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On a question of Xianghui Shi

For $A \subset \aleph_\omega$ let $\alpha(A)$ be the least $\alpha > \aleph_\omega$ such that $L_\alpha(H_{\aleph_\omega}, A) \models ZF^-$ *); we call $L_{\alpha(A)}(H_{\aleph_\omega}, A)$ the A-<u>model</u>, and denote it by M_A. For $A, B \subset \aleph_\omega$, we write $A \leq B$ iff $A \in M_B$ (iff $M_A \subset M_B$). X. Shi asked if $\leq$ is a prewellordering on $\mathcal{P}(\aleph_\omega)$ inside $L[E]$, $L[E]$ being a pure extender model.

We show:

<u>Theorem</u>. Assume that no transitive model of ZF^- has an inner model with a Woodin cardinal. Let $L[E]$ be a fully iterable pure extender model. Then $L[E] \models$ "$\leq$ is a prewellordering on $\mathcal{P}(\aleph_\omega)$."

Proof. We first verify the following.

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*) By ZF^-, we mean a sufficiently rich fragment of ZF which allows us to prove the relevant facts about the core model.

<u>Claim.</u> Let M be a transitive model of ZF^- with $H_{\aleph_\omega} = K^M | \aleph_\omega \in M$. Then $M \models$ "$\mathcal{P}(\aleph_\omega) \subset K$."

To see this, fix $A \in \mathcal{P}(\aleph_\omega) \cap M$. Let $e : \aleph_\omega \to K^M | \aleph_\omega$ be bijective and definable over $K^M | \aleph_\omega$. Then $(e^{-1}(A \cap \aleph_n) : n < \omega) \in {}^\omega(\aleph_\omega) \cap M$. By Covering inside M, there is some $D \in K^M$, $D \in [\aleph_\omega]^{<\aleph_\omega}$, $D \supset \{e^{-1}(A \cap \aleph_n) : n < \omega\}$. Let $f : \gamma \to D$ be the Mostowski collapse of D (so that $f \in K^M$ and $\gamma < \aleph_\omega$). As $H_{\aleph_\omega} = K^M | \aleph_\omega \in M$, $f^{-1}{}'' \{e^{-1}(A \cap \aleph_n) : n < \omega\} \in K^M$, so that $\{e^{-1}(A \cap \aleph_n) : n < \omega\} = f''(f^{-1}{}''\{e^{-1}(A \cap \aleph_n) : n < \omega\}) \in K^M$ which implies $\{A \cap \aleph_n : n < \omega\} \in K^M$ by $e \in K^M$ and hence $A = \bigcup \{A \cap \aleph_n : n < \omega\} \in K^M$.

The Claim implies that inside $L[E]$, if $A \in \mathcal{P}(\aleph_\omega)$, then $A \in K^{M_A} | \aleph_\omega^{+K^{M_A}}$. By Weak Covering, $\aleph_\omega^{+K^{M_A}} = \aleph_\omega^{+M_A}$.

Let us now work in $L[E]$. For $A \subset \aleph_\omega$ write $\beta(A) = \aleph_\omega^{+M_A}$. By absoluteness, $K^{M_A}|\beta(A)$ is iterable in $L[E]$ (see e.g. Lemma 2.1 of [Cl-Sch]) which implies that $K^{M_A}|\beta(A) = L[E]|\beta(A)$ by Lemma 3.5 of [Gi-Sch-Sh].

Now let $A, B \subset \aleph_\omega$. If $\beta(A) \leq \beta(B)$, then $A \in K^{M_A}|\beta(A)$ (by the Claim)

$$= L[E]|\beta(A) = K^{M_B}|\beta(B) \subset M_B, \text{ i.e.,}$$

$A \in M_B$. So $A \leq B$ iff $\beta(A) \leq \beta(B)$ and $\leq$ is a prewellordering. $\dashv$

[Gi-Sch-Sh] M. Gitik, R. Schindler, S. Shelah, "Pcf theory and Woodin cardinals."

