and set

$$f_{a,b}(z) := \prod_j \frac{1-z\cdot\prod_i t_i^{a_{ij}}}{1-z\cdot\prod_i t_i^{b_{ij}}} .$$

We let $\mathbb{A}^n_{(f)}$ denote the complement of the zeroes and poles of f on $\mathbb{A}_F^n = \mathrm{Spec}(F[t_1,\ldots,t_n])$. We also let $S_{(f)}$ denote the intersection of $\mathbb{A}^n_{(f)}$ with

$$S = \{\prod_i t_i(t_i-1) = 0\} .$$

In the language of §2 we wish to consider the element $\{f_{a,b}(z),t_1,\dots,t_n\}_A \in H_A^{n+1}(\mathbb{A}^n_{(f)};\mathbb{Q}(n+1))$; but we have to make sure that for a suitable choice of a and b the hypotheses of §2 are met. We use the following elementary lemma.

(3.1) Lemma: Let F be any field of characteristic zero.

For every $n \geq 1$ there are matrices of integers $a = (a_{ij})$ and $b = (b_{ij})$ of size $n \times 2^{n-1}$ such that:

(i) $a_{ij}, b_{ij} \geq 2$ for all i and j ;

(ii) for every i there is a permutation $\sigma = \sigma_i \in \Sigma_{2^{n-1}}$ such that $a_{kj} = b_{k\sigma(j)}$ for all $k \neq i$ and all j ;

(iii) $C_{a,b} := \sum_j (\prod_i a_{ij}^{-1} - \prod_i b_{ij}^{-1}) \neq 0$.

Proof: We proceed by induction. For $n=1$ put (e.g.)

$$a(1) := 2 \quad \text{and} \quad b(1) := 3 .$$

For $n > 1$ define

$$a(n) := \begin{pmatrix} a(n-1) & b(n-1) \\ n\dots n & n+1\dots n+1 \end{pmatrix} \text{and} \quad b(n) := \begin{pmatrix} a(n-1) & b(n-1) \\ n+1\dots n+1 & n\dots n \end{pmatrix}.$$

It is easy to see that (i) and (ii) hold. The number $C_{a,b}$ in (iii) is equal to $\frac{1}{6} \cdot \prod_{i=2}^{n} (\frac{1}{i} - \frac{1}{i+1}) \neq 0$.

(3.2) Corollary: For any choice of a and b satisfying the conditions of the previous lemma the rational function

$$\frac{1-f_{a,b}(z)}{t_1\cdot\ldots\cdot t_n}$$

is regular on $\mathbb{A}^n_{(f)}$, $f = f_{a,b}(z)$, and lies in the ideal $(\prod_i t_i(t_i-1))$.

Proof: Condition (i) of (3.1) ensures that the rational function $1-f$ vanishes of order ≥ 2 on the intersection of $\mathbb{A}^n_{(f)}$ with the coordinate hyperplanes $t_i = 0$. Using condition (ii) we have for any i

$$f = \prod_j \frac{1-(z\cdot\prod_{k\neq i} t_k^{a_{kj}})\cdot t_i^{a_{ij}}}{1-(z\cdot\prod_{k\neq i} t_k^{b_{k\sigma(j)}})\cdot t_i^{b_{i\sigma(j)}}} = \prod_j \frac{1-c_{ij}t_i^{a_{ij}}}{1-c_{ij}t_i^{b_{i\sigma(j)}}}$$

so that $1-f$ also vanishes on the intersection of $\mathbb{A}^n_{(f)}$ with the hyperplanes $t_i = 1$.

Fixing matrices a and b as in (3.1) we put $f := f_{a,b}(z)$ and $C := C_{a,b} \neq 0$. We may now form the symbol

$$l_{a,b}(z) := C^{-1}_{a,b}\cdot\{f_{a,b}(z),t_1,\dots,t_n\}_A \in H^{n+1}_A(\mathbb{A}^n_{(f)},S_{(f)};\mathbb{Q}(n+1)) .$$

By the same symbol we shall denote the inverse image in $H^{n+1}_A(\mathbb{A}^n_{(f)},S_{(f)}.;\mathbb{Q}(n+1))$. Here $S_{(f)}.$, resp. $S.$, is the simplicial scheme over $\mathbb{A}^n_{(f)}$, resp. $\mathbb{A}^n_F$, obtained "by resolution of singularities of the divisor with normal crossings" $S_{(f)}$ in $\mathbb{A}^n_{(f)}$, resp. S in $\mathbb{A}^n_F$: We have

$$S. : S_o \leftleftarrows S_1 \;\substack{\leftarrow\\ \leftarrow\\ \leftarrow}\; S_2 \dots$$

where

$$S_o := \text{normalization of } S \text{ and}$$
$$S_p := S_o \underset{\mathbb{A}^n_F}{\times} \dots \underset{\mathbb{A}^n_F}{\times} S_o$$

and similarly for $S_{(f)}.$. We shall see in the next section that $l_{a,b}(z)$ is independent of the choice of a and b, in a sense to be made precise.

We now wish to apply the crucial lemma (2.4). In that formula we shall take as relative n-cycle

$$Z := \{0 \le t_i \le 1 \quad \text{for all } i\}$$

To see that this is legitimate we have to convince ourselves that for any embedding $\alpha: F \to \mathbb{C}$ the relative cycle Z actually lies in $\mathbb{A}^n_{\mathbb{C}(\alpha f)}$. It is here that the assumptions on z are used. Indeed, assume by way of contradiction that there is a $t \in Z$ with $\alpha f(t) = 0$. (The case where $\alpha f(t) = \infty$ is similar.) Then there is a j with

$$\prod_i t_i^{a_{ij}} = \alpha z^{-1}$$

from which we conclude that

$$\left|\prod_i t_i^{a_{ij}}\right| = |\alpha z^{-1}| = 1 \;,\; \text{i.e., } t_i = 1 \quad \text{for all } i$$

so that $z = 1$, contrary to our hypothesis $z \neq 1$.

