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$\neg$CH and (*)

Let T be a tree of size $\aleph_1$.

<u>Claim</u> 1. Assume $\neg$CH. Then T has no new cofinal branch in $V^{Col(\omega_1, 2^{\aleph_1})}$.

Proof. Otherwise we may let $p \in Col(\omega_1, 2^{\aleph_1})$ force that $\dot\tau$ is a new cofinal branch. We may easily construct $(p_s, t_s : s \in {}^{<\omega}2)$ s.t. $p \geq p_s \geq p_t$ for all $s \subset t$, $t_s <_T t_t$ for $s \subset t$, $p_s \Vdash \check{t}_s \in \dot\tau$, and $t_{s^\frown 0} \perp_T t_{s^\frown 1}$ for all s.

As $p \Vdash \dot\tau$ is cofinal and $Col(\omega_1, 2^{\aleph_1})$ is ω-closed, for each $x \in {}^{\omega}2$ we may pick p_x and t_x s.t. $p_x \leq p_{x \restriction n}$, $t_x >_T t_{x\restriction n}$, $p_x \Vdash \check{t}_x \in \dot\tau$ for all $n < \omega$. But then $\{t_x : x \in {}^{\omega}2\}$ is an antichain in T of size $2^{\aleph_0}$, contradicting $\neg$CH and the fact that T has size $\aleph_1$.

$\dashv$ (Claim 1)

Let us now suppose that T has at most

$\aleph_1$ cofinal branches, $(b_i : i < \omega_1)$, to begin with. Let $t_i \in b_i \setminus \bigcup \{b_j : j < i\}$. Pick $t_i^* \geq_T t_i$ s.t. $t_i^* \in b_i \setminus \bigcup \{b_j : t_j <_T t_i\}$.

Both $i \mapsto t_i$ and $i \mapsto t_i^*$ are injective.

<u>Claim 2.</u> If $t_j^* <_T t_i^*$, then $t_i^* \notin b_j$.

Proof. Say $t_i^* \in b_j$, $j \neq i$. As $t_i^* \geq_T t_i$, we must have $j > i$ by the choice of t_i. But now $t_j \leq_T t_j^* < t_i^*$ contradicts the choice of t_i^*.

$\dashv$ (Claim 2)

Let $\overline{T} = \{t \in T : \forall i < \omega_1 \, (t \in b_i \rightarrow t \leq_T t_i^*)\}$.

<u>Claim 3.</u> $\overline{T}$ doesn't have any cofinal branches.

Proof. Let b_i be a cofinal branch thru $\overline{T}$. For all $t \in \overline{T} \cap b_i$, $t \leq_T t_i^*$. Contradiction.

$\dashv$ (Claim 3)

Claim 4. There is a c.c.c. forcing $\mathbb{P}$ which adds some $f : \overline{T} \to \omega$ s.t. $s <_{\overline{T}} t \Rightarrow f(s) \neq f(t)$.

Corollary. In the extension by $\mathbb{P}$ there is some $g : T \to \omega$ s.t. if $s <_T t, t'$ and $g(s) = g(t) = g(t')$, then $t \leq_T t'$ or $t' \leq_T t$.

Proof of the Corollary from Claim 4. If $t \in T \setminus \overline{T}$, there is then some cofinal b_i with $t \in b_i$ and $t_i^* < t$. By Claim 2, this b_i is unique.

Define $g(t) = \begin{cases} f(t) & \text{if } t \in \overline{T} \\ f(t_i^*) & \text{if } t \in T \setminus \overline{T},\ t \in b_i,\ t_i^* < t \end{cases}$

Assume $s <_T t, t'$, $g(s) = g(t) = g(t')$. We must then have $t, t' \in T \setminus \overline{T}$. Let b_i be s.t. $t \in b_i$, $t_i^* < t$, and let b_j be s.t. $t' \in b_j$, $t_j^* < t'$.

If t, t' are incomparable in T, then $i \neq j$, so that by $g(t) = f(t_i^*) = f(t_j^*) = g(t')$ we must have that t_i^*, t_j^* are incomparable elements of $\overline{T}$.
But then $s < t_i^*, t_j^*$ even though $f(s) = f(t_i^*) = f(t_j^*)$. Contradiction!
$\dashv$ (Cor.)

