Comparison of the regulators of Beilinson and of Borel

M. Rapoport

This contribution is the fruit of a collaboration with U.Stuhler. Although he in the end refused to have his name appear as an author, he played an important part in the preparation of this paper, which I consider as the result of a joint effort. Our aim here is to give an exposition of A.Beilinson's proof ([1], App. to §2) that in the case of a number field the Beilinson regulator map coincides with the Borel regulator map up to a non-zero rational factor. This compatibility is needed in order to regard Borel's results [5] as a confirmation of the Beilinson conjecture in this particular case, and thereby enters into Beilinson's proof of the Gross conjecture in the cyclotomic case (compare [1], [17]). (In fact, Borel's results are stronger in a certain sense, compare our remarks at the end of §1).

The Beilinson regulator map is defined under very general circumstances and behaves functorially in a pleasant way. However, it is hard to calculate explicitly, if only because the source of this map is closely related to K-groups which are very poorly understood in general. In the case of a number field, k, the (rational) K-groups can be explicitly determined, thanks to our knowledge of the cohomology of discrete arithmetically defined groups [4]. Furthermore, the periods of an explicitly defined differential form over homology cycles coming from the K-group can be calculated [5]. This then leads to the definition of the Borel regulator map and to Borel's theorem that the co-volume of the image of the n-th regulator map equals $\zeta_k^*(1-n)$. We refer the reader to [5] where in the introduction Borel gives a lucid exposition of his proof.

We now explain the plan of this article. In §1 we recall the definition of the Beilinson regulator map in this special case and give the definition of the Borel regulator map which uses continuous cohomology; a slightly different way of constructing the Borel regulator map appears at the end of §4. In §2 it is proved that the normalization functor induces an equivalence of categories of reduced <u>small</u> co-simplicial algebras and reduced <u>small</u> differential graded algebras. The concept of smallness introduced by Beilinson was invented for the proof of the comparison theorem. Recall that in the simplicial context D.Quillen (Rational homotopy theory, Ann. of Math.

90, 1969, p. 205-295) has proved that the normalization functor from the category of reduced simplicial commutative algebras to the category of reduced commutative differential graded algebras (over a field of characteristic zero) induces an equivalence of the corresponding homotopy categories. By putting the smallness restriction on the algebras in question (and working in the co-simplicial context) no homotopies are needed. This result is used to define a kind of "de Rham complex" of a small differential graded algebra, and thereby also to give a sufficiently canonical definition of the Weil algebra. We also recall the definition of Chern classes by means of the Weil algebra. In §3 we present the second main ingredient of Beilinson's proof, namely the interpretation of the van Est isomorphism in continuous cohomology as a restriction map to the cohomology of an infinitesimal version of the classifying space, namely the largest small simplicial subscheme. As L.Illusie pointed out to us it seems that again the concept of smallness allows one to avoid complications which occur in Quillen's formal categories [16]. In §4 we then prove that the two regulator maps essentially coincide.

In Beilinson's manuscript there is also a description of theorems of Bloch, Beilinson, Tsygan, and Feigin on additive K-theory, for which we refer the reader to the original source. Thus we have tried here to give an account of the remaining 3 1/2 pages. Even though these have expanded into some 20 pages we are not confident that we have done them justice; one of the reasons is that, not being topologists ourselves, we do not have sufficient insight into the deeper topological significance of Beilinson's proof. On the other hand, we tried to fill some gaps in Beilinson's argument.

We wish to express our gratitude to F.Grunewald, T.Zink, and especially R.Weissauer for very helpful conversations on these topics.

The notations used here are in conformity with the notation used elsewhere in this volume, compare esp. [13].

§1. Definition of the two regulators

Let k be a finite number field and let $X = \operatorname{Spec} k$. We wish to recall the definition of Beilinson's regulator map in this special case,

$$r : H^1_{\mathcal{A}}(X, \mathbb{Q}(n)) \longrightarrow H^1_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(n)) \ .$$

Here $n \geq 1$, and $H^1_{\mathcal{A}}(X, \mathbb{Q}(n))$ is a certain piece of the K-group $K_{2n-1}(k) \otimes \mathbb{Q}$ ([14].).

We remark that the complex $\mathbb{R}(n)_{\mathcal{D}}$ on $\operatorname{Spec} \mathbb{C}$ reduces for $n > 0$ to

$$\mathbb{R}(n) \to \mathbb{C} \quad \text{(degrees 0 and 1)}$$

and is isomorphic via $\pi_{n-1} : \mathbb{C} \to \mathbb{R}(n-1)$ to $\mathbb{R}(n-1)[-1]$. It follows that

$$H^j_{\mathcal{D}}(\operatorname{Spec} \mathbb{C}, \mathbb{R}(n)) = \begin{cases} \mathbb{R}(n-1) & j = 1 \\ 0 & j \neq 1 \end{cases} .$$

We consider the morphism of simplicial schemes given by evaluation

$$e : \operatorname{Spec} \mathbb{C} \times B.GL_N(\mathbb{C}) \longrightarrow B.GL_N/\mathbb{C} \ .$$

Here on the left the simplicial set $B.GL_N(\mathbb{C})$ is considered as a scheme (disjoint union of points) and the morphism is the obvious one. The n-th Chern class $c_n \in H^{2n}_{\mathcal{D}}(B.GL_N, \mathbb{Q}(n))$ ([13]) defines an element $c_n \in H^{2n}_{\mathcal{D}}(B.GL_N, \mathbb{R}(n))$ which yields by restriction and using the Künneth formula (legitimate since the coefficient system is a vector space over $\mathbb{R}$)

$$\begin{aligned} e^*(c_n) \ &\in \ H^{2n}_{\mathcal{D}}(\operatorname{Spec} \mathbb{C} \times B.GL_N(\mathbb{C}), \mathbb{R}(n)) \simeq \\ &\simeq \ H^1_{\mathcal{D}}(\operatorname{Spec} \mathbb{C}, \mathbb{R}(n)) \otimes H^{2n-1}(B.GL_N(\mathbb{C}), \mathbb{R}) \\ &\xrightarrow[\sim]{\pi_{n-1}} \ H^{2n-1}(B.GL_N(\mathbb{C}), \mathbb{R}(n-1)) = H^{2n-1}(GL_N(\mathbb{C}), \mathbb{R}(n-1)). \end{aligned}$$

We note that the element obtained is invariant under the simultaneous action of $\operatorname{Gal}(\mathbb{C}/\mathbb{R})$ on the discrete group $GL_N(\mathbb{C})$ and the coefficient system $\mathbb{R}(n-1)$. This construction is compatible with increasing N which is to be taken large compared to n. We now note that

$$H^{2n-1}(GL_N(\mathbb{C}), \mathbb{R}(n-1)) = \operatorname{Hom}(H_{2n-1}(GL_N(\mathbb{C}), \mathbb{Z}), \mathbb{R}(n-1)) \ .$$

Composing with the Hurewicz map we obtain finally a map

$$\begin{aligned} \text{(1.1.)} \quad K_{2n-1}(\mathbb{C}) = \pi_{2n-1}(BGL(\mathbb{C})^+) \to H_{2n-1}(GL(\mathbb{C}), \mathbb{Z}) \to \mathbb{R}(n-1) = \\ = H^1_{\mathcal{D}}(\operatorname{Spec} \mathbb{C}, \mathbb{R}(n)) \ . \end{aligned}$$

Returning to our number field k, we write $X_{/\mathbb{C}} = \operatorname{Spec} k \otimes \mathbb{C} = \coprod_{\sigma : k \to \mathbb{C}} \mathbb{C}$ and find using the previous calculation

$$H^1_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(n)) = \left[\bigoplus_{\sigma : k \to \mathbb{C}} H^1_{\mathcal{D}}(\operatorname{Spec} \mathbb{C}, \mathbb{R}(n)) \right]^{\operatorname{Gal}(\mathbb{C}/\mathbb{R})}$$

$$= \left[\bigoplus_{\sigma : k \to \mathbb{C}} \mathbb{R}(n-1) \right]^{\operatorname{Gal}(\mathbb{C}/\mathbb{R})}$$

(The dimension of this vector space is thus equal to r_2, resp. $r_1 + r_2$ according as n is even or odd, where r_1 and r_2 have the customary meaning.) The regulator map is now defined by first extending scalars from $\mathbb{Q}$ to $\mathbb{C}$ and then taking in every component the map (1.1.) defined by the n-th Chern class above. As a matter of fact, the Chern character (which really is the regulator) is the product of this map with the rational number $(-1)^{n-1}/(n-1)!$; but this will be of no importance to us.

We next give the definition of Borel's regulator [5]. We base ourselves on [15]. We need some preparation. Let G be a Lie group and V a continuous G-module. There is a cohomology theory of continuous G-modules which is defined just as Eilenberg-MacLane cohomology for discrete G-modules, but using continuous cochains. In practice these cohomology groups can in fact be computed using C^∞-cochains ([6], p.276).