We observe that since Z is simply connected and since the restriction of αf to the boundary of Z is identically 1 we may choose a branch of the logarithm $\log\alpha f$ which satisfies the hypothesis of (2.4). We therefore obtain the following formula.

<u>(3.3) Lemma</u>: The image of $l_{a,b}(z)$ under the regulator map

$$r_{\mathcal{D}}: H_{\mathcal{A}}^{n+1}(\mathbb{A}^n_{(f)}, S_{(f)}; \mathbb{Q}(n+1)) \to H_{\mathcal{D}}^{n+1}(\mathbb{A}^n_{(f)\mathbb{R}}, S_{(f)\mathbb{R}}; \mathbb{R}(n+1))$$

has the property that, for any embedding $\alpha: F \to \mathbb{C}$,

$$\langle Z, \alpha r_{\mathcal{D}}(l_{a,b}(z))\rangle = c_{a,b}^{-1} \cdot \int_Z \log f_{a,b} \operatorname{dlog} t_1 \wedge \ldots \wedge \operatorname{dlog} t_n$$

$$= L_{n+1}(\alpha z) \in \mathbb{C}/\mathbb{R}(n+1) .$$

Proof: Only the last identity has to be checked. The changes of variables $t_i \mapsto t_i^{a_{ij}}$ and $t_i \mapsto t_i^{b_{ij}}$ show that

$$c^{-1} \cdot \int_Z \log\alpha f \operatorname{dlog} t_1 \wedge \ldots \wedge \operatorname{dlog} t_n = \int_Z \log(1 - \alpha(z) t_1 \cdot \ldots \cdot t_n) \operatorname{dlog} t_1 \wedge \ldots \wedge \operatorname{dlog} t_n .$$

The right hand side is the classical integral representation of the polylogarithm function $L_{n+1}(\alpha z)$ (which may be proved by induction on n).

<u>§4. Reduction to the Main Lemma</u>

The second crucial result whose proof will be the content of the last two sections is the following fact.

<u>(4.1) Main lemma:</u> If $\zeta \neq 1$ is a root of unity then the symbols $l_{a,b}(\zeta)$ are contained in the image of the restriction map

$$H_A^{n+1}(\mathbb{A}_F^n, S.; \mathbb{Q}(n+1)) \overset{\text{res}}{\to} H_A^{n+1}(\mathbb{A}^n_{(f)}, S_{(f)}.; \mathbb{Q}(n+1)) \ .$$

In fact, taking this for granted it is not difficult to actually view our symbols $l_{a,b}(\zeta)$ as elements of the absolute cohomology group $H_A^1(X, \mathbb{Q}(n+1))$ where $X := \operatorname{Spec}(F)$ (viewed as an affine $\mathbb{Q}$-scheme).

<u>(4.2) Proposition:</u> We have a canonical isomorphism

$$H_A^1(X, \mathbb{Q}(n+1)) \cong H_A^{n+1}(\mathbb{A}_F^n, S.; \mathbb{Q}(n+1)) \ .$$

Proof: We first consider the standard spectral sequence

$$E_1^{p,q} = H_A^q(S_p, \mathbb{Q}(n+1)) \Rightarrow H_A^{p+q}(S., \mathbb{Q}(n+1))$$

of the simplicial scheme $S.$. Every S_p is a disjoint union of affine spaces $\mathbb{A}_F^i$ so that, by homotopy invariance, we have

$$H_A^q(S_p) = H_A^q(X)^{\pi_o(S_p)}$$

where, for notational simplicity, we omit the coefficients in the following. Therefore, $E_1^{\cdot q}$ is the complex

$$H_A^q(X)^{\pi_o(S_o)} \xrightarrow{\partial_o} H_A^q(X)^{\pi_o(S_1)} \xrightarrow{\partial_1} H_A^q(X)^{\pi_o(S_2)} \xrightarrow{\partial_2} \dots$$

which clearly is the standard cochain complex with coefficients in $H_A^q(X)$ for the boundary ∂Z of the n-cube Z , i.e., for the (n-1)-sphere Σ^{n-1} . We consequently obtain

$$E_2^{pq} = H^p(\Sigma^{n-1}, H_A^q(X)) = \begin{cases} H_A^q(X) & \text{for } n>1 \text{ and } p=0,n-1 \\ H_A^q(X) \oplus H^q(X) & \text{for } n=1 \text{ and } p=0, \\ 0 & \text{otherwise.} \end{cases}$$

We furthermore take into account that

$$H_A^q(X) = 0 \quad \text{for} \quad q \neq 1$$

(compare Part I §1). Both facts together obviously imply

$$H_A^n(S.) = \begin{cases} H_A^1(X) & \text{if} \quad n > 1 \ , \\ H_A^1(X) \oplus H_A^1(X) & \text{if} \quad n = 1 \ . \end{cases}$$

On the other hand, if we apply the second fact together with the homotopy invariance to the relative cohomology

$$\ldots \to H^n_A(S.) \to H^{n+1}_A(\mathbb{A}^n_F, S.) \to H^{n+1}_A(\mathbb{A}^n_F) \to \ldots$$

we obtain

$$H^n_A(S.) = H^{n+1}_A(\mathbb{A}^n_F, S.) \quad \text{if} \quad n > 1 \ ,$$

resp. the exact sequence

$$0 \to H^1_A(X) \to H^1_A(S.) \to H^2_A(\mathbb{A}^1_F, S.) \to 0 \ .$$

Combining both results gives in either case the required canonical isomorphism. q.e.d.