Proof of Claim 4: We let $\mathbb{P}$ be the set of all finite $p : \overline{T} \rightarrowtail \omega$ s.t. if $s, t \in \mathrm{dom}(p)$, $s <_{\overline{T}} t$ or $t <_{\overline{T}} s$, then $p(s) \neq p(t)$, ordered by $p \leq q$ iff $p \supset q$. It suffices to verify that $\mathbb{P}$ has the c.c.c.

Suppose that $A \subset \mathbb{P}$ is an antichain of size $\aleph_1$. By the Δ-lemma we may then find some $B \subset A$ of size $\aleph_1$ s.t. there is a finite "root" $r \subset \overline{T}$ and some $n < \omega$ s.t.

(a) $\mathrm{dom}(p) \cap \mathrm{dom}(q) = r$ for $p, q \in B$, $p \neq q$,

(b) $\underline{p \restriction r = q \restriction r}$, ~~and~~ for $p, q \in B$, and

(c) $\overline{\mathrm{dom}(p) \setminus r} = n$ for $p \in B$.

Write $B = (p_i : i < \omega_1)$. Let $U \subset \mathcal{P}(\omega_1)$ be a uniform ultrafilter on ω_1.

<u>Subclaim.</u> There are $i \neq j$ such that all elements of $\mathrm{dom}(p_i) \setminus r$ are $\overline{T}$-incompatible with all elements of $\mathrm{dom}(p_j) \setminus r$.

The subclaim shows that B, hence A, is not an antichain after all, giving a contradiction.

Proof of the subclaim.

Write $\mathrm{dom}(p_i)\setminus r = \{t^i_0, \dots, t^i_{n-1}\}$. For $j<\omega_1$ and $k, l < n$ write

$$Y^{j,l}_k = \{ i < \omega : t^j_l \leq_T t^i_k \vee t^i_k \leq_T t^j_l \}.$$

Let us assume that the subclaim is false. Then for every $j<\omega_1$,

$$\bigcup_{l=0}^{n} \bigcup_{k=0}^{n} Y^{j,l}_k = \omega_1,$$

so that we may pick ~~[illegible]~~ $l = l_j$ and $k = k_j$ with $Y^{j,l}_k \in \mathcal{U}$.

There is some $k<n$ s.t. $X = \{ j<\omega_1 : k_j = k\}$ is uncountable (in fact in $\mathcal{U}$).

We claim that $\{ t^j_{l_j} : j \in X\}$ consists of pairwise $\bar{T}$-comparable nodes in $\bar{T}$, so that this set gives rise to a cofinal branch thru $\bar{T}$ which produces the desired contradiction.

Let $j, j' \in X$. Let $i \in Y^{j,l_j}_k \cap Y^{j',l_{j'}}_k \in \mathcal{U}$ be sufficiently large. We then have that

t^i_k is $\bar{T}$-comparable with both $t^j_{l_j}$ and $t^{j'}_{l_{j'}}$,

so that as i is sufficiently large,

$t^j_{l_j} <_{\bar{T}} t^i_k$ as well as $t^{j'}_{l_{j'}} <_{\bar{T}} t^i_k$; in

particular, $t^j_{l_j}, t^{j'}_{l_{j'}}$ are indeed $\bar{T}$-comparable.

$\dashv$ (Claim 4)

We produced:

Theorem. (Baumgartner) MA for σ-closed $*$ c.c.c. forcings of size $2^{\aleph_1}$ proves that every tree T of size $\aleph_1$ is weakly special in the sense that there is some $f: T \to \omega$ s.t. if $s <_T t, t'$ and $f(s) = f(t) = f(t')$, then $t \leq_T t'$ or $t' \leq_T t$.

We aim to use this to ~~pr~~ show the following result:

Theorem (Woodin) Assume $\neg CH$ and there is some $g \subset \mathbb{P}_{max}$ which is $L(\mathbb{R})$-generic. Then $(*)$ holds true.