Let $K \subset G$ be a maximal compact subgroup. Denoting as usual by $S^{\cdot}(G/K)$ the de Rham complex with real C^∞-coefficients we obtain a homomorphism of complexes in the category of continuous G-modules

(1.2.) $$\mathbb{R} \to S^{\cdot}(G/K) = [S^0(G/K) \to S^1(G/K) \to ...]$$

Using the fact that G/K is diffeomorphic to a euclidean space, (1.2.) may be shown to be a resolution in a suitable sense ([6],p.279). On the other hand, G-modules like $S^i(G/K)$ are injective in a strong sense ([6], p.278) so that the continuous cohomology of $S^{\cdot}(G/K)$ may be computed by simply taking G-invariants, so that[1)]

(1.3.) $$\begin{aligned} H^*_{cont}(G, \mathbb{R}) &= H^*_{cont}(G, S^{\cdot}(G/K)) = H^*(S^{\cdot}(G/K)^G) \\ &= H^*(\mathrm{g}, \mathrm{k}; \mathbb{R}) \ . \end{aligned}$$

Here the last term is the relative Lie algebra cohomology group which we now proceed to recall.

Let g be a Lie algebra over a field of characteristic zero and let V be a g-module. The Lie algebra complex $C^{\cdot}(\mathrm{g}, V)$ is

$$C^q(\mathrm{g}, V) = \mathrm{Hom}(\Lambda^q \mathrm{g}, V) = \Lambda^q \mathrm{g}' \otimes V \ , \quad q = 0, 1, ...$$

(g' = dual vector space), with differential $d : C^q(\mathrm{g}, V) \to C^{q+1}(\mathrm{g}, V)$ given as follows:

$$\begin{aligned} df(x_0, ..., x_q) = &\sum_i (-1)^i x_i f(x_0, ..., \hat{x}_i, ..., x_q) + \\ &\sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_0, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_q). \end{aligned}$$

The relative cohomology groups for a sub Lie algebra $\mathrm{k} \subset \mathrm{g}$ are the cohomology groups of the sub-complex

$$C^q(\mathrm{g}, \mathrm{k}; V) = \mathrm{Hom}_{\mathrm{k}}(\Lambda^q(\mathrm{g}/\mathrm{k}), V) \ ,$$

1) All this holds for more general G-modules than the trivial module $\mathbb{R}$.

where the action of $\mathbf{k}$ on $\Lambda^q(\mathbf{g}/\mathbf{k})$ is induced by the adjoint action. In the above chain of isomorphisms we used the isomorphism

$$S^{\cdot}(G/K)^G \simeq C^{\cdot}(\mathbf{g},\mathbf{k};\mathbb{R})$$

given by assigning to a differential form its value at the identity. The isomorphism between continuous cohomology and relative Lie algebra cohomology is called the van Est isomorphism.

We need to calculate the relative Lie algebra cohomology in the case of interest. Let G be a reductive Lie group. Let $\mathbf{g} = \mathbf{k} \oplus \mathbf{p}$ be the Cartan decomposition corresponding to $\mathbf{k}$ and let $\mathbf{g}_u = \mathbf{k} \oplus i\mathbf{p} \subset \mathbf{g} \otimes \mathbb{C}$ be the compact form. Extension of scalars defines canonical isomorphisms (in the middle is Lie algebra cohomology over $\mathbb{C}$ with trivial coefficients)

$$H^*(\mathbf{g},\mathbf{k};\mathbb{R}) \otimes \mathbb{C} \simeq H^*(\mathbf{g}_{\mathbb{C}},\mathbf{k}_{\mathbb{C}}) \simeq H^*(\mathbf{g}_u,\mathbf{k};\mathbb{R}) \otimes \mathbb{C} \ . \tag{1.4.}$$

Denoting by G_u the Lie group corresponding to $\mathbf{g}_u$ we have as before

$$\begin{aligned} H^*(\mathbf{g}_u,\mathbf{k};\mathbb{R}) &\simeq H^*(S^{\cdot}(G_u/K)^{G_u}) = H^*(S^{\cdot}(G_u/K)) = \\ &\simeq H^*_{Betti}(G_u/K;\mathbb{R}) \ . \end{aligned} \tag{1.5.}$$

Here the second isomorphism is obtained by an average argument using the fact that G_u is compact, and the third isomorphism is the de Rham isomorphism. Similarly $H^*(\mathbf{g}_u,\mathbb{R}) \cong H^*_{Betti}(G_u,\mathbb{R})$.

Combining now (1.3.) – (1.5.) we obtain a canonical isomorphism

$$\gamma : H^*_{Betti}(G_u/K;\mathbb{R}) \otimes \mathbb{C} \xrightarrow{\sim} H^*_{cont}(G,\mathbb{R}) \otimes \mathbb{C} \ . \tag{1.6.}$$

It does not carry the $\mathbb{R}$-cohomology into one another since under the isomorphism (1.4.) the $\mathbb{R}$-cohomology in degree m is carried into $i^m \cdot H^m(\mathbf{g}_u,\mathbf{k};\mathbb{R})$. This is due to the fact that in the definition of $\mathbf{g}_u$ there is an i standing in front of the $\mathbf{p}$ so that $H^*(\mathbf{g}_u,\mathbf{k};\mathbb{R}) = H^*(\mathrm{Hom}_{\mathbf{k}}(\Lambda i \cdot \mathbf{p},\mathbb{R}))$.

We now apply these considerations to the case where $G = GL_N(\mathbb{C})$, with maximal compact subgroup $K = U_N$, the unitary group. In this case we may identify G_u with $U_N \times U_N$, with U_N embedded diagonally. Explicitly, denoting by $\sigma : X \longrightarrow -{}^t\overline{X}$ the Cartan involution on $\mathbf{g}$ with respect to $\mathbf{k}$ we obtain a $\mathbb{C}$-linear isomorphism

$$\begin{aligned} \mathbf{g} \otimes \mathbb{C} &\longrightarrow \mathbf{g} \oplus \mathbf{g} \\ X \otimes \lambda &\longmapsto (\lambda X, \lambda\sigma(X)) \ . \end{aligned} \tag{1.7.}$$

In terms of this identification, the action of complex conjugation with respect to the real form $\mathbf{g}$ of $\mathbf{g} \otimes \mathbb{C}$ becomes

$$\overline{(X_1,X_2)} = (\sigma(X_2),\sigma(X_1))$$

and the Cartan involution

$$\sigma(X_1,X_2) = (X_2,X_1) \ .$$

Therefore $\mathbf{g}_u$, which is the fixed space under the product of these two involutions is

$$\mathbf{g}_u = \mathbf{k} \oplus \mathbf{k} \subset \mathbf{g} \oplus \mathbf{g} \ ,$$

with $\mathbf{k}$ embedded diagonally.

We identify

$$\begin{aligned} G_u/K &\xrightarrow{\sim} U_N \\ (x,y) &\longmapsto x\cdot y^{-1} \quad . \end{aligned}$$

We note that the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on $H^*(\mathfrak{g},\mathfrak{k};\mathbb{R})$ which under the van Est isomorphism corresponds to the obvious action on $H^*_{cont}(G,\mathbb{R})$ is the one induced by conjugation on $\mathfrak{p}$. This action corresponds to the action by conjugation on $\mathfrak{p}_u$ and hence also by conjugation on $U_N = G_u/K$. Therefore under the isomorphism

$$H^*_{cont}(G,\mathbb{C}) \simeq H^*(U_N,\mathbb{C})$$

the actions of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ by simultaneous conjugation correspond to one another.

The cohomology of U_N, say with coefficients in $\mathbb{Q}$ is the free exterior algebra generated by the cohomology classes of odd spheres, coming from the action of U_N on $\mathbb{C}^N$ (e.g., [3], 9.1.)

$$H^*_{Betti}(U_N,\mathbb{Q}) = \bigwedge_{\mathbb{Q}}(u_1,u_3,\ldots,u_{2N-1}) \quad .$$

The action of U_N on $\mathbb{C}^N$ is compatible with complex conjugation, i.e. $\overline{x\cdot v} = \overline{x}\cdot\overline{v}$ and complex conjugation on $\mathbb{C}^n$ induces a homeomorphism of degree $(-1)^n$ on S^{2n-1}. It follows that

(1.8.) $$\overline{u}_{2n-1} = (-1)^n \cdot u_{2n-1} \quad .$$

We consider

$$(2\pi i)^n u_{2n-1} \in H^{2n-1}_{Betti}(U_N,\mathbb{R}(n)) \subset H^{2n-1}_{Betti}(U_N,\mathbb{C}) \quad .$$

Its image under the isomorphism γ (1.6.) lies in $H^{2n-1}_{cont}(G,\mathbb{R}(n-1))$ and is invariant under the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$:

(1.9.) $$b_{2n-1} = \gamma((2\pi i)^n\cdot u_{2n-1}) \in H^{2n-1}_{cont}(GL_N(\mathbb{C}),\mathbb{R}(n-1)) \quad .$$

Its image under the natural map from continuous to discrete cohomology (= forget the topology)

$$H^{2n-1}_{cont}(GL_N(\mathbb{C}),\mathbb{R}(n-1)) \longrightarrow H^{2n-1}(GL_N(\mathbb{C}),\mathbb{R}(n-1))$$

is the Borel regulator element. Just as for the Beilinson regulator it defines a homomorphism

$$K_{2n-1}(\mathbb{C}) \longrightarrow \mathbb{R}(n-1)$$

and, in case k is a finite number field, the <u>Borel regulator map</u>

(1.10.) $$K_{2n-1}(k)\otimes\mathbb{Q} \longrightarrow H^1_{\mathcal{D}}(X_{/\mathbb{R}},\mathbb{R}(n)) \quad .$$

We refer to the end of §4 for a slightly different construction of the Borel regulator map.