Exactly the same argument works in Deligne cohomology so that we also get a canonical isomorphism

$$H^1_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(n+1)) \cong H^{n+1}_{\mathcal{D}}(\mathbb{A}^n_{F\mathbb{R}}, S._{\mathbb{R}}; \mathbb{R}(n+1)) \ .$$

The key diagram which we have to study now is the following:

$$\begin{tikzcd}
H^1_A(X, \mathbb{Q}(n+1)) \arrow[r, "r_{\mathcal{D}}"] \arrow[d, "\cong"'] & H^1(X_{\mathbb{R}}, \mathbb{R}(n+1)) \arrow[r, "\subseteq"] \arrow[d, "\cong"] & \prod\limits_{\alpha} \mathbb{C}/\mathbb{R}(n+1) \arrow[ddd, equal] \\
H^{n+1}_A(\mathbb{A}^n_F, S.; \mathbb{Q}(n+1)) \arrow[r, "r_{\mathcal{D}}"] \arrow[d, "\mathrm{res}"'] & H^{n+1}_{\mathcal{D}}(\mathbb{A}^n_{F\mathbb{R}}, S._{\mathbb{R}}; \mathbb{R}(n+1)) \arrow[d, "\mathrm{res}"] & \\
H^{n+1}_A(\mathbb{A}^n_{(f)}, S_{(f)}; \mathbb{Q}(n+1)) \arrow[r, "r_{\mathcal{D}}"] & H^{n+1}_{\mathcal{D}}(\mathbb{A}^n_{(f)\mathbb{R}}, S_{(f)\cdot\mathbb{R}}; \mathbb{R}(n+1)) & \\
H^{n+1}_A(A^n_{(f)}, S_{(f)}; \mathbb{Q}(n+1)) \arrow[r, "r_{\mathcal{D}}"] \arrow[u] & H^{n+1}_{\mathcal{D}}(\mathbb{A}^n_{(f)\mathbb{R}}, S_{(f)\mathbb{R}}; \mathbb{R}(n+1)) \arrow[r, "\langle Z, . \rangle"] \arrow[u, "\cong"'] & \prod\limits_{\alpha} \mathbb{C}/\mathbb{R}(n+1)
\end{tikzcd}$$

The right lower vertical arrow is an isomorphism by the very definition of Deligne cohomology (via smooth simplicial resolutions). All the left rectangles are commutative by the general functorial properties of the regulator map $r_{\mathcal{D}}$.

(4.3) Lemma: The right rectangle in the above diagram is commutative.

Proof: This is a statement purely about the Betti cohomology of schemes over $\mathrm{Spec}(\mathbb{R})$. We can assume, for the purposes of this proof, that $F = \mathbb{R}$ and we then have to show the commutativity of the diagram

$$\begin{array}{ccccccc}
H^{o}(X(\mathbb{C}),\mathbb{R}(n)) & & & & & = & \mathbb{R}(n) \\
\downarrow{\scriptstyle (*)} & & & & & & \\
H^{n-1}(S.(\mathbb{C}),\mathbb{R}(n)) & = & H^{n-1}(S(\mathbb{C}),\mathbb{R}(n)) & = & H^{n-1}(S(\mathbb{R}),\mathbb{R}(n)) & \xrightarrow{<\partial Z,.>} & \mathbb{R}(n) \\
\downarrow & & & & & & \\
H^{n}(\mathbb{A}^{n}(\mathbb{C}),S.(\mathbb{C});\mathbb{R}(n)) & = & H^{n}(\mathbb{A}^{n}(\mathbb{C}),S(\mathbb{C});\mathbb{R}(n)) & & & \xrightarrow{<Z,.>} & \mathbb{R}(n)
\end{array}$$

where the starred arrow is constructed as in the proof of (4.2). The commutativity of the lower rectangle is a formal property of the connecting homomorphism ∂ . On the other hand, ∂Z is a fundamental cycle for $S(\mathbb{R})$ so that the map $<\partial Z,.>$ is the usual trace map. If we go back to the proof of (4.2) we easily see that $(*)$ by construction is a section of the trace map.

(4.4) Lemma: The two restriction maps in the above diagram are injective.

Proof: By Borel's theorem ((1.3) in Part I) the regulator map $r_{\mathcal{D}}$ in the two upper rows of the diagram are injective. Therefore it suffices to prove the assertion for the $\mathcal{D}$-cohomology where it is an immediate consequence of (4.3).

Now let $\mu(F)$ denote the subset of roots of unity in F . The above discussion shows that, for $\zeta \in \mu(F)\setminus\{1\}$, our symbol $l_{a,b}(\zeta)$ has a unique preimage $\tilde{l}_{a,b}(\zeta) \in H^1_A(X,\mathbb{Q}(n+1))$. Furthermore, the above commutative diagram together with (3.3) implies that

$$\alpha r_{\mathcal{D}}(\tilde{l}_{a,b}(\zeta) = L_{n+1}(\alpha\zeta) \in \mathbb{C}/\mathbb{R}(n+1) .$$

Since the regulator map $r_{\mathcal{D}}$ is injective by Borel we see that the element $\tilde{l}_{a,b}(\zeta)$ does not depend on the particular choice of the matrices a and b . By setting

$$\begin{aligned}\varepsilon_{n+1}: \mu(F)\setminus\{1\} &\to H^1_A(X,\mathbb{Q}(n+1))\\ \zeta &\mapsto \varepsilon_{n+1}(\zeta) := \tilde{l}_{a,b}(\zeta)\end{aligned}$$

we therefore get a natural map such that

$$\alpha r_{\mathcal{D}}(\varepsilon_{n+1}(\zeta)) = L_{n+1}(\alpha\zeta) \quad \text{for any} \quad \alpha: F \to \mathbb{C} .$$

Hence Theorem (1.1) will be established once we have proved (4.1).