Proof. We need to see that $\mathcal{P}(\omega_1) \subset L(\mathbb{R})[g]$.

We have that $HC \subset L(\mathbb{R})[g] \subset V$.

Let $h \in L(\mathbb{R})[g]$ be such that

$$h : \omega_2^{L(\mathbb{R})[g]} \longrightarrow {}^{<\omega_1}2$$

is onto. By $\omega_2^{L(\mathbb{R})[g]} \leq \omega_2$ and $\neg CH$, we must have $\omega_2^{L(\mathbb{R})[g]} = \omega_2$.

Let us fix some $\chi : \omega_1 \longrightarrow 2$. Pick $\eta < \omega_2$ s.t. $\chi \restriction \xi \in h''\eta$ for all $\xi < \omega_1$. Let

$$T = \{ t \in {}^{<\omega_1}2 : \exists i < \eta \ t \subset h(i) \}.$$

So T is a tree of size $\aleph_1$.

<u>Claim 5</u>. The conclusion of Baumgartner's theorem holds in $L(\mathbb{R})[g]$.

Assuming that Claim 5 holds true, let $f : T \longrightarrow \omega$, $f \in L(\mathbb{R})[g]$, be as in the conclusion of Baumgartner's theorem. We have that $(\chi \restriction \xi : \xi < \omega_1)$ is a cofinal branch thru T, so that there is some $\xi < \omega_1$ and some $n < \omega$ such that for all $t \in {}^{<\omega_1}2$,

$t = \chi \upharpoonright \xi'$ for some $\xi' < \omega_1$ iff

there is some $t' >_T \chi \upharpoonright \xi$ with $f(t') = n = f(\chi \upharpoonright \xi)$.

But then $(\chi \upharpoonright \xi : \xi < \omega_1) \in L(\mathbb{R})[g]$, so that

$\chi \in L(\mathbb{R})[g]$.

It remains to show Claim 5. It obviously suffices to prove the following.

<u>Claim</u> 6. Let φ be a Σ_1 formula. Assume ZFC proves that for all $T \in H_{\omega_2}$ there is some stationary set preserving forcing $\mathbb{Q} = \mathbb{Q}_T$ s.t. $V^{\mathbb{Q}} \models \varphi(T)$. Then $L(\mathbb{R})[g] \models \forall T \in H_{\omega_2}\ \varphi(T)$.

Claim 6 formulates a form of Π_2 maximality for $\mathbb{P}_{max}$. Claim 6 implies Claim 5, as the proof of Baumgartner's theorem shows that (in ZFC) for every tree of size $\aleph_1$ there is a σ-closed * c.c.c. (hence stationary set preserving) forcing which weakly specializes T.

Proof of Claim 6. Suppose that $T \in H_{\omega_2}$, $T = \dot{T}^g$, $p \in g$ is such that $p \Vdash \neg\varphi(\dot{T})$, and $t \in p$ is such that t gets mapped to T by the generic iteration of p of length ω_1+1 as being given by g.

It suffices to find $q <_{\mathbb{P}_{max}} p$ with $q \models \varphi(i(t))$, where $i : p \to p^*$, $i \in q$ witnesses $q <_{\mathbb{P}_{max}} p$.

Let $x \in \mathbb{R}$ code p, and let $N = M_1^\#(x)$. Let us use the bottom measurable ~~Woodin~~ of N to force the existence of a precipitous ideal I on ω_1, let $N[h]$ denote the generic extension. Let $i : p \to p^*$ witness $N[h] <_{\mathbb{P}_{max}} p$. By our hypothesis, there is some $\mathbb{Q} \in N[h]$ which forces $\varphi(i(t))$; we may assume $\mathbb{Q}$ is smaller than the Woodin of $N[h]$, so that we may use this Woodin to produce an precipitous ideal I^* on ω_1 which is compatible with I. Let $N[h][h_1][h_2]$ denote the final extension. Then $N[h][h_1][h_2]$ $<_{\mathbb{P}_{max}} p$ and $N[h][h_1][h_2] \models \varphi(i(t))$. $\dashv$