We conclude this section with two remarks. It follows from the localization sequence and the fact that the higher K-groups of a finite field are finite that for $n > 1$

$$K_{2n-1}(o_k) \otimes \mathbb{Q} \simeq K_{2n-1}(k) \otimes \mathbb{Q} \quad .$$

On the other hand, Borel [5] has shown that for $n > 1$ the homomorphism (1.10.) is injective, defines a $\mathbb{Q}$-structure on $H^1_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(n))$ and that the co-volume of its image is equal (modulo $\mathbb{Q}^*$) to $\zeta_k^*(1-n)$[1)], whereas Beilinson's conjecture predicts the corresponding facts for a suitable piece $H^1_{\mathcal{A}}(X, \mathbb{Q}(n))$ of the K-group. However, it follows from the compatibility of the Chern character with the Adams operators (compare [18]) that Borel's result implies that

$$K_{2n-1}(k) \otimes \mathbb{Q} = H^1_{\mathcal{A}}(X, \mathbb{Q}(n)) \quad .$$

This reasoning (i.e., using [18] as a reference for the compatibility of the regulator map with the Adams operators) presupposes of course that the Beilinson regulator and the Borel regulator coincide. The result also holds for $n = 1$.

[1)] In fact, Borel's result is stated differently but boils down to the above.

§2 Some auxiliary considerations

So as not to interrupt the later discussions we collect here some definitions and facts which will be needed. Let k be a field of characteristic zero and let $\mathcal{A}$ be a k-linear $\otimes$-category with unit object $\mathbb{1}$. We assume that $\operatorname{End}\mathbb{1} = k$. One has the concepts of algebra objects in $\mathcal{A}$ (an object X with morphisms $\mathbb{1} \to X$, $X \otimes X \to X$ satisfying certain obvious axioms), of graded objects of $\mathcal{A}$, of complexes in $\mathcal{A}$ (here always with differentials of degree 1) of differential graded algebras (DGA for short - always in non-negative degree).

Let $X^\cdot$ be a co-simplicial algebra in $\mathcal{A}$ (i.e., a co-simplicial object in the category of algebras in $\mathcal{A}$). We have the standard cup product (e.g., [16], p.7, but translate from simplicial to co-simplicial)

$$\begin{array}{ccc} X^p \otimes X^q & \longrightarrow & X^{p+q} \\ x, y & \longmapsto & x \cup y = d^{p+q} \circ \ldots \circ d^{p+1}(x) \cdot d^0 \circ \ldots \circ d^0(y) \end{array}$$

This product is associative. The normalization $NX^\cdot$ is the DGA

$$NX^p = \cap \operatorname{Ker} s^i : X^p \longrightarrow X^{p-1} \quad ,$$

with differential induced from $d = \sum(-1)^i d^i : X^p \to X^{p+1}$ and product induced from the cup product. Even when $X^\cdot$ is a commutative co-simplicial algebra, $NX^\cdot$ is not necessarily a graded-commutative DGA. This leads to the following definitions.

2.1. Definitions (Beilinson):

a) A co-simplicial algebra $X^\cdot$ in $\mathcal{A}$ is called small if $X^\cdot$ is a commutative algebra with unit which is generated by X^0 and X^1 (in the sense of the cup product) and such that the ideal $\operatorname{Ker} s^0 \subset X^1$ *has square zero.*

b) A DGA *$Y^\cdot$ is called small, if $Y^\cdot$ is graded-commutative and is generated by Y^0 and Y^1 (as an algebra).*

For simplicity we shall only consider reduced co-simplicial algebras $X^\cdot$ (i.e., $X^0 = \mathbb{1}$, so that $d^0 = d^1 : X^0 \to X^1$ and $X^1 = X^0 \oplus NX^1$) and reduced DGA $Y^\cdot$ (i.e., $Y^0 = \mathbb{1}$, so that $d^0 = 0 : Y^0 \to Y^1$).

2.2. Proposition:

The normalization functor induces an equivalence of categories between the category of reduced small co-simplicial algebras in $\mathcal{A}$ and the category of reduced small DGA *in $\mathcal{A}$.*

For simplicity of exposition we shall do as if $\mathcal{A}$ was the category of k-vector spaces, in particular calculate with elements. We shall need the following lemmas.

2.3. Lemma:

Let $X^\cdot$ be a reduced commutative co-simplicial algebra in $\mathcal{A}$. Then $X^\cdot$ is generated by X^1 if and only if $NX^\cdot$ is generated by NX^1.

We postpone the proof for a while and explain first the strategy of the proof of 2.2.. We shall need some facts about co-simplicial objects X and their associated complex $X^{\sim}$ and normalized subcomplex $NX \subset X^{\sim}$. We translate from simplicial theory into co-simplicial theory. Denote by $\Delta(X \times Y)$ the diagonal co-simplicial object in the bi-co-simplicial object $X \times Y$ ($= X^p \otimes Y^q$ in degree (p,q), with obvious morphisms). There are natural homomorphisms of complexes, the Alexander-Whitney map and the shuffle map (comp. [16], p.7, or [10], VI, §12)

(2.4.) $$X^{\sim} \otimes X^{\sim} \xrightarrow{AW} \Delta(X \times X)^{\sim}$$

(2.5.) $$\Delta(X \times X)^{\sim} \xrightarrow{S} X^{\sim} \otimes X^{\sim}$$

Here AW is given componentwise by

$$\begin{array}{ccc} X^p \otimes X^q & \longrightarrow & X^{p+q} \otimes X^{p+q} \\ x \otimes y & \longmapsto & d^{p+q} \ldots d^{p+1}(x) \otimes d^0 \ldots d^0(y) \end{array} ,$$

and S is given componentwise, for each (p,q) with $p+q=n$, by

$$\begin{array}{ccc} X^n \otimes X^n & \longrightarrow & X^p \otimes X^q \\ x \otimes y & \longmapsto & \sum\limits_{(\mu,\nu)} \varepsilon(\mu,\nu) s^{\nu_q} \ldots s^{\nu_1} x \otimes s^{\mu_p} \ldots s^{\mu_1} y \end{array} .$$

The sum ranges over all (p,q)-shuffles (μ,ν) with sign $\varepsilon(\mu,\nu)$. Both maps respect the normalized complexes, i.e., for 2.4. $NX \otimes NX$ is carried into $N\Delta(X \times X)$ and similarly for 2.5..Using the Alexander-Whitney map the definition of the cup product may now be rephrased as follows. A co-simplicial algebra in $\mathcal{A}$ is a co-simplicial object X together with a morphism

$$\Delta(X \times X) \longrightarrow X$$

satisfying certain conditions. The morphism induces homomorphisms of complexes

$$X^{\sim} \otimes X^{\sim} \xrightarrow{AW} \Delta(X \times X)^{\sim} \longrightarrow X^{\sim} ,$$

i.e., an algebra structure on $X^{\sim}$, and similarly an algebra structure on NX. This is the cup product. We shall use the shuffle map to make a co-simplicial algebra out of a DGA. To this end we shall use the Dold-Puppe theory which shows that N induces an equivalence of categories between the category of co-simplicial objects in $\mathcal{A}$ and the category of complexes in $\mathcal{A}$ concentrated in non-negative degree. A quasi-inverse K is given as follows. It associates to a complex Y the co-simplicial object

$$(KY)^n = \bigoplus_{f:[0,n] \twoheadrightarrow [0,p]} Y^p_f$$

where the sum is indexed by the surjective monotonic maps f and where $Y^p_f = Y^p$. If $u : [0,m] \to [0,n]$ is a monotonic map then to each diagram

$$\begin{tikzcd} {[0,m]} \arrow[r, "u"] \arrow[d, "g"'] & {[0,n]} \arrow[d, "f"] \\ {[0,q]} \arrow[r, hook, "j"] & {[0,p]} \end{tikzcd}$$

where f and g are surjective and j injective (note that j is uniquely determined by f and g) we associate the morphism

$$K(u)_{g,f} : Y^q_g \longrightarrow Y^p_f$$

equal to the identity if $p = q$, to the differential $d : Y^{p-1} \to Y^p$ if $j = d^0$ and equal to zero in all other cases. For pairs (g,f) not occurring in such a diagram the component $K(u)_{g,f}$ is put equal to zero.

It is easy to see ([11], p.222) that $NKY = Y$ where Y^p lies in $(KY)^p$ as Y^p_{id}. Furthermore, calling $D^p \subset (KY)^{\sim p}$ the remaining direct summands we have, as one checks easily ([11], 3.18)

$$D^p = \sum_{i=0}^{p-1} \operatorname{Im} d^i \quad .$$

Since K is an equivalence of categories we have for all co-simplicial objects X

$$X^{\sim} = D^{\cdot} \oplus N^{\cdot} \quad , \tag{2.6.}$$

a decomposition into subcomplexes. With respect to the cup product D is a left ideal in $X^{\sim}$. Indeed this follows from the formula for $x \in X^p$ and $y \in X^{q-1}$

$$\begin{aligned} x \cup d^i(y) &= (d^{p+q} \circ \ldots \circ d^{p+1}(x)) \cdot ((d^0)^p d^i(y)) = \\ &= (d^{i+p} d^{p+q-1} \circ \ldots \circ d^{p+1}(x)) \cdot (d^{i+p}(d^0)^p(y)) \\ &= d^{i+p}(x \cup y) \end{aligned} \tag{2.7}$$

We now show how to produce a co-simplicial algebra from a DGA. Let Y be a DGA,

$$Y \otimes Y \longrightarrow Y \quad .$$

Applying the functor K we obtain homomorphisms of complexes

$$N\Delta(KY \times KY) \xrightarrow{S} NKY \otimes NKY = Y \otimes Y \longrightarrow Y = NKY$$

which since N is an equivalence of categories corresponds to a morphism of co-simplicial objects

$$\Delta(KY \times KY) \longrightarrow KY \quad .$$

The properties of the shuffle map (comp. [10] VI, §12) show that the associativity law holds and furthermore (in contrast to the functor N) that KY is a commutative co-simplicial algebra if Y is graded-commutative.