§5. Proof of the Main Lemma for $n = 1$

In this case the proof is very simple. For $n = 1$, S consists of two points and therefore $S.$ is the constant simplicial scheme, so that

$$H^*_{\mathcal{A}}(\mathbb{A}^1_F, S; \mathbb{Q}(*)) = H^*_{\mathcal{A}}(\mathbb{A}^1_F, S.; \mathbb{Q}(*)) .$$

We have to show that $l_{a,b}(\zeta)$ lies in the image of the restriction map

$$H^2_{\mathcal{A}}(\mathbb{A}^1_F, S; \mathbb{Q}(2)) \to H^2_{\mathcal{A}}(\mathbb{A}^1_{(f)}, S_{(f)}; \mathbb{Q}(2)) .$$

Using the Gysin sequence this amounts to the statement that the image of $l_{a,b}(\zeta)$ under the connecting homomorphism

$$H^2_{\mathcal{A}}(\mathbb{A}^1_{(f)}, S_{(f)}; \mathbb{Q}(2)) \to \oplus\, H^1_{\mathcal{A}}(y, \mathbb{Q}(1))$$

is zero; here y ranges over the zeros and poles of f and we have made use of the fact that S is disjoint from the support of the divisor of f . Recall that

$$f = \frac{1 - \zeta t^a}{1 - \zeta t^b} \qquad \text{with} \quad a,b \geq 2 \quad \text{and} \quad a \neq b$$

where we have abbreviated t_1 to t . We will see that already the restriction of $l_{a,b}(\zeta)$ to $\mathbb{A}^1_{(f)} \smallsetminus S_{(f)}$ is zero. Both f and t are invertible on $\mathbb{A}^1_{(f)} \smallsetminus S_{(f)}$ and therefore this restriction simply is the Steinberg symbol

$$l_{a,b}(\zeta)\,|\mathbb{A}^1_{(f)} \smallsetminus S_{(f)} = \{f\} \cup \{t\} = \{f,t\} \in H^2_{\mathcal{A}}(\mathbb{A}^1_{(f)} \smallsetminus S_{(f)}; \mathbb{Q}(2)) .$$

If $\zeta^N = 1$ we compute using the Steinberg relation

$$\begin{aligned} \{f,t\} &= \tfrac{1}{a}\{1 - \zeta t^a, t^a\} - \tfrac{1}{b}\{1 - \zeta t^b, t^b\} \\ &= -\tfrac{1}{a}\{1 - \zeta t^a, \zeta\} + \tfrac{1}{b}\{1 - \zeta t^b, \zeta\} \\ &= -\tfrac{1}{Na}\{1 - \zeta t^a, 1\} + \tfrac{1}{Nb}\{1 - \zeta t^b, 1\} = 0 . \end{aligned}$$

§6. Proof of the Main Lemma for $n \geq 2$

Curiously enough the proof will proceed by reducing our purely K-theoretic assertion to a certain assertion about

Deligne cohomology which then is established via Hodge theoretic arguments.

First step: We introduce the following notations. Let

$$T^n := \{(t_1,\dots,t_n) : \prod t_i \neq 0\}$$

be the torus which is the open complement of the coordinate hyperplanes in $\mathbb{A}^n_F$. As already in previous sections we denote by an index the intersection with an open subset, as e.g. in $S^n_T := S \cap T^n$. Also note that for clarity's sake we use an index n to indicate in which affine space we are working.

(6.1) Lemma: To prove the main lemma it is sufficient to prove that the restriction homomorphism

$$H^{n+1}_A(\mathbb{A}^n_{(f)}, S^n_{(f)\cdot}; \mathbb{Q}(n+1)) \to H^{n+1}_A(T^n_{(f)}, S^n_{T(f)\cdot}; \mathbb{Q}(n+1))$$

maps $l_{a,b}(\zeta)$ to zero.

Proof: Since the support of the divisor of f is contained in T^n we have $\mathbb{A}^n_F = \mathbb{A}^n_{(f)} \cup T^n$. The corresponding Mayer-Vietoris sequence has the form

$$H^{n+1}_A(\mathbb{A}^n_F, S^n_\cdot; \mathbb{Q}(n+1)) \to H^{n+1}_A(\mathbb{A}^n_{(f)}, S^n_{(f)\cdot}; \mathbb{Q}(n+1)) \oplus H^{n+1}_A(T^n, S^n_{T\cdot}; \mathbb{Q}(n+1))$$
$$\to H^{n+1}_A(T^n_{(f)}, S^n_{T(f)\cdot}; \mathbb{Q}(n+1))$$

which shows that

$$\ker(H^{n+1}_A(\mathbb{A}^n_{(f)}, S^n_{(f)\cdot}; \mathbb{Q}(n+1)) \to H^{n+1}_A(T^n_{(f)}, S^n_{T(f)\cdot}; \mathbb{Q}(n+1)))$$
$$\subseteq \operatorname{im}(H^{n+1}_A(\mathbb{A}^n_F, S^n_\cdot; \mathbb{Q}(n+1)) \to H^{n+1}_A(\mathbb{A}^n_{(f)}, S^n_{(f)\cdot}; \mathbb{Q}(n+1))) .$$

From this the claim follows.

Second step: The restriction of $l_{a,b}(\zeta)$ in $H^{n+1}_A(T^n_{(f)}, S^n_{T(f)\cdot}; \mathbb{Q}(n+1))$ may be described as follows. Let $T^{n-1}_i := T^n \cap \{t_i = 1\}$, so that $S^n_T = \bigcup_{i=1}^n T^{n-1}_i$. Since t_i is invertible on T^n and equal to 1 on T^{n-1}_i it defines an element

$$\{t_i\} \in H^1_A(T^n, T^{n-1}_i; \mathbb{Q}(1))$$

and the cup-product

$$H^1_A(T^n,T^{n-1}_1;\mathbb{Q}(1)) \times\ldots\times H^1_A(T^n,T^{n-1}_n;\mathbb{Q}(1)) \xrightarrow{\cup} H^n_A(T^n,S^n_T;\mathbb{Q}(n))$$

yields an element $\{t_1,\ldots,t_n\} \in H^n_A(T^n,S^n_{T^\bullet};\mathbb{Q}(n))$. Restricting $\{t_1,\ldots,t_n\}$ to $(T^n_{(f)},S^n_{(f)^\bullet})$ and cupping with

$$\{f\} \in H^1_A(T^n_{(f)},\mathbb{Q}(1))$$

we obtain that restriction as a sum of Steinberg symbols

$$\begin{aligned} 1_{a,b}(\zeta)\,|\,(T^n_{(f)},S^n_{T(f)^\bullet}) &= C^{-1}\{f,t_1,\ldots,t_n\} \\ &= C^{-1}\cdot\sum_j\{1-\zeta\prod_i t_i^{a_{ij}},t_1,\ldots,t_n\} \\ &\quad - C^{-1}\cdot\sum_j\{1-\zeta\prod_i t_i^{b_{ij}},t_1,\ldots,t_n\}\ . \end{aligned}$$

Consider

$$\dot{T}^{n-1} := \{(t_1,\ldots,t_n)\ :\ \zeta\prod_i t_i = 1\} \subseteq T^n$$

and let $U^n := T^n \setminus \dot{T}^{n-1}$.