[Cl-Sch] B. Claverie, R. Schindler, "Woodin's axiom (*), bounded forcing axioms, and precipitous ideals on ω_1."

Let us discuss a generalization of the above theorem.

Let κ be a singular strong limit cardinal. For $A \subset \kappa$ let $\alpha(A)$ be the least α such that $\alpha > \kappa$ and $L_\alpha(H_\kappa, A) \models ZF^-$; we again call $L_{\alpha(A)}(H_\kappa, A)$ the A-<u>model</u>, and denote it by M_A. For $A, B \subset \kappa$, we write $A \leq B$ iff $A \in M_B$ (iff $M_A \subset M_B$).

We say that $\leq$ is a prewellordering on a cone iff there is some $A_0 \subset \kappa$ such that $\leq \restriction \{A \subset \kappa : A_0 \leq A\}$ is a prewellordering on $\{A \subset \kappa : A_0 \leq A\}$.

<u>Theorem.</u> Assume that no transitive model of ZF^- has an inner model with a Woodin cardinal. Let $L[E]$ be a fully iterable pure extender model. Let κ be a singular cardinal of $L[E]$. Then $L[E] \models$ "$\leq$ is a prewellordering on a cone."

which is not a limit of measurable cardinals

Proof. Let $\gamma_0 \in (\kappa, \kappa^{+L[E]})$ be such that $\rho_\omega(L[E] \| \gamma_0) = \kappa$ and $L[E] \| \gamma_0 \models$ "κ is singular." Let $h : \kappa \longrightarrow L[E] \| \gamma_0$ be a bijection which is definable over $L[E] \| \gamma_0$. h induces some $A_0 \subset \kappa$ which codes $L[E] \| \gamma_0$.

In particular $L[E] \| \gamma_0 \in M_A$ for all $A \subset \kappa$, $A_0 \leq A$. But then inside $L[E]$,

$$(*) \qquad L[E] \| \gamma_0 \lhd K^{M_A} \quad \text{for all } A \subset \kappa,\ A_0 \leq A$$

by Lemma 2.1 of [Cl-Sch] and Lemma 3.5 of [Gi-Sch-Sh].

In particular, $K^{M_A} \models$ "κ is singular," which implies, as κ is not a limit of measurables, that inside M_A every $X \in [\kappa]^{<\kappa}$ may be covered by an element of $[\kappa]^{<\kappa} \cap K^{M_A}$ by the proof from [Mi-Sch]. We may then prove in much the same way as before:

<u>Claim</u>. Inside $L[E]$, if $A \subset \kappa$, $A_0 \leq A$, then $M_A \models$ "$\mathcal{P}(\kappa) \subset K^{M_A}$."

The rest is then as before.

[Mi-Sch] Mitchell-Schimmerling, "Covering at limit cardinals."

Let us now construct an example of an $L[E]$ which has a singular cardinal κ and $A, B \subset \kappa$ with $A \not\leq B$ and $B \not\leq A$.

Let us fix $M = L[E]$, fully iterable, and some κ such that

(a) κ is ~~regular~~ inaccessible in M, and

there are $\nu < \mu$ such that

(b) E^M_ν has critical point κ and κ as its only generator,

(c) E^M_ν is total on $M|\mu$, and

(d) E^M_μ has critical point $> \nu$ and $\text{crit}(E^M_\mu)$ as its only generator, and $\rho_1(M||\mu) > \kappa$.

E.g., work with some $L[E]$ which has two measurable cardinals, $\kappa < \text{crit}(E^M_\mu)$.

Let $Q \overset{\sigma}{\cong} X = \text{Hull}^{M||\mu}_{\Sigma_1}(\kappa) \prec_{\Sigma_1} M||\mu$, where Q is transitive. Then $Q \in M|\mu$ by (d), $\rho_1(Q) = \kappa$, $\kappa^{+Q} < \kappa^{+M||\mu}$, ~~$\rho_1$~~ and in fact $Q = M||\bar\mu$, where $\bar\mu$ is least such that

$\kappa^{+Q} < \bar{\mu} < \kappa^{+M\|\mu}$ and $\rho_\omega(M\|\bar{\mu}) \leq \kappa$.