2.8. Lemma:

a) If X is a reduced small co-simplicial algebra in $\mathcal{A}$, then NX is a reduced small DGA *in $\mathcal{A}$.*

b) If Y is a reduced small DGA *in $\mathcal{A}$, then KY is a reduced small co-simplicial algebra in $\mathcal{A}$ and $NKY = Y$.*

Assuming this lemma as well for the moment we prove the proposition 2.2. as follows. By lemma 2.8. we know that K is a fully faithful functor between reduced small algebras and it remains to show the essential surjectivity. Let $\Delta(X \times X) \xrightarrow{m} X$ be a reduced small co-simplicial algebra. Since $K \circ N$ is isomorphic to the identity functor of the category of co-simplicial objects we obtain, neglecting this isomorphism, a new structure of a reduced small co-simplicial algebra on X which induces the same algebra structure by cup product on NX as the algebra structure with which we started (use lemma 2.8.),

$$\Delta(X \times X) \xrightarrow{m'} X \quad .$$

We show first that the two cup product structures on $X^\sim$ coincide. Since $X^\sim$ is generated by X^1 (both in the sense of the first and the second cup product) and using the associativity laws we need to show only that the linear maps

$$X^1 \otimes ... \otimes X^1 \underset{\mu'=\cup'}{\overset{\mu=\cup}{\rightrightarrows}} X^p$$

are identical for all p. We decompose X^1 as $X^1 = \mathbb{1} \oplus N^1$ (a special case of the decomposition 2.6.), which then defines a decomposition of the tensor product above. The restriction of μ or μ' to the summand $N^1 \otimes ... \otimes \mathbb{1} \otimes ... \otimes N^1$ (factor $\mathbb{1}$ in the positions $i_1 < ... < i_r$) is given by

(2.9.) $$\mu(x_1 \otimes ... \otimes 1 \otimes ... \otimes x_p) = d^{i_r-1} d^{i_{r-1}-1} ... d^{i_1-1}(x_1 \cup ... \cup x_p) \quad ,$$

as follows by a repeated application of the identity (2.7.). An obvious induction shows that $\mu = \mu'$, so that the two cup product structures on $X^\sim$ are identical. To show that the two co-simplicial algebra structures on X coincide and since X^1 generates X, it suffices to show that the two maps

$$(X^1 \otimes ... \otimes X^1) \otimes (X^1 \otimes ... \otimes X^1) \underset{m'(\cup',\cup')}{\overset{m(\cup,\cup)}{\rightrightarrows}} X^p$$

(twice p factors) coincide. This follows from the bi-multiplicativity of the cup product (which holds since X is commutative for either algebra structure)

$$(x \cup y) \cdot (x' \cup y') = (x \cdot x') \cup (y \cdot y')$$

and the obvious fact that on X^1 both algebra structures coincide.

It remains to prove lemmas 2.3. and 2.8..

Proof of 2.3.: The argument is similar to the above. We have to consider

$$\mu : X^1 \otimes ... \otimes X^1 \longrightarrow X^p$$

and its restriction $\overline{\mu}$ to the summand $N^1 \otimes ... \otimes N^1$ in the direct sum decomposition of the tensor product induced from the decomposition $X^1 = \mathbb{1} \oplus N^1$. Then the image of $\overline{\mu}$ lies in N^p. From the identity (2.9.) we conclude that μ maps all direct summands other than $N^1 \otimes ... \otimes N^1$ into D^p. Therefore μ is surjective if and only if the image of $\overline{\mu}$ is all of N^p.

Proof of 2.8.: a) It is obvious that NX is reduced and by 2.3. NX is generated by N^1. It remains to show that NX is commutative. Since N^1 generates NX it suffices to show that $x \cup x = 0$ for $x \in N^1$. Consider

$$\mu : X^1 \otimes X^1 \xrightarrow{\cup} X^2 \quad .$$

This map is surjective. Since $(\sum(-1)^i d^i)(x) \in N^2$, it is by the argument in the proof of 2.3. the image of an element $v \in N^1 \otimes N^1$,

$$d^1(x) = d^0(x) + d^2(x) + \mu(v) \quad .$$

Since N^1 is an ideal of square zero we get

$$0 = d^1(x^2) = (d^1(x))^2 = 2 \cdot x \cup x \quad .$$

b) That KY is reduced is obvious, that it is commutative follows from the properties of the shuffle map which have been mentioned earlier, and that KY is generated by KY^1 follows from 2.3.. Therefore KY is a small co-simplicial algebra, since an easy calculation using the definition of the algebra structure on KY shows that NKY^1 has square zero. The last assertion follows from the commutative diagram

$$\begin{array}{ccc}
Y \otimes Y & \longrightarrow & Y \\
s \uparrow & & \| \\
N\Delta(KY \times KY) & \longrightarrow & NKY \\
AW \uparrow & & \| \\
NKY \otimes NKY & \longrightarrow & NKY \\
\| & & \| \\
Y \otimes Y & \longrightarrow & Y
\end{array}$$

in which the first and last horizontal arrows are identical since on the normal cochains the composition $S \circ AW$ is the identity morphism ([12], II, Thm. 2.1.a)).

In what follows we shall be interested in only two $\otimes$-categories.

1) The category Vec_k of k-vector spaces. In this case the concepts of co-simplicial algebras and DGA are the usual ones. We shall call them c-algebras resp. d-algebras.

2) The category $C_{\geq 0}(\mathrm{Vec}_k)$ of complexes of vector spaces in degree ≥ 0. In this case we shall call a co-simplicial algebra a cd-algebra and a DGA a dd-algebra. It is clear that a cd-algebra is simply a co-simplicial DGA, whereas a dd-algebra is a bigraded algebra in $A^{*,\cdot}$ with two differentials d^* and d of degree $(1,0)$ and $(0,1)$ respectively, with $dd^* = d^*d$ and which are graded derivations with respect to the first resp. the second degree.

There is an obvious functor

$$C_{\geq 0}(\mathrm{Vec}_k) \longrightarrow \mathrm{Vec}_k$$

which assigns to a complex its zero'th component. Correspondingly we have functors

$$\begin{array}{c}
\text{reduced small } cd\text{-algebras} \longrightarrow \text{reduced small } c\text{-algebras} \\
\text{reduced small } dd\text{-algebras} \longrightarrow \text{reduced small } d\text{-algebras}
\end{array}$$

These functors have left-adjoints, to be denoted by $\bar{\Omega}$.

Explicitly, if $R^{\cdot}$ is a reduced small c-algebra one associates as follows to $R^{\cdot}$ a reduced small cd-algebra $\bar{\Omega}^*(R^{\cdot})$ with a homomorphism $R^{\cdot} \to \bar{\Omega}^*(R^{\cdot})$ with the required universal property.

$$\bar{\Omega}^*(R^{\cdot}) = \Omega^*(R^{\cdot})/cd - \text{ideal spanned by} \\ [\operatorname{Ker} s^0 : \Omega^*(R^1) \to \Omega^*(R^0)]^2 \quad .$$

Here $\Omega^*(R^\cdot) = \Omega^*_{R^\cdot/k}$ is the de Rham complex of $R^\cdot$. From the fact that N is an equivalence of categories (Proposition 2.2.) we obtain a natural isomorphism of functors

$$N\bar{\Omega} = \bar{\Omega}N \quad .$$

We shall now apply these concepts to an explicit example. Let g be a Lie algebra over k and let $C^\cdot(\mathrm{g}) = C^\cdot(\mathrm{g}, k)$ be the Lie algebra complex with values in the trivial representation (see §1). Then $C^\cdot(\mathrm{g})$ is a reduced small d-algebra. We consider the complex which is concentrated in degrees 1 and 2,

$$d^* = id : \mathrm{g}' \longrightarrow \mathrm{g}' \quad ,$$

where as before g' denotes the dual vector space. We form the free graded-commutative DGA,

$$W = \Lambda(\mathrm{g}' \to \mathrm{g}') = S(\mathrm{g}') \otimes \Lambda(\mathrm{g}') \quad .$$

Then W is graded in an obvious way (total degree), but also graded according to the word length in terms of generators, i.e., the direct summand

$$W^{i,j} = S^i(\mathrm{g}') \otimes \Lambda^{j-i}(\mathrm{g}')$$

has total degree $2i + j - i = i + j$, and word length $i + (j - i) = j$. The differential d^* extends uniquely as a graded derivation w.r.t. the first degree. It is the Koszul differential ([7], §9,3.) $d^* : W^{i,j} \to W^{i+1,j}$,

$$d^*(x_1 ... x_i \otimes y_1 \wedge ... \wedge y_{j-i}) = \sum (-1)^s y_s x_1 ... x_i \otimes y_1 \wedge ... \wedge \hat{y}_s \wedge ... \wedge y_{j-i}$$

On the other hand there is a canonical embedding $\Lambda^\cdot \mathrm{g}' \hookrightarrow W^{0,\cdot}$ and, as one checks easily, the differential in the Lie complex extends in a unique way into a differential $d : W^{*,\cdot} \to W^{*,\cdot+1}$ such that $dd^* = d^*d$, and indeed d induces on $W^{i,\cdot}$ the differential in the Lie complex $C^\cdot(\mathrm{g}, S^i(\mathrm{g}'))$. It is obvious from the construction that $W^{*,\cdot}$ has a universal property[1)] which implies that

$$W^{*,\cdot} = \bar{\Omega}^*(C^\cdot(\mathrm{g})) \quad .$$

$W^{*,\cdot}$ is called the Weil algebra.