<u>(6.2) Lemma</u>: It suffices to prove that the Steinberg symbol

$$\{1-\zeta\prod_i t_i,t_1,\ldots,t_n\} \in H^{n+1}_A(U^n,S^n_{U^\bullet};\mathbb{Q}(n+1))$$

is zero.

Proof: Each of the summands above,

$$C^{-1}\{1-\zeta\prod_i t_i^{a_{ij}},t_1,\ldots,t_n\} = C^{-1}\prod_i a_{ij}^{-1}\{1-\zeta\prod_i t_i^{a_{ij}},t_1^{a_{1j}},\ldots,t_n^{a_{nj}}\}$$

is (up to a rational factor) the inverse image under the morphism

$$(T^n_{(f)},S^n_{T(f)^\bullet}) \rightarrow (U^n,S^n_{U^\bullet})$$

given by $t_i \mapsto t_i^{a_{ij}}$ of the Steinberg symbol in the assertion.

<u>Third step</u>: We now reduce to the corresponding assertion in Deligne cohomology.

<u>(6.3) Lemma</u>: It suffices to prove that the corresponding symbol

$$\{1-\zeta\prod_i t_i,t_1,\ldots,t_n\} \in H^{n+1}_{\mathcal{D}}(U^n_{\mathbb{R}},S^n_{U^\bullet\mathbb{R}};\mathbb{R}(n+1))$$

in Deligne cohomology is zero.

This is an immediate consequence of the following result.

<u>(6.4) Proposition:</u> The regulator map

$$r_{\mathcal{D}} : H^{\bullet}_{\mathcal{A}}(U^n, S^n_{U^\bullet}; \mathbb{Q}(*)) \to H^{\bullet}_{\mathcal{D}}(U^n_{\mathbb{R}}, S^n_{U^\bullet\mathbb{R}}; \mathbb{R}(*))$$

is injective.

The proof requires some preparation.

<u>(6.5) Lemma:</u> The regulator map $r_{\mathcal{D}}$ induces isomorphisms

$$H^{\bullet}_{\mathcal{A}}(T^n, S^n_{T^\bullet}; \mathbb{Q}(*)) \otimes \mathbb{R} \xrightarrow{\cong} H^{\bullet}_{\mathcal{D}}(T^n_{\mathbb{R}}, S^n_{T^\bullet\mathbb{R}}; \mathbb{R}(*)) .$$

Proof: The regulator map $r_{\mathcal{D}}$ induces a morphism between the spectral sequences for a simplicial scheme

$$\begin{array}{ccc} H^q_{\mathcal{A}}(S^n_{T_p}, \mathbb{Q}(*)) \otimes \mathbb{R} & \Longrightarrow & H^{p+q}_{\mathcal{A}}(S^n_{T^\bullet}, \mathbb{Q}(*)) \otimes \mathbb{R} \\ \downarrow{\scriptstyle r_{\mathcal{D}}} & & \downarrow{\scriptstyle r_{\mathcal{D}}} \\ H^q_{\mathcal{D}}(S^n_{T_p\mathbb{R}}, \mathbb{R}(*)) & \Longrightarrow & H^{p+q}_{\mathcal{D}}(S^n_{T^\bullet\mathbb{R}}, \mathbb{R}(*)) \end{array}$$

as well as a morphism between the relative cohomology sequences

$$\begin{array}{ccc} \downarrow & & \downarrow \\ H^q_{\mathcal{A}}(T^n, S^n_{T^\bullet}; \mathbb{Q}(*)) \otimes \mathbb{R} & \xrightarrow{r_{\mathcal{D}}} & H^q_{\mathcal{D}}(T^n_{\mathbb{R}}, S^n_{T^\bullet\mathbb{R}}; \mathbb{R}(*)) \\ \downarrow & & \downarrow \\ H^q_{\mathcal{A}}(T^n, \mathbb{Q}(*)) \otimes \mathbb{R} & \xrightarrow{r_{\mathcal{D}}} & H^q_{\mathcal{D}}(T^n_{\mathbb{R}}, \mathbb{R}(*)) \\ \downarrow & & \downarrow \\ H^q_{\mathcal{A}}(S^n_{T^\bullet}; \mathbb{Q}(*)) \otimes \mathbb{R} & \xrightarrow{r_{\mathcal{D}}} & H^q_{\mathcal{D}}(S^n_{T^\bullet\mathbb{R}}, \mathbb{R}(*)) \\ \downarrow & & \downarrow \end{array} .$$

Since each $S^n_{T_p}$ as well as T^n is a disjoint union of tori $\mathbb{G}^i_m$ over F we are therefore reduced to proving that the regulator map $r_{\mathcal{D}}$ induces isomorphisms

$$H^{\bullet}_{\mathcal{A}}(\mathbb{G}^i_m, \mathbb{Q}(*)) \otimes \mathbb{R} \xrightarrow{\cong} H^{\bullet}_{\mathcal{D}}(\mathbb{G}^i_{m\mathbb{R}}, \mathbb{R}(*)) .$$