We may and shall assume w.l.o.g. that $\nu \in X$. Let $\bar{\nu} = \sigma^{-1}(\nu)$.

Let $\bar{U} = \{X \in Q \cap \mathcal{P}(\kappa) : \{\{\xi\} : \xi \in X\} \in (E^M_{\bar{\nu}})_{\{\kappa\}}\}$ be the normal measure component of $E^M_{\bar{\nu}}$, and let $U = \{X \in M|\nu \cap \mathcal{P}(\kappa) : \{\{\xi\} : \xi \in X\} \in (E^M_\nu)_{\{\kappa\}}\}$ be the normal measure component of E^M_ν.

By $\sigma : Q \longrightarrow_{\Sigma_1} M\|\mu$, $\sigma(E^M_{\bar{\nu}}) = E^M_\nu$, we have that $\bar{U} \subset U$; also $M\|\mu \models$ "$\mathrm{Card}(\bar{U}) = \kappa$." Furthermore, $E^M_{\bar{\nu}}$ is total on $M|\bar{\mu}$.

Now let $Y \prec_{\Sigma_\omega} M\|\mu$ be countable, $Y \in M$, and $\{\bar{\nu}, \bar{\mu}, \nu\} \subset Y$. Let $Y^+ = \mathrm{Hull}^{M\|\mu}_{\Sigma_\omega}(Y \cup \gamma)$, where $\gamma = \sup(Y \cap \kappa)$.

By (a), if $\bar{\gamma} \in Y \cap \kappa$ and $s \in [Y]^{<\omega}$, then for each Skolem term τ,

$\sup\{\tau^{M\|\mu}(\xi, \bar{\gamma}, s) : \xi < \bar{\gamma}\} \in Y \cap \kappa$;

this implies that $\gamma \leq \sup(Y^{+} \cap \kappa) = \sup(Y \cap \kappa) = \gamma$, so that $\gamma = Y^{+} \cap \kappa$.

Let $P \overset{\sigma^*}{\cong} Y^{+} \prec_{\Sigma_\omega} M\|\mu$. Then $\mathrm{crit}(\sigma^*) = \gamma$, $\sigma^*(\gamma) = \kappa$, and γ is a (limit of) cardinal(s) in $M\|\mu$ (equivalently, in P), by (a), and so we may conclude that $H_\gamma^P = H_\gamma^M$. Also, $P \trianglelefteq M$.

Let $\bar{\nu}_0, \bar{\mu}_0, \nu_0 = \sigma^{*-1}(\bar{\nu}, \bar{\mu}, \nu)$ and $\mu_0 = P \cap OR$.

If $\bar{U}_0 \,/\, U_0$ is the normal measure component of $E^M_{\bar{\nu}_0} \,/\, E^M_{\nu_0}$, resp., then $\bar{U}_0 \subset U_0$; moreover, $M\|\mu_0 \models$ "$\mathrm{Card}(\bar{U}_0) = \gamma$," and $\bar{U}_0$ is total on $M|\bar{\mu}_0$.

Let $(X_i : i<\gamma) \in M|\mu_0$ be such that $\bar{U}_0 = \{X_i : i<\gamma\}$, and let $X = \underset{i<\gamma}{\Delta} X_i$. By $\bar{U}_0 \subset U_0$ and the normality of U_0, $X \in U_0$ and $X \setminus i+1 \subset X_i$ for all $i<\gamma$.

As $\gamma = \sup(Y \cap \kappa)$, $\bar{\bar{Y}} = \aleph_0$, γ has cofinality

ω (in M). Let $(\alpha_n : n<\omega)$ be strictly increasing and cofinal in γ. Let $\kappa_n = \min(X \setminus \alpha_n)$, $n<\omega$. If $\alpha_n > i$, then $\kappa_n \in X_i$. In other words, for every $Y \in \bar{U}_0$, a tail end of $\{\kappa_n : n<\omega\}$ is contained in Y.