The Koszul complex $W^{*,\cdot}$ is acyclic in degree > 0 and a resolution of k in degree 0 ([7], §9,3, Prop. 3.). We therefore obtain the first statement in the following lemma (cohomology of the simple complex associated to a double complex).

2.10. Lemma:

a) $H^j(W^{*,\cdot}) = 0$ *for* $j > 0$ *and* $H^0(W^{*,\cdot}) = k$.

b) $H^{2n}(W^{\geq n,\cdot}) = (S^n \mathrm{g}')^{\mathrm{g}}$.

[1)] In fact, $W^{*,\cdot}$ has the universal property within all graded-commutative DGA, not just reduced small ones.

Proof of b): Since $W^{i,j} = 0$ for $i < j$ we obtain

$$\begin{aligned} H^{2n}(W^{\geq n,\cdot}) &= \mathrm{Ker}(S^n(\mathfrak{g}') \otimes \Lambda^0 \mathfrak{g}' \xrightarrow{d} S^n(g') \otimes \Lambda^1 \mathfrak{g}') \\ &= H^0(\mathfrak{g}, S^n(g')) = S^n(\mathfrak{g}')^{\mathfrak{g}} \ . \end{aligned}$$

From now on G is a reductive algebraic group over k with Lie algebra $\mathfrak{g}$. Extending scalars from k to $\mathbb{C}$ and using the existence of a compact form we deduce from the results of §1 that

(2.11.) $$H^*_{DR}(G) \xrightarrow{\sim} H^*(\mathfrak{g})$$

The isomorphism is induced by restricting a differential form to the identity.

2.12. Lemma:

There is one and only one ring homomorphism

$$H^{2*}_{DR}(B.G) \longrightarrow S^*(\mathfrak{g}')^{\mathfrak{g}}$$

which is functorial in G and such that for $G = \mathbf{G}_m$ it identifies $\mathfrak{g}' = S^1(\mathfrak{g}')^{\mathfrak{g}}$ with the space of invariant differentials on $G = B_1G$ (first component of $B.G$). This homomorphism is an isomorphism.

Proof: The last condition means the following. The edge homomorphism

$$H^2_{DR}(B.G) \longrightarrow H^1_{DR}(G)$$

which appears in the Eilenberg-Moore spectral sequence ([9], 9.1.5)

$$E_1^{pq} = H^q_{DR}(G^p) \Longrightarrow H^{p+q}_{DR}(B.G)$$

is an isomorphism for $G = \mathbf{G}_m$. Under the identification of $H^2_{DR}(B.\mathbf{G}_m)$ with $H^1_{DR}(\mathbf{G}_m)$ the morphism in the statement of 2.12. becomes the isomorphism $H^1_{DR}(\mathbf{G}_m) \xrightarrow{\sim} H^0(\mathbf{G}_m, \Omega^1) = \mathfrak{g}'$. Let $T \subset G$ be a maximal torus. The first assertion follows from the diagram in which W is the Weyl group (not the Weil algebra!) (e.g. [9], 6.1.6.)

$$\begin{array}{ccc} H^{2*}_{DR}(B.G) & \rightarrow & H^{2*}_{DR}(B.T) \\ \downarrow & & \downarrow \\ S^*(\mathfrak{g}')^{\mathfrak{g}} = S^*(\mathfrak{t}')^W & \hookrightarrow & S^*(\mathfrak{t}') \end{array} .$$

The second assertion follows from the strong form of the splitting principle which identifies $H^*_{DR}(B.G)$ with the W-invariants in $H^*_{DR}(B.T)$ (loc.cit.).

We next consider the following chain of homomorphisms defined via 2.10.. Let $n > 0$.

$$\begin{aligned} (S^n\mathfrak{g}')^{\mathfrak{g}} = H^{2n}(W^{\geq n,\cdot}) = H^{2n-1}(W^{*,\cdot}/W^{\geq n,\cdot}) \rightarrow H^{2n-1}(W^{*,\cdot}/W^{\geq 1,\cdot}) = \\ = H^{2n-1}(\mathfrak{g}) \ . \end{aligned}$$

Let

$$P^{2*}(\mathfrak{g}) = S^*(\mathfrak{g}')^{\mathfrak{g}}/(S^{\geq 1}(\mathfrak{g}')^{\mathfrak{g}} \cdot S^{\geq 1}(\mathfrak{g}')^{\mathfrak{g}})$$

be the factor space of indecomposable elements and let $\mathrm{Prim}^*(\mathfrak{g})$ be the sub-space of primitive elements in $H^*(\mathfrak{g})$. The following theorem is due to H.Cartan [8]; for a proof we refer to [14], 6.14.

2.13. Theorem:

The above homomorphism induces an isomorphism

$$P^{2n}(\mathrm{g}) \xrightarrow{\sim} \mathrm{Prim}^{2n-1}(\mathrm{g}) \quad .$$

We now apply the results to $G = GL_N$ over $\mathbb{Q}$. In this case, as is well known, the Chern class $c_n \in H^{2n}_{DR}(B.G_{/\mathbb{Q}})$ defines via 2.12. a generator of the vector space $P^{2n}(\mathrm{g}_{/\mathbb{Q}})$ which thus has to go under the isomorphism in 2.13. to a generator v_{2n-1} of $\mathrm{Prim}^{2n-1}(\mathrm{g})$, which via 2.11. is also a generator of $\mathrm{Prim}^{2n-1}_{DR}(G_{/\mathbb{Q}}) \subset H^{2n-1}_{DR}(G_{/\mathbb{Q}})$. Under the comparison isomorphism

$$H^*_{DR}(G_{/\mathbb{C}}) = H^*_{Betti}(G(\mathbb{C}), \mathbb{C})$$

the subspace $P^{2n-1}_{DR}(G_{/\mathbb{Q}})$ goes into $H^{2n-1}_{Betti}(G(\mathbb{C}), \mathbb{Q}(n))$. Indeed, this follows from [5], 4.3., using the restriction isomorphism

$$H^*_{Betti}(G(\mathbb{C}), \mathbb{C}) \xrightarrow{\sim} H^*_{Betti}(K, \mathbb{C}) \quad .$$

Here as in §1, $K = U_N$ is the maximal compact subgroup of $GL_N(\mathbb{C})$. Making use of the element u_{2n-1} introduced in §1, at least up to a rational factor, we conclude

2.14. Corollary:

Let G be $GL_{N/\mathbb{C}}$. The image v_{2n-1} of the n'th Chern class $c_n \in H^{2n}_{DR}(B.G)$ under the homomorphism

$$H^{2n}_{DR}(B.G) \overset{2.12.}{\simeq} S^n(\mathrm{g}')^{\mathrm{g}} \xrightarrow{2.13.} H^{2n-1}(\mathrm{g}) = H^{2n-1}(\mathrm{k}, \mathbb{C})$$

is equal to a non-zero rational multiple of the image of $(2\pi i)^n u_{2n-1} \in H^{2n-1}_{Betti}(K, \mathbb{C})$ under the comparison isomorphism $H^*_{DR}(K) \otimes \mathbb{C} \simeq H^*_{Betti}(K, \mathbb{C})$.

We conclude this section with two questions which are raised by the preceding considerations. Let G be a reductive algebraic group over $\mathbb{C}$. Then the edge homomorphism in the Eilenberg-Moore spectral sequence ([9], 9.1.5.) gives for $n > 0$

$$H^{2n}_{Betti}(B.G(\mathbb{C}), \mathbb{Q}) \longrightarrow E_1^{1,2n-1} = H^{2n-1}_{Betti}(G(\mathbb{C}), \mathbb{Q}) \quad .$$

Does this homomorphism correspond under the comparison isomorphism between Betti cohomology and de Rham cohomology to the map defined in 2.14.?

Let G be a reductive group over an arbitrary field of characteristic zero. Is the isomorphism in (2.13.) the inverse of the transgression homomorphism associated to the Leray spectral sequence in de Rham cohomology of the universal G-bundle over $B.G$? This is stated without proof by Beilinson, [1], A 3.1.

§3. Beilinson's version of the van Est isomorphism

Let G be an algebraic group over $\mathbb{R}$. Let $B.G^{(1)}$ be the largest simplicial closed subscheme of $B.G$ with first component the first infinitesimal neighbourhood of the identity, $G^{(1)}$.

$$pt \leftleftarrows G^{(1)} \;\overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}}\; B_2G^{(1)} \;\begin{smallmatrix}\longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow\end{smallmatrix}\; \dots$$

We thus have inductively (inverse images inside of B_pG)

$$B_pG^{(1)} = \bigcap_{i=0}^{p} (d^i)^{-1}(B_{p-1}G^{(1)}) \ .$$

Clearly $B_pG^{(1)} \subset G^{(1)} \times \dots \times G^{(1)}$, but this is in general a strict inclusion. On the other hand, $B.G^{(1)}$ contains the first infinitesimal neighbourhood of the identity in each component. The simplicial scheme $B.G^{(1)}$ is the largest small simplicial subscheme in $B.G$. Beilinson's interpretation of the Borel regulator is based on the following lemma.