This is a consequence of Borel's theorem ((1.3) in Part I) once we show that the $\mathcal{A}$- , resp. $\mathcal{D}$- , cohomology of the torus $\mathbb{G}^i_m$ can be expressed completely in terms of the respective cohomology of the base field in a way which is compatible with the regulator map. Since the argument in both cases is the same we only treat the $\mathcal{D}$-cohomology. Let X be a smooth scheme of finite type over $\mathbb{R}$ and let $j: \mathbb{G}_{mX} \to \mathbb{A}^1_X$ be the complement of the zero section $X \to \mathbb{A}^1_X$. From homotopy invariance we conclude that the map

$$H^{\bullet}_{\mathcal{D}}(X, \mathbb{R}(*)) = H^{\bullet}_{\mathcal{D}}(\mathbb{A}^1_X, \mathbb{R}(*)) \xrightarrow{j^*} H^{\bullet}_{\mathcal{D}}(\mathbb{G}_{mX}, \mathbb{R}(*))$$

is canonically split. Therefore the Gysin sequence for the zero section $X \to \mathbb{A}^1_X$ gives canonical isomorphisms

$$H^{\bullet}_{\mathcal{D}}(\mathbb{G}_{mX}, \mathbb{R}(*)) = H^{\bullet}_{\mathcal{D}}(X, \mathbb{R}(*)) \oplus H^{\bullet-1}_{\mathcal{D}}(X, \mathbb{R}(*-1)) .$$

The compatibility of the regulator map with that decomposition is a consequence of the Riemann-Roch theorem without denominators ([23],[24],[14],[11]). Our claim then is achieved inductively. q.e.d.

We note that $\dot{T}^{n-1}$ is the image of the closed immersion

$$\begin{aligned} i_n : T^{n-1} &\to T^n \\ (t_1,\dots,t_{n-1}) &\mapsto (t_1,\dots,t_{n-1},(\zeta \prod_{i=1}^{n-1} t_i)^{-1}) . \end{aligned}$$

Put

$$\tilde{S}^{n-1}_T := S^{n-1}_T \cup \dot{T}^{n-2} .$$

Then i_n induces an isomorphism

$$(T^{n-1},\tilde{S}^{n-1}_T) \xrightarrow{\cong} (\dot{T}^{n-1},\dot{T}^{n-1} \cap S^n_T) .$$

Furthermore the intersection is transversal, and $\tilde{S}^{n-1}_T$ is a divisor with normal crossings in T^{n-1} and in fact the union of the smooth divisors $\{t_i=1\}$ for $i=1,\dots,n-1$ whose union is S^{n-1}_T and of the smooth divisor $\dot{T}^{n-2}$. We therefore are entitled to form the simplicial resolution $\tilde{S}^{n-1}_{T\bullet}/T^{n-1}$ (compare §3, after (3.2)).

<u>(6.6) Lemma</u>: The regulator map $r_{\mathcal{D}}$ induces isomorphisms

$$H^{\bullet}_{\mathcal{A}}(T^{n-1},\tilde{S}^{n-1}_{T\bullet};\mathbb{Q}(*)) \otimes \mathbb{R} \xrightarrow{\cong} H^{\bullet}_{\mathcal{D}}(T^{n-1},\tilde{S}^{n-1}_{T\bullet}; \mathbb{R}(*)) .$$

Proof: This will be proved by induction on n. For $n=1$ it is Borel's theorem. So assume it to be true for $n-1$. We consider the commutative diagram (for simplicity we leave out the coefficients)

$$\begin{tikzcd}
{} \arrow[d] & {} \arrow[d] \\
H_A^{q-1}(T^{n-1},\tilde{S}^{n-1}_{T^\cdot}) \otimes \mathbb{R} \arrow[r, "r_{\mathcal{D}}"] \arrow[d] & H_{\mathcal{D}}^{q-1}(T^{n-1}_{\mathbb{R}},\tilde{S}^{n-1}_{T^\cdot\mathbb{R}}) \arrow[d] \\
H_A^{q}(T^{n},\tilde{S}^{n}_{T^\cdot}) \otimes \mathbb{R} \arrow[r, "r_{\mathcal{D}}"] \arrow[d] & H_{\mathcal{D}}^{q}(T^{n}_{\mathbb{R}},\tilde{S}^{n}_{T^\cdot\mathbb{R}}) \arrow[d] \\
H_A^{q}(T^{n},S^{n}_{T^\cdot}) \otimes \mathbb{R} \arrow[r, "r_{\mathcal{D}}"] \arrow[d] & H_{\mathcal{D}}^{q}(T^{n}_{\mathbb{R}},S^{n}_{T^\cdot\mathbb{R}}) \arrow[d] \\
{} & {}
\end{tikzcd}$$

where the columns are obtained from the relative cohomology sequence and the isomorphism

$$(T^{n-1},\tilde{S}^{n-1}_{T^\cdot}) \overset{\cong}{\to} (\dot{T}^{n-1},\dot{T}^{n-1} \cap S^{n}_{T^\cdot}) .$$

The lower horizontal arrow is an isomorphism by (6.5) and the upper one by assumption. The five lemma then implies that the middle horizontal arrow is an isomorphism, too. q.e.d.

Our proposition (6.4) now is easily established: By the Riemann-Roch theorem without denominators the regulator map $r_{\mathcal{D}}$ is compatible with the Gysin map for the smooth divisor $\dot{T}^{n-1}$ in T^n, i.e., we have a commutative diagram (again we leave out the coefficients)

$$\begin{tikzcd}
{} \arrow[d] & {} \arrow[d] \\
H_A^{q+1}(T^{n},S^{n}_{T^\cdot}) \otimes \mathbb{R} \arrow[r, "r_{\mathcal{D}}"] \arrow[d] & H_{\mathcal{D}}^{q+1}(T^{n}_{\mathbb{R}},S^{n}_{T^\cdot\mathbb{R}}) \arrow[d] \\
H_A^{q+1}(U^{n},S^{n}_{U^\cdot}) \otimes \mathbb{R} \arrow[r, "r_{\mathcal{D}}"] \arrow[d] & H_{\mathcal{D}}^{q+1}(U^{n}_{\mathbb{R}},S^{n}_{U^\cdot\mathbb{R}}) \arrow[d] \\
H_A^{q}(T^{n-1},\tilde{S}^{n-1}_{T^\cdot}) \otimes \mathbb{R} \arrow[r, "r_{\mathcal{D}}"] \arrow[d] & H_{\mathcal{D}}^{q}(T^{n-1}_{\mathbb{R}},\tilde{S}^{n-1}_{T^\cdot\mathbb{R}}) \arrow[d] \\
{} & {}
\end{tikzcd} .$$

Now apply the two lemmas above together with the five lemma.