Now, as $\bar{U}_0$ is total on $M|\bar{\mu}_0$ and $E^M_{\bar{\mu}_0}$ has critical point $> \bar{\nu}_0$, $\bar{U}_0$ is a total measure in $L[M\|\bar{\nu}_0]$. But then $(\kappa_n : n<\omega)$ is Prikry generic over $L[M\|\bar{\nu}_0]$ w.r.t. the measure $\cancel{M}$. $\bar{U}_0 \approx E_{\bar{\nu}_0}$. Recall $C \in M$.

Let $C_0 = \{\kappa_{2n} : n<\omega\}$ and $C_1 = \{\kappa_{2n+1} : n<\omega\}$.

Let α least such that $\alpha > \bar{\nu}_0$ and $L_\alpha[M\|\bar{\nu}_0] \models ZF^-$. Then $L_\alpha[M\|\bar{\nu}_0, C_0, C_1] \models ZF^-$, so that

$M_{C_0} \subset L_\alpha[M\|\bar{\nu}_0, C_0]$ and $M_{C_1} \subset L_\alpha[M\|\bar{\nu}_0, C_1]$.

But $C_0 \notin L_\alpha[M\|\bar{\nu}_0, C_1]$ and $C_1 \notin L_\alpha[M\|\bar{\nu}_0, C_0]$, so $C_0 \not\leq C_1$ and $C_1 \not\leq C_0$.

Again, for $A \subset OR$ let $\alpha(A)$ be the smallest $\alpha > \sup(A)$ s.t. $L_\alpha[A] \models ZF^-$ (where ZF^- is a suff. rich fragment of ZF), and let $M_A = L_{\alpha(A)}[A]$.

<u>Claim 1.</u> If $\kappa \geq \sup(A)$, $\kappa \in M_A$, then $cf^V(\kappa^{+M_A}) = \omega$.

Proof. Let $X = Hull_{\Sigma_\omega}^{M_A}(\emptyset) \prec M_A$, and let $Y = Hull_{\Sigma_\omega}^{M_A}(X \cup (\kappa+1))$. If τ is a Skolem term and $s \in [X]^{<\omega}$, then $\sup\{\tau^{M_A}(\vec{\xi}, s) : \vec{\xi} \in \kappa+1\} \cap \kappa^{+M_A} \in X \cap \kappa^{+M_A}$; hence $\sup(Y \cap \kappa^{+M_A}) = \sup(X \cap \kappa^{+M_A})$, so that $cf^V(Y \cap \kappa^{+M_A}) = \omega$. However, by the choice of M_A, $Y = M_A$, so that $cf^V(\kappa^{+M_A}) = \omega$. $\dashv$

Let us now assume that κ is a singular cardinal with $cf(\kappa) = \omega$ and that there is $(\kappa_n : n < \omega)$ increasing and cofinal in κ s.t. each κ_n is a measurable cardinal. Let $A \subset \kappa$ be such that $H_\kappa \cup \{(\kappa_n : n<\omega)\} \subset M_A$. Let U_n be a measure on κ_n, $n<\omega$.

<u>Claim 2.</u> In V there is some sequence $(\lambda_n : n<\omega)$ s.t. for all $(X_n : n<\omega) \in M_A$ with $X_n \in \mathcal{U}_n$ for all $n<\omega$, $\lambda_n \in X_n$ for all sufficiently big $n<\omega$.

Proof. Let $\eta_n \nearrow \kappa^{+M_A}$ (which has countable cofinality by Claim 1), and let $f_n : \kappa \to \eta_n$, $f_n \in M_A$, be bijective, $n<\omega$. Let $\bar{X}_{m,n} = f_n{}'' \kappa_m$, $n,m<\omega$, and write $\tilde{X}_n = \bigcup\{\bar{X}_{m,\bar{n}} : m, \bar{n} \le n\}$, $n<\omega$. Then $\tilde{X}_n \subset \tilde{X}_{n+1}$, each $\tilde{X}_n$ is in M_A and of size κ_n there, and $\kappa^{+M_A} = \bigcup\{\tilde{X}_n : n<\omega\}$.