3.1. Lemma:

There is a canonical isomorphism of reduced DGA

$$NH^0(B.G^{(1)}, S^0) = NH^0(B.G^{(1)}, \mathcal{O}) \cong C^{\cdot}(\mathrm{g})$$

Proof: We are dealing here with the cohomology of (simplicial) analytic spaces over $\mathbb{R}$ ([13], 2.1.). Clearly $H^0(B_1G^{(1)}, \mathcal{O}) = H^0(B_1G^{(1)}, S^0) = \mathbb{R} \oplus \mathrm{g}'$, with g' being the augmentation ideal of square zero. Therefore both sides coincide in degrees 0 and 1, and we shall show that there is a unique extension to an isomorphism of DGA. Uniqueness is clear.

Let $\hat{\mathcal{O}}$ be the completion of the local ring at the identity element of G, so that $\mathbb{R} \oplus \mathrm{g}' = \hat{\mathcal{O}}/\mathrm{m}^2$, with m denoting the maximal ideal. Choose an isomorphism $\hat{\mathcal{O}} \cong R[[X_1, \dots, X_n]]$ such that the formal group law of G is described by n power series in $2n$ variables $G_1(\underline{X}, \underline{Y}), \dots, G_n(\underline{X}, \underline{Y})$ with

$$G_i(\underline{X}, \underline{Y}) = X_i + Y_i + \sum c_i^{jk} X_j X_k \bmod \deg 3 \ .$$

By definition, the affine ring of $B_2G^{(1)}$ is the ring

$$(\hat{\mathcal{O}}/\mathrm{m}^2 \otimes \hat{\mathcal{O}}/\mathrm{m}^2)/J_2$$

where J_2 is the image of $d^1(\mathrm{m}^2)$, i.e., of the ideal generated by the products $G_i \cdot G_j$ in $\hat{\mathcal{O}} \otimes \hat{\mathcal{O}} = R[[\underline{X}, \underline{Y}]]$. But

$$\begin{aligned} G_i \cdot G_j &= (X_i + Y_i + \sum c_i^{k\ell} X_k Y_\ell)(X_j + Y_j + \sum c_j^{k\ell} X_k Y_\ell) \bmod \deg 3 \\ &= X_i Y_j + Y_i X_j \ \text{ in } \ \hat{\mathcal{O}}/\mathrm{m}^2 \otimes \hat{\mathcal{O}}/\mathrm{m}^2 \ . \end{aligned}$$

For the normalization we obtain

$$\begin{aligned} N^2H^0(B.G^{(1)}, S^0) = N^2H^0(B.G^{(1)}, \mathcal{O}) &= \mathrm{g}' \otimes \mathrm{g}'/J_2 = \\ &= \mathrm{g}' \otimes \mathrm{g}' / \operatorname{Sym}^2 \mathrm{g}' = \Lambda^2 \mathrm{g}' \ . \end{aligned}$$

Recalling the definition of the face and degeneracy operators in higher degree we see that the left hand side in 3.1. in degree p is the factor space of $\mathrm{g}'^{\otimes p}$ by the symmetrizer subspace of two consecutive variables, i.e., $\Lambda^p \mathrm{g}'$. To see that the vector space isomorphism thus obtained respects the DGA structures it suffices to show that it is multiplicative in degree 1 and that the first differentials coincide. The multiplicative structure on $C^{\cdot}(\mathrm{g})$ is given by the wedge product. To $x, y \in \mathrm{g}'$ there is associated the alternating bilinear form on g

$$x \wedge y(\xi, \eta) = x(\xi) \cdot y(\eta) - x(\eta) \cdot y(\xi) \quad .$$

The multiplicative structure on $N\mathcal{O}(B.G^{(1)})$ is induced by the cup product, so that for $f = x \in \mathrm{g}', g = y \in \mathrm{g}'$ we obtain the element $f \cup g = x \otimes y \in \mathrm{g}' \otimes \mathrm{g}'/J_2$, to which corresponds the alternating bilinear form $x \wedge y$ on g. Thus multiplicativity is clear. It remains to compare the first differentials and for this we use the following formula for the Lie bracket in g (comp. [20], exercise 1.44., p.20). Let $\xi_i \in \mathrm{g}$ be the dual basis corresponding to the coordinates X_i. Then

$$[\xi_k, \xi_\ell] = \sum_i (c_i^{k\ell} - c_i^{\ell k}) \cdot \xi_i \quad .$$

The differential on the left side

$$d : \mathrm{g}' \longrightarrow \mathrm{g}' \otimes \mathrm{g}'/J_2$$

is given by

$$\begin{aligned} d(X_i) &= X_i - (X_i + Y_i + \sum c_i^{k\ell} X_k Y_\ell) + Y_i \bmod J_2 \\ &= -\sum c_i^{k\ell} X_k Y_\ell \quad , \end{aligned}$$

to which corresponds the alternating bilinear form on g

$$\begin{aligned} d(X_i)(\xi_k, \xi_\ell) &= -(c_i^{k\ell} - c_i^{\ell k}) \\ &= -X_i([\xi_k, \xi_\ell]) \quad . \end{aligned}$$

Recalling the definition of the differential in the Lie complex (§1) this proves the assertion.

We now return to the definition of the Borel regulator which was based on the continuous group cohomology $H^*_{cont}(G(\mathbb{R}), \mathbb{R})$.(In fact we shall take $\mathrm{g} = R_{\mathbb{C}/\mathbb{R}} GL_N$). Since continuous cohomology may be calculated by C^∞-cochains we have

$$H^*_{cont}(G(\mathbb{R}), \mathbb{R}) = H^*(B.G, S^0) \quad . \tag{3.2.}$$

Here on the right side is the cohomology of the simplicial scheme over $\mathbb{R}$ with values in the sheaf of real-valued C^∞-functions (cf. [13]). The restriction homomorphism and lemma 3.1. now define a homomorphism

$$H^*_{cont}(G(\mathbb{R}), \mathbb{R}) = H^*(B.G, S^0) \longrightarrow H^*(B.G^{(1)}, S^0/J) = H^*(\mathrm{g}, \mathbb{R}) \quad .$$

Here $J \subset S^0$ denotes the simplicial ideal generated by m^2 in degree 1, so that $S^0/J = S^0_{B.G^{(1)}}$.

3.3. Theorem:

The above homomorphism is the composition of the van Est isomorphism and the canonical map from the relative Lie algebra cohomology to the absolute Lie algebra cohomology.

By construction, this composition is obtained as follows. Consider the homomorphism of complexes

$$\mathbb{R} \longrightarrow [S^0(G) \longrightarrow S^1(G) \longrightarrow ...] \ .$$

It induces a homomorphism in continuous cohomology,

$$\begin{aligned} H^*_{cont}(G(\mathbb{R}), \mathbb{R}) \to H^*_{cont}(G(\mathbb{R}), S^\cdot(G)) &= H^*(S^\cdot(G)^{G(\mathbb{R})}) \\ &= H^*(\mathrm{g}, \mathbb{R}) \ . \end{aligned}$$

If M is a continuous $G(\mathbb{R})$-module we associate to it a simplicial S^0-module sheaf on $B.G$, to be denoted by $\tilde{M}$. Explicitly, let

$$\tilde{M}^p = \mathrm{Map}_{C^\infty}(G^p, M)$$

(or rather "sheafification of"), with arrows $d^j : \tilde{M}^p \to \tilde{M}^{p+1}$ given by

$$d^j(f)(g_1, ..., g_{p+1}) = \begin{cases} f(g_2, ..., g_{p+1}) & j = 0 \\ f(g_1, ..., g_j g_{j+1}, ..., g_{p+1}) & 1 \le j \le p \\ g_{p+1}^{-1} \cdot f(g_1, ..., g_p) & j = p+1 \end{cases}$$

Then just as in (3.2.) we have $H^*_{cont}(G(\mathbb{R}), M) = H^*(B.G, \tilde{M})$.

We denote by $\tilde{M}^{(1)}$ the sheaf $\tilde{M} \otimes S^0/J$ on $B.G^{(1)}$. We need the following generalization of lemma 3.1..

3.4. Lemma:

$NH^0(B.G^{(1)}, \tilde{M}^{(1)}) = C^\cdot(\mathrm{g}, M)$.

Proof: Both sides are modules in the differential graded sense over the DGA $NH^0(B.G^{(1)}, \mathcal{O}) = C^\cdot(\mathrm{g})$ and are generated by their zero'th component, which is M in both cases. Furthermore, as graded vector spaces we have

$$\begin{aligned} NH^0(B.G^{(1)}, \tilde{M}^{(1)}) &= NH^0(B.G^{(1)}, \mathcal{O}) \otimes M \\ C(\mathrm{g}, M) &= C(\mathrm{g}) \otimes M \ . \end{aligned}$$

Hence both sides are isomorphic as graded vector spaces. Since the differentials on both sides agree in degree zero, the two modules are isomorphic.