<u>Last step</u>: Here we finally show that

$$\{1-\zeta\prod_i t_i,t_1,\ldots,t_n\} = 0 \quad \text{in} \quad H_{\mathcal{D}}^{n+1}(U^n_{\mathbb{R}},S^n_{U^\cdot\mathbb{R}};\mathbb{R}(n+1)) .$$

We make a careful study of the maps in the commutative exact diagram

$$\begin{tikzcd}
H^n_{\mathcal{D}}(T^n_{\mathbb{R}}, S^n_{T\cdot\mathbb{R}}; \mathbb{R}(n)) \arrow[r, "i^*_n"] & H^n_{\mathcal{D}}(T^{n-1}_{\mathbb{R}}, \tilde{S}^{n-1}_{T\cdot\mathbb{R}}; \mathbb{R}(n)) & \\
& H^{n+1}_{\mathcal{D}}(U^n_{\mathbb{R}}, S^n_{U\cdot\mathbb{R}}; \mathbb{R}(n+1)) \arrow[u, "\partial"] \arrow[r] & H^{n+1}_{\mathcal{D}}(U^n_{\mathbb{R}}, \mathbb{R}(n+1)) \\
& H^{n+1}_{\mathcal{D}}(T^n_{\mathbb{R}}, S^n_{T\cdot\mathbb{R}}; \mathbb{R}(n+1)) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

where the column is obtained from the Gysin sequence and the isomorphism $(T^{n-1},\tilde{S}^{n-1}_{T\cdot}) \overset{\cong}{\to} (\dot{T}^{n-1},\dot{T}^{n-1} \cap S^n_{T\cdot})$.

<u>(6.7) Lemma</u>: $H^q(T^n(\mathbb{C}),S^n_{T\cdot}(\mathbb{C});\mathbb{Q}) = \mathbb{Q}(-n)$ (as a mixed Hodge structure) for $q=n$ and is zero otherwise.

Proof: Since $(T^n,S^n_T) \cong (\mathbb{G}_m,\{1\})^n$ is the n-fold fibre product the Künneth formula reduces the proof to the case $n=1$ which is clear.

<u>(6.8) Lemma</u>: The restriction map

$$H^{n+1}_{\mathcal{D}}(T^n_{\mathbb{R}}, S^n_{T\cdot\mathbb{R}}; \mathbb{R}(n+1)) \to H^{n+1}_{\mathcal{D}}(U^n_{\mathbb{R}}, \mathbb{R}(n+1))$$

is injective.

Proof: Since the dimension of the varieties involved is $\leq n$ the $\mathcal{D}$-cohomology groups may be identified with the corresponding Betti cohomology groups (in degree and twist one less). Fixing an (arbitrary) embedding $\alpha: F \to \mathbb{C}$ we therefore have to show that the map

$$H^n(T^n(\mathbb{C}),S^n_{T\cdot}(\mathbb{C});\mathbb{R}(n)) \to H^n(U^n(\mathbb{C}),\mathbb{R}(n))$$

is injective. We consider the commutative exact diagram of mixed Hodge structures

$$\begin{tikzcd}
H^{n-2}(T^{n-1}(\mathbb{C}),\mathbb{Q}(-1)) \arrow[r] & H^n(T^n(\mathbb{C}),\mathbb{Q}) \arrow[r] & H^n(U^n(\mathbb{C}),\mathbb{Q}) \\
& H^n(T^n(\mathbb{C}),S^n_{T\cdot}(\mathbb{C});\mathbb{Q}) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

where the row is obtained from the Gysin sequence. Using the Künneth formula as in the proof of (6.7) we see that on the one hand the vertical arrow is an isomorphism of mixed Hodge structures both isomorphic to $\mathbb{Q}(-n)$ and that on the other hand $H^{n-2}(T^{n-1}(\mathbb{C}),\mathbb{Q}(-1)) \cong \mathbb{Q}(1-n)\oplus\ldots\oplus\mathbb{Q}(1-n)$. The oblique arrow consequently is injective.

(6.9) Lemma: The map

$$i_n^*: H^n_{\mathcal{D}}(T^n_{\mathbb{R}}, S^n_{T^\cdot\,\mathbb{R}}; \mathbb{R}(n)) \to H^n_{\mathcal{D}}(T^{n-1}_{\mathbb{R}}, \tilde{S}^{n-1}_{T^\cdot\,\mathbb{R}}; \mathbb{R}(n))$$

is the zero map.

Proof: We equivalently have to show that the connecting homomorphism

$$H^n_{\mathcal{D}}(T^{n-1}_{\mathbb{R}}, \tilde{S}^{n-1}_{T^\cdot\,\mathbb{R}}; \mathbb{R}(n)) \xrightarrow{\partial} H^{n+1}_{\mathcal{D}}(T^n_{\mathbb{R}}, \tilde{S}^n_{T^\cdot\,\mathbb{R}}; \mathbb{R}(n))$$

in the corresponding relative cohomology sequence is injective. For that we use the commutative exact diagram

$$\begin{tikzcd}
H^n_{\mathcal{D}}(T^{n-1}_{\mathbb{R}}, \tilde{S}^{n-1}_{T^\cdot\,\mathbb{R}}; \mathbb{R}(n)) \arrow[r, "\partial"] & H^{n+1}_{\mathcal{D}}(T^n_{\mathbb{R}}, \tilde{S}^n_{T^\cdot\,\mathbb{R}}; \mathbb{R}(n)) \\
H^{n-1}_{\mathrm{Betti}}(T^{n-1}_{\mathbb{R}}, \tilde{S}^{n-1}_{T^\cdot\,\mathbb{R}}; \mathbb{C}/\mathbb{R}(n)) \arrow[u, "\cong"] \arrow[r, "\partial_{\mathrm{Betti}}"] & H^n_{\mathrm{Betti}}(T^n_{\mathbb{R}}, \tilde{S}^n_{T^\cdot\,\mathbb{R}}; \mathbb{C}/\mathbb{R}(n)) \arrow[u] \\
 & F^n H^n_{DR}(T^n_{\mathbb{R}}, \tilde{S}^n_{T^\cdot\,\mathbb{R}}) \arrow[u, "\pi"]
\end{tikzcd}$$