Let $((X^i_n : n<\omega) : i<\kappa^{+M_A}) \in M_A$ be an enumeration of all $(X_n : n<\omega) \in M_A$ s.t. $X_n \in \mathcal{U}_n$ f.a. $n<\omega$. Let $Z_n = \{(X^i_n : n<\omega) : i \in \tilde{X}_n\}$. Let $Y_n = \bigcap\{X^i_n : i \in \tilde{X}_{n-1}\} \in \mathcal{U}_n$.

Let $\lambda_n \in Y_n$ for all $n<\omega$.

We claim that $(\lambda_n : n<\omega)$ satisfies the conclusion of Claim 2. To see this, let $(X_n : n<\omega) \in M_A$ be s.t. $X_n \in U_n$ for all n. There is some $n_0<\omega$ s.t. $(X_n : n<\omega) \in Z_m$ for all $m \geq n_0$. Then if $m > n_0$, $Y_m \subset X_m$, and hence $\lambda_m \in X_m$ for all $m > n_0$. $\dashv$

Claim 2 implies that $(\lambda_n : n<\omega)$ is $\mathbb{P}_{(U_n : n<\omega)}$-generic over M_A, where $\mathbb{P}_{(U_n : n<\omega)}$ denotes the diagonal Prikry forcing using the measures U_n, $n<\omega$.

But then also $((\lambda_{2n} : n<\omega), (\lambda_{2n+1} : n<\omega))$ is $\mathbb{P}_{(U_{2n} : n<\omega)} \times \mathbb{P}_{(U_{2n+1} : n<\omega)}$-generic over M_A and $M_A[(\lambda_{2n} : n<\omega)] = M_{A,(\lambda_{2n} : n<\omega)}$ and $M_A[(\lambda_{2n+1} : n<\omega)] = M_{A,(\lambda_{2n+1} : n<\omega)}$. Hence $(\lambda_{2n+1} : n<\omega) \notin M_{A,(\lambda_{2n} : n<\omega)}$ and $(\lambda_{2n} : n<\omega) \notin M_{A,(\lambda_{2n+1} : n<\omega)}$, so that $A \oplus (\lambda_{2n} : n<\omega)$ and $A \oplus (\lambda_{2n+1} : n<\omega)$ are $\leq$-incomparable.

We now show an optimal version of the theorem on p. 4.

<u>Thm.</u> Assume that no transitive model of ZF$^-$ has an inner model with a Woodin cardinal. Let $L[E]$ be a fully iterable pure extender model. Let κ be a singular cardinal of $L[E]$ which is not ~~as~~ equal to $\sup(\kappa_n : n<\omega)$, where each κ_n is measurable. Then $L[E] \models$ "$\leq$ is a prewellordering on a cone."

Proof. Let $A \subset \kappa$ be s.t. $L[E]|\kappa \in M_A$ and $M_A \models$ "κ is singular of cofinality $cf^V(\kappa)$".
By hypothesis, $cf^V(\kappa) > \omega$ or else $\{\mu<\kappa : \mu$ is measurable in $L[E]\}$ is bounded below κ (here, $V = L[E]$). To verify the theorem, it suffices to prove that $\mathcal{P}(\kappa) \cap M_A \subset K^{M_A}$, for which in turn it suffices to prove that if $X \in [\kappa]^{cf(\kappa)} \cap M_A$, then there is some

$Y \in K^{M_A}$, $\bar{\bar{Y}} < \kappa$, $Y \supset X$.

Let us work in M_A. Fix $X \in [\kappa]^{\varphi(\kappa)}$.

Let $\pi : H \longrightarrow M_A | \theta$, some "big" θ, where H is transitive, ${}^{\omega}H \subset H$, $\mathrm{ran}(\pi) \supset X$, and $\mathrm{Card}(H) = \bar{\bar{X}}^{\aleph_0} < \kappa$.