Denoting as before by $\hat{\mathcal{O}}$ the completion of the local ring at the identity we put

$$\hat{M} = M \otimes_{S^0(G)} \hat{\mathcal{O}} \ .$$

Then there is a natural identification ([20], p.21)

$$\hat{S}^j(G) = \mathrm{Hom}(\mathcal{U}(\mathrm{g}), \Lambda^j \mathrm{g}') \quad ,$$

where $\mathcal{U}(\mathrm{g})$ denotes the universal enveloping algebra. It follows that (Shapiro isomorphism)

$$H^*(\mathrm{g}, \hat{S}^j(G)) = H^*(\mathrm{g}, \mathrm{Hom}(\mathcal{U}(\mathrm{g}), \Lambda^j \mathrm{g}')) = \Lambda^j \mathrm{g}'[0] \quad .$$

The assertion now follows from the commutativity of the following diagram (here commutativity means that isomorphisms going the "wrong way" have to be inverted).

$$\begin{tikzcd}
H^*_{cont}(G(\mathbb{R}), \mathbb{R}) \arrow[r] \arrow[d] & H^*_{cont}(G(\mathbb{R}), S^0(G) \to S^1(G) \to \ldots) \arrow[r] \arrow[d, "\sim"] & H^*(S^\cdot(G)^{G(\mathbb{R})}) \arrow[d, equal] \\
H^*(B.G, S^0) \arrow[r] \arrow[d] & H^*(B.G, \tilde{S}^0(G) \to \tilde{S}^1(G) \to \ldots) \arrow[d] & H^*(\mathrm{g}, \mathbb{R}) \arrow[dddddl, equal, bend left] \\
H^*(B.G^{(1)}, S^0/J) \arrow[r] \arrow[d, "N"] & H^*(B.G^{(1)}, \tilde{S}^0(G)^{(1)} \to \tilde{S}^1(G)^{(1)} \to \ldots) \arrow[d, "N"] & \\
H^*(C^\cdot(\mathrm{g}, \mathbb{R})) \arrow[r] \arrow[dddr, equal, bend right] & H^*(C^\cdot(\mathrm{g}, S^0(G)) \to C^\cdot(\mathrm{g}, S^1(G)) \to \ldots) \arrow[d] & \\
& H^*(C^\cdot(\mathrm{g}, \hat{S}^0(G)) \to C^\cdot(\mathrm{g}, \hat{S}^1(G)) \to \ldots) \arrow[d] & \\
& H^*(C^\cdot(\mathrm{g}, \mathrm{Hom}(\mathcal{U}(\mathrm{g}), \mathbb{R})) \to C^\cdot(\mathrm{g}, \mathrm{Hom}(\mathcal{U}(\mathrm{g}), \Lambda^1 \mathrm{g}')) \to \ldots) & \\
& H^*(\mathbb{R} \to \Lambda^1 \mathrm{g}' \to \ldots) \arrow[u, "\sim"] &
\end{tikzcd}$$

§4. Equality of the two regulator maps

We continue with the set-up of §3. On $B.G^{(1)}$ we consider the following analogue of the Deligne complex

$$\bar{A}(n)_{\mathcal{D}} = (\bar{\Omega}^{\geq n} \oplus A(n) \longrightarrow \bar{\Omega}^*)[1] \quad .$$

Here $\bar{\Omega}^* = \bar{\Omega}^*(\mathcal{O}(B.G^{(1)}))$ is the value of the functor $\bar{\Omega}^*$ on the reduced small c-algebra $\mathcal{O}(B.G^{(1)})$, which we may also consider as a complex of sheaves on $B.G^{(1)}$. Clearly $H^j(B.G^{(1)}, A(n)) = 0$ for $j > 0$. On the other hand we deduce from lemma 3.1. that

$$\begin{aligned} H^j(B.G^{(1)}, \bar{\Omega}^*) = H^j(N\bar{\Omega}^*) = H^j(\bar{\Omega}^*(N\mathcal{O}(B.G^{(1)})) = H^j(\bar{\Omega}^*(C^{\cdot}(\mathrm{g}))) = \\ = H^j(W^{*,\cdot}) = 0 \quad \text{for} \quad j > 0 \end{aligned}$$

by the results of §2. We thus obtain for $n > 0$ an isomorphism

$$H^{2n}(B.G^{(1)}, \bar{A}(n)_{\mathcal{D}}) \xrightarrow{\sim} H^{2n}(B.G^{(1)}, \bar{\Omega}^{\geq n}) \quad .$$

We now have a commutative diagram (the upper horizontal arrow is an isomorphism (compare [13]))

$$\begin{array}{ccccc} H^{2n}_{\mathcal{D}}(B.G, A(n)) & \longrightarrow & H^{2n}(B.G, F^n) & = & H^{2n}_{DR}(B.G) \\ \downarrow & & \downarrow & & \downarrow \phi \\ H^{2n}(B.G^{(1)}, \bar{A}(n)_{\mathcal{D}}) & \longrightarrow & H^{2n}(B.G^{(1)}, \bar{\Omega}^{\geq n}) & = & H^{2n}(W^{\geq n,\cdot}) \end{array} \quad .$$

Here the vertical arrows are defined using the maps (comp. §2)

$$\begin{aligned} \Omega^*_{BG} \to \Omega^*_{BG^{(1)}} \to \bar{\Omega}^* = \Omega^*_{BG^{(1)}}/cd - \text{ideal generated by} \\ [\operatorname{Ker} s^0 : \Omega^*(\mathcal{O}(G^{(1)})) \to \mathbb{R}]^2 \quad . \end{aligned}$$

4.1. Proposition:

The composition of $\phi : H^{2n}_{DR}(B.G) \to H^{2n}(W^{\geq n,\cdot})$ with the isomorphism of 2.10., $H^{2n}(W^{\geq n,\cdot}) \simeq S^n(\mathrm{g}')^{\mathrm{g}}$, *coincides with the isomorphism of lemma* 2.12.

Proof: Indeed, the construction of ϕ is clearly functorial and for $G = \mathbf{G}_m$ is the required isomorphism.

4.2. Corollary:

The Beilinson regulator and the Borel regulator are identical maps (up to a factor in $\mathbb{Q}^*$)

$$K_{2n-1}(\mathbb{C}) \longrightarrow \mathbb{R}(n-1) \quad .$$

In this proof we shall have to distinguish whether we consider the Lie algebra g of $G = GL_{N/\mathbb{C}}$ as a real or complex Lie algebra; correspondingly we write $\mathrm{g}_{/\mathbb{R}}$ or g. By proposition 4.1. the composition of the following homomorphisms coincides with the map considered before 2.13.

$$\begin{aligned} H^{2n}_{DR}(B.G) \xrightarrow{\phi} H^{2n}(W^{\geq n,\cdot}) \xrightarrow{\sim} H^{2n-1}(W^{*,\cdot}/W^{\geq n,\cdot}) \to H^{2n-1}(W^{*,\cdot}/W^{\geq 1,\cdot}) = \\ = H^{2n-1}(\mathrm{g}) = H^{2n-1}_{DR}(G) \end{aligned}$$

Therefore the image of the n-th Chern class is the canonical primitive element $v_{2n-1} \in H^{2n-1}(\mathfrak{g})$. We consider the commutative diagram where as usual $\pi_{n-1} : \mathbb{R}(n)_{\mathcal{D}} \to \tilde{\mathbb{R}}(n)_{\mathcal{D}}$ denotes the natural homomorphism to the "real version" of the Deligne complex ([13])

$$\begin{tikzcd}
H^{2n}_{\mathcal{D}}(B.G, \mathbb{R}(n)) \arrow[r, "\pi_{n-1}"] \arrow[ddd, bend right, "\phi"'] \arrow[dd] & H^{2n}(B.G, \tilde{\mathbb{R}}(n)_{\mathcal{D}}) \arrow[r] & H^{2n-1}(B.G, S^{\cdot}(n-1)/F^{n}) \arrow[r] & H^{2n-1}(B.G, S^{0}(n-1)) \arrow[d] \\
 & & H^{2n-1}(\mathfrak{g}_{/\mathbb{R}}, \mathbb{R}(n-1)) \arrow[r, phantom, "="] & H^{2n-1}(B.G^{(1)}, S^{0}/J(n-1)) \\
H^{2n}(B.G^{(1)}, \overline{\mathbb{R}(n)}_{\mathcal{D}}) \arrow[r] \arrow[d] & H^{2n-1}(B.G^{(1)}, \bar{\Omega}^{\cdot}/\bar{\Omega}^{\geq n} + \mathbb{R}(n)) \arrow[r] & H^{2n-1}(B.G^{(1)}, \mathcal{O}/\mathbb{R}(n)) \arrow[u, "\pi_{n-1}"] \arrow[r, phantom, "="] & H^{2n-1}(B.G^{(1)}, \mathcal{O}) \\
H^{2n}(W^{\geq n, \cdot}) \arrow[r] & H^{2n-1}(W^{\bullet, \cdot}/W^{\geq n, \cdot}) \arrow[r] & H^{2n-1}(W^{\bullet, \cdot}/W^{\geq 1, \cdot}) \arrow[u] \arrow[r, phantom, "="] & H^{2n-1}(\mathfrak{g}).
\end{tikzcd}$$