;

here the columns are part of the long exact sequence which connects $\mathcal{D}$-cohomology with Betti and deRham cohomology (compare [10] 2.10 c)). Since in the left column the dimension of the varieties is $<n$ we have an isomorphism there. By (6.7), $\partial_{\mathrm{Betti}}$ clearly is injective. It remains to show that

$$\operatorname{im} \partial_{\mathrm{Betti}} \cap \operatorname{im} \pi = 0 .$$

We will use Hodge theory. If $\alpha: F \to \mathbb{C}$ is any fixed embedding, it follows by induction from (6.7) and the relative cohomology sequences

$$\ldots \to H^{q-1}(T^{n-1}(\mathbb{C}), \tilde{S}^{n-1}_{T^\cdot}(\mathbb{C}); \mathbb{Q}) \to H^q(T^n(\mathbb{C}), \tilde{S}^n_{T^\cdot}(\mathbb{C}); \mathbb{Q})$$

$$\to H^q(T^n(\mathbb{C}), S^n_{T^\cdot}(\mathbb{C}); \mathbb{Q}) \to \ldots$$

that the factors in the weight filtration of $H^{n-1}(T^{n-1}(\mathbb{C}), \tilde{S}^{n-1}_{T^\cdot}(\mathbb{C}); \mathbb{Q})$ are $\mathbb{Q}(-j)$ for $0 \leq j \leq n-1$ (and similarly for $H^n(T^n(\mathbb{C}), \tilde{S}^n_{T^\cdot}(\mathbb{C}); \mathbb{Q})$). We already see that in $H^n(T^n(\mathbb{C}), \tilde{S}^n_{T^\cdot}(\mathbb{C}); \mathbb{C})$ we have $\operatorname{im} \partial_{\mathrm{Betti}} \cap F^n H^n = 0$. In order to guarantee that this remains true in $H^n(T^n(\mathbb{C}), \tilde{S}^n_{T^\cdot}(\mathbb{C}); \mathbb{C}/\mathbb{R}(n))$ we have to establish that the n-th step F^n of the Hodge filtration is defined over $\mathbb{R}$ (even over $\mathbb{Q}$ as we actually will show).

A generator of F^n is the differential form

$$(2\pi\sqrt{-1})^{-n}\ \mathrm{dlog}t_1\wedge\ldots\wedge\mathrm{dlog}t_n$$

and it suffices to prove that its periods over rational cycles are rational. Recall $\alpha\zeta = \exp(2\pi\sqrt{-1}\,a/N)$ with $1 \leq a < N$. Consider the unit cube in $\mathbb{R}^n$ and subdivide it into $n+1$ pieces by the hyperplanes $\Sigma x_i = j - \frac{a}{N}$, $j = 1,\ldots,n$. It is easy to see that the images of these pieces under the map $t_i = \exp(2\pi\sqrt{-1}\,x_i)$ form a basis of $H^n(T^n(\mathbb{C}),\tilde{S}^n_T(\mathbb{C});\mathbb{Q})$. The integral of our form over such a piece is just the volume of the piece - which is rational.

<u>Remark:</u> The above proof actually shows that $H^q(T^n(\mathbb{C}),\tilde{S}^n_{T}.(\mathbb{C});\mathbb{Q}) \cong \mathbb{Q} \oplus \mathbb{Q}(-1) \oplus\ldots\oplus \mathbb{Q}(-n)$ (as mixed Hodge structures) for $q = n$ and is zero otherwise.

We now go back to our diagram. Since $\dot{T}^{n-1}$ is defined inside T^n by the equation $1 - \zeta \prod_i t_i = 0$ we obtain from the usual property of the Gysin sequence (compare the proof of (2.2))

$$\partial\{1 - \zeta \prod_i t_i, t_1,\ldots,t_n\} = i_n^*\{t_1,\ldots,t_n\}$$

and consequently, by (6.9),

$$\partial\{1 - \zeta \prod_i t_i, t_1,\ldots,t_n\} = 0\ .$$

If we show that the restriction of $\{1-\zeta\prod_i t_i,t_1,\ldots,t_n\}$ to $U^n_{\mathbb{R}}$ is zero, too, then (6.8) implies that $\{1-\zeta\prod_i t_i,t_1,\ldots,t_n\}$ itself is zero. But that restriction is the cup-product of the invertible functions $1 - \zeta\prod_i t_i$, $t_1,\ldots,t_n$ on the affine scheme U^n , also to be denoted by $\{1-\zeta\prod_i t_i,t_1,\ldots,t_n\}$. It is zero since

$$\begin{aligned}\{1-\zeta\prod_i t_i,t_1,\ldots,t_n\} &= \{1-(\zeta t_1)t_2\ldots t_n,\zeta t_1,t_2,\ldots,t_n\}\\ &\quad - \{1-(\zeta t_1)t_2\ldots t_n,\zeta,t_2,\ldots,t_n\}\end{aligned}$$

and $\{a,\zeta,b,\ldots\} = \frac{1}{N}\{a,\zeta^N,b,\ldots\} = \frac{1}{N}\{a,1,b,\ldots\} = 0$ and $\{1-x_1\ldots x_n,x_1,\ldots,x_n\} = 0$. The last equality follows immediately from the Steinberg identity $\{1-x,x\} = 0$ and from $\{x,x\} = 0$.

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