Write $\bar{K} = \pi^{-1}(L[E]|\kappa)$. There is a tree $\mathcal{T}$ on K^{M_A} s.t. $\mathcal{M}_\infty^{\mathcal{T}} \trianglerighteq \bar{K}$.

If the generators of the extenders used on the main branch of $\mathcal{T}$ are bounded in $\bar{\kappa} = \pi^{-1}(\kappa)$, then we get the desired covering set. So let us suppose otherwise.

Let $\alpha \in [0, \infty)_{\mathcal{T}}$ be s.t. $\bar{\kappa} \in \mathrm{ran}(\pi^{\mathcal{T}}_{\alpha\infty})$ (and no drop). Write $\kappa' = \pi^{\mathcal{T}\,-1}_{\alpha\infty}(\bar{\kappa})$. If $\beta \in [0, \infty)_{\mathcal{T}} \setminus \alpha$, then $\mathrm{crit}(\pi^{\mathcal{T}}_{\beta\infty}) \leq \pi^{\mathcal{T}}_{\alpha\beta}(\kappa')$.

If we always have $\mathrm{crit}(\pi^{\mathcal{T}}_{\beta\infty}) < \pi^{\mathcal{T}}_{\alpha\beta}(\kappa')$, then for each such β there must be some $\gamma > \beta$ on the main branch with $\mathrm{crit}(\pi^{\mathcal{T}}_{\gamma\infty}) > \pi^{\mathcal{T}}_{\beta\gamma}(\mathrm{crit}(\pi_{\beta\infty}))$; but then $\bar{\kappa}$ is a limit of measurables in $\overline{K}$, hence (by elementarity) κ is in $L[E]|\kappa$.

Let $\{\beta_n : n<\omega\}$ on the main branch be such that $crit(\pi_{\beta_{n+1}\infty}) > \pi_{\beta_n\beta_{n+1}}(crit(\pi_{\beta_n\infty}))$, and $\sup \beta_n < \infty$. Write $\lambda_n = crit(\pi_{\beta_n\infty})$. Then $(\lambda_n : n<\omega) \in \overline{K}$ is diagonally Prikry generic over $\mathcal{M}_\infty^{\mathcal{I}}$, hence over $\overline{K}$, hence $(\pi(\lambda_n) : n<\omega) \in L[E]$ would be diagonally Prikry generic over $L[E]$ which is all nonsense. (If $(\pi(\lambda_n) : n<\omega) \in ran(\pi)$ didn't have this property, $ran(\pi)$ would contain a counterexample.)

Hence in this case the β_n's must be cofinal in the length of $[0,\infty)_{\mathcal{I}}$ and the λ_n's are cofinal in $\bar{\kappa}$, so that κ is a singular limit of measurables of cofinality ω.

Now suppose that $crit(\pi^{\mathcal{I}}_{\beta\infty}) = \pi^{\mathcal{I}}_{\alpha\beta}(\kappa')$ from some point on. Let $(\beta_n : n<\omega)$ enumerate the first ω such β, and again write $\lambda_n = \mathrm{crit}(\pi^{\mathcal{I}}_{\beta_n \infty})$. Again, $(\lambda_n : n<\omega)$ would be generic over $M^{\mathcal{I}}_\infty$, hence over $\bar{K}$, so that $(\pi(\lambda_n) : n<\omega) \in L[E]$ would be generic over K^{M_A} for one or the other version of Prikry forcing. This only makes sense if $(\pi(\lambda_n) : n<\omega)$ is cofinal in κ.

We may assume that on a tail end we use the measure of Mitchell order 0 on λ_n to form $\pi^{\mathcal{I}}_{\beta_n\, \mathcal{I}\text{-succ}(\beta_n)}$, as o.w. κ gets a limit of measurables again of cofinality ω. But then $(\lambda_n : n<\omega)$ is in fact Prikry generic for the order 0 measure on $\bar{\kappa}$ in $M^{\mathcal{I}}_\infty$ which gives as in [Mi-Sch] that κ is measurable in K^{M_A}. But we assumed that $K^{M_A} \models$ "κ is singular."