Here the interesting arrows come by projecting $\bar{\Omega}^{*}$ resp. $S^{\cdot}$ on its zero'th component. Now the n-th Chern class c_n may be considered as an element in $H^{2n}_{\mathcal{D}}(B.G, \mathbb{R}(n))$. By what we saw already the image of c_n in $H^{2n-1}(\mathfrak{g})$ is the element v_{2n-1}. On the other hand, the homomorphism

$$e^{*} : H^{2n}_{\mathcal{D}}(B.G, \mathbb{R}(n)) \longrightarrow H^{2n-1}(B.GL_N(\mathbb{C}), \mathbb{R}(n-1))$$

may be factored as the composition of

$$H^{2n}_{\mathcal{D}}(B.G, \mathbb{R}(n)) \longrightarrow H^{2n-1}(B.G, S^{0}(n-1))$$

and the forget-the-topology-map

$$H^{2n-1}_{cont}(GL_N(\mathbb{C}), \mathbb{R}(n-1)) \longrightarrow H^{2n-1}(GL_N(\mathbb{C}), \mathbb{R}(n-1)) \quad ,$$

where the first map is the composition of the maps in the upper horizontal line in the diagram above. Therefore, to conclude the proof, since the map

$$H^{*}(\mathfrak{g}, \mathfrak{k}; \mathbb{R}) = H^{*}(\mathfrak{gl}_N(\mathbb{C}), \mathfrak{u}_N; \mathbb{R}) \to H^{*}(\mathfrak{gl}_N(\mathbb{C}), \mathbb{R}) = H^{*}(\mathfrak{g}_{/\mathbb{R}}, \mathbb{R})$$

is injective ([5] 5.2.), it suffices to show that the image $\tilde{b}_{2n-1}$ of v_{2n-1} under $\pi_{n-1} : H^{2n-1}(\mathfrak{g}) \to H^{2n-1}(\mathfrak{g}_{/\mathbb{R}}, \mathbb{R}(n-1))$ coincides with the image of the Borel regulator element b_{2n-1} under the injection $H^{2n-1}(\mathfrak{g}, \mathfrak{k}; \mathbb{R}(n-1)) \to H^{2n-1}(\mathfrak{g}, \mathbb{R}(n-1))$.

The first map is the composition of the map

$$\phi : H^{*}(\mathfrak{g}) \longrightarrow H^{*}(\mathfrak{g}_{/\mathbb{R}}; \mathbb{C}) \quad ,$$

which comes about by considering a $\mathbb{C}$-multilinear form on $\mathfrak{g}$ as an $\mathbb{R}$-multilinear form, and the projection π_{n-1} on the coefficients. In terms of the isomorphism (1.7.)

$$\mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{g} \oplus \mathfrak{g}$$

and identifying $H^*(\mathfrak{g}_{/\mathbb{R}}; \mathbb{C})$ with $H^*(\mathfrak{g}_{\mathbb{C}})$, the map ϕ is induced by the projection on the first factor, i.e., under the Künneth isomorphism

$$H^*(\mathfrak{g}_{\mathbb{C}}) \simeq H^*(\mathfrak{g} \oplus \mathfrak{g}) \cong H^*(\mathfrak{g}) \otimes H^*(\mathfrak{g})$$

the map ϕ sends x to $x \otimes 1$. We were unable to show the equality of b_{2n-1} and $\tilde{b}_{2n-1}$, but less is required to prove 4.2.. Indeed, the image of the Hurewicz map $\pi_{2n-1}(BGL_N(\mathbb{C})^+) \otimes \mathbb{R} \to H_{2n-1}(GL_N(\mathbb{C}), \mathbb{R})$ is contained in the primitive subspace (compare [18]) which is dual to the factor space of indecomposable elements in the cohomology. It therefore remains to show that the images of b_{2n-1} and $\tilde{b}_{2n-1}$ in the indecomposable quotient coincide.

On $H^*(\mathfrak{g}_{\mathbb{C}})$ there is a canonical rational structure induced from the Künneth decomposition. The algebra map induced from the diagonal $\mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$,

$$d : H^*(\mathfrak{g}_{\mathbb{C}}) \longrightarrow H^*(\mathfrak{g}) \quad ,$$

is defined over $\mathbb{Q}$ and is surjective. Furthermore, there is an exact sequence of spaces of indecomposables in (relative) Lie algebra cohomology (compare [5], 5.2.).

$$0 \to P^*(\mathfrak{g}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}}) \to P^*(\mathfrak{g}_{\mathbb{C}}) \xrightarrow{d} P^*(\mathfrak{g}) \to 0 \quad .$$

In degree $2n-1$ the middle space has dimension 2 and the space on the right dimension 1. Our claim therefore follows from the following lemma.

4.3. Lemma:

The images of b_{2n-1} and $\tilde{b}_{2n-1}$ in $P^*(\mathfrak{g}_{\mathbb{C}})$ are both rational, non-zero, and lie in the kernel of d.

Proof: We first consider b_{2n-1}. Applying the comparison isomorphism, its image in $P^*(\mathfrak{g}_{\mathbb{C}})$ comes from the composition of maps

$$\begin{array}{ccc} H^*_{Betti}(G_u/K, \mathbb{C}) & \hookrightarrow & H^*_{Betti}(G_u, \mathbb{C}) \\ \uparrow \wr & & \\ H^*_{Betti}(K, \mathbb{C}) & & \end{array}$$

Since these are induced from continuous maps they preserve rational cohomology. Therefore the rationality of b_{2n-1} follows just as in 2.14. from [5], 4.3. The other assertions about b_{2n-1} are trivially true.

We now consider $\tilde{b}_{2n-1}$. This also gives another description of the Borel regulator. The involution on $H^*(\mathfrak{g}_{/\mathbb{R}}, \mathbb{C})$ induced by complex conjugation on the coefficient system corresponds to the involution on $H^*(\mathfrak{g}_{\mathbb{C}})$ which is induced by the involution on the complex $\Lambda^* \mathfrak{g}'_{\mathbb{C}}$ which sends a multilinear form f on $\mathfrak{g}_{\mathbb{C}}$ to the multi-linear form cf with

$$(cf)(X_1, ..., X_p) = \overline{f(\iota(X_1), ..., \iota(X_p))} \quad ,$$

where $\iota : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ is induced from $X \otimes \lambda \to X \otimes \bar{\lambda}$, i.e., in terms of the isomorphism (1.7.),

$$(X_1, X_2) \longrightarrow (-{}^t\bar{X}_2, -{}^t\bar{X}_1) \quad .$$

Let $\tau : X \to -{}^tX$ be the canonical outer automorphism of $\mathfrak{g}$. It is defined over $\mathbb{Q}$. Since on $\mathbf{k}$ we have $\bar{X} = \tau(X)$, we conclude from (1.8.) and (2.14.) that

$$\tau(v_{2n-1}) = (-1)^n \cdot v_{2n-1} \quad .$$

Since v_{2n-1} is real we therefore obtain

$$c(v_{2n-1} \otimes 1) = 1 \otimes \tau(v_{2n-1}) = (-1)^n \cdot 1 \otimes v_{2n-1}$$

and hence, since $\pi_{n-1} = Id + (-1)^{n-1}c$,

$$\begin{aligned} \tilde{b}_{2n-1} &= (Id + (-1)^{n-1}c)(v_{2n-1} \otimes 1) \\ &= v_{2n-1} \otimes 1 - 1 \otimes v_{2n-1} \quad . \end{aligned} \tag{4.4.}$$

Now all required properties of $\tilde{b}_{2n-1}$ are obvious.

Bibliography

[1] A.Beilinson: Higher regulators and values of L functions; Sovremennye Problemy Matematiki, 24, VINITI, Moscow (1984), p. 181-238. (in Russian)

[2] S.Bloch: Higher regulators, algebraic K-theory and zeta functions of elliptic curves. Lecture notes U.C. Irvine (1978)

[3] A.Borel: Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts; Ann. Math. **57**, 1953, p. 115-207

[4] A.Borel: Stable real cohomology of arithmetic groups; Ann. Sci. ENS **7**, 1974, p. 235-272

[5] A.Borel: Cohomologie de SL_n et valeurs de fonctions zêta, Ann. Scuola Normale Superiore, **7**, 1974, p. 613-636

[6] A.Borel, N.Wallach: Continuous cohomology, discrete subgroups and representations of reductive groups; Ann. Math. Studies **94**, 1980

[7] N.Bourbaki: Algèbre; chap. 10, Algèbre homologique; Masson, Paris 1980

[8] H.Cartan: a) Notions d'algèbre différentielle, applications aux groupes de Lie...

b) La transgression dans un groupe de Lie et dans un espace fibré principal; Coll. de topologie algébrique, Bruxelles, 1950, p. 16-27, 57-71

[9] P.Deligne: Theorie de Hodge III, Pub. Math. IHES, **44**, 1974, p. 5-77

[10] A.Dold: Lectures on algebraic topology, Springer Verlag 1972

[11] A.Dold, D.Puppe: Homologie nicht-additiver Funktoren. Anwendungen. Ann. Inst. Fourier, Grenoble, **11**, 1961, p. 201-312

[12] S.Eilenberg, S.MacLane: On the groups $H(\Pi, n)$, I, II. Ann. of Math. **58**, 1953, p. 55-106; **60**, 1954, p. 49-139

[13] H.Esnault, E.Viehweg: Deligne-Beilinson cohomology. These proceedings.

[14] W.Greub, S.Halperin, R.Vanstone: Connections, curvature and cohomology, vol. III. Academic Press, New York, 1976

[15] B.Gross: On the values of Artin L-functions. Unpublished manuscript. 1979

[16] L.Illusie: Complexe cotangent et déformations I, II. LNM 239, 283, Springer Verlag 1971, 1972

[17] J.Neukirch: The Beilinson conjecture for algebraic number fields. These proceedings.

[18] P.Schneider: Introduction to the Beilinson conjectures. These proceedings.

[19] C.Soulé: Régulateurs. Sem. Bourbaki No. 644, 1984/85

[20] T.Zink: Cartiertheorie kommutativer formaler Gruppen. Teubner-Texte zur Mathematik 68. Leipzig 1984.