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THE DELFINO PROBLEM # 12

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Let PM :≡ every projective set of reals is Lebesgue measurable, PC :≡ every projective set of reals has the property of Baire.

Let PU :≡ every projective subset of $\mathbb{R} \times \mathbb{R}$ admits a projective uniformization; i.e. if

$A \subset \mathbb{R} \times \mathbb{R}$ is projective there is

$F \subset A$ also projective s.t.

$$\forall x \; (\exists y \; (x,y) \in A \longrightarrow \exists \text{ exactly one } y' \; (x,y') \in F) .$$

Let PD :≡ projective determinacy.

Classical: PD ⇒ PM, PC, PU.

Question: (Woodin, ca. 1981 ; Delfino Problem # 12)

Does PM ∧ PC ∧ PU ⟹ PD ?

Rmk.: was open for at least 16 years!

Thm. (Steel, 1997) No!

We'll also determine the consistency strength of PM $\wedge$ PC $\wedge$ PU ($\wedge$ Ω subtle).

In order to formulate the results:

Def. κ is α-strong iff for all $X \in H_{\alpha^+}$ there is $\pi : V \longrightarrow M$ with M an inner model, $X \in M$, and $\kappa = \text{c.p.}(\pi)$.

We say the #12 hypothesis holds iff there are cardinals κ_i, $i < \omega$, with $\kappa_i < \kappa_{i+1}$ and limit λ s.t.

$\forall i$ κ_i is λ-strong.

Thm. 1 (Steel, 1997) If there is a model of ZFC + the #12 hypothesis then there is one of ZFC + PM + PC + PU.

<u>Thm.</u> 2 (Schindler, 1997) If there is a model of ZFC + Ω subtle + PM + PC + PU then there is one of ZFC + Ω subtle + the # 12 hypothesis.

<u>Corollary</u>. Equiconsistent are:

(1) ZFC $\wedge$ Ω subtle $\wedge$ PM $\wedge$ PC $\wedge$ PU,

(2) ZFC $\wedge$ Ω subtle $\wedge$ # 12 hypothesis.

Rmk.: On the other hand, PD is much stronger, namely $\forall x \in \mathbb{R}\ \forall n < \omega\ \exists$ inner model containing x and having n Woodins.

Later, we'll also discuss problems inspired by the Delfino Problem # 12.

Proof of thm. 1 :

We'll construct a model of ZFC + PM + PC + PU + $\neg \Delta^1_2$-det. [Rmk.: $\underset{\sim}{\Pi}^1_1$-det. will necessarily hold by a thm. of Woodin.]
the model is essentially due to Woodin, who was able to prove that in it PM + PC hold (and much more).

Solovay's lemma. Let $A \subset \mathbb{R}$. Let M be a transitive model of ZFC, let $\phi(u, \vec{v})$ be a formula, and let $\vec{S} \in M$. Suppose that

$$x \in A \leftrightarrow M[x] \models \phi(x, \vec{S}).$$

(1) If $\bigcup \{ B :$ B is a Borel set with code in M, and B is null $\}$ is null, then A is Lebesgue measurable,

(2) if $\bigcup \{ B :$ B is a Borel set with code in M, and B is meager $\}$ is meager, then A has the property of Baire.

<u>Corollary</u> to Solovay's lemma: If $\mathbb{R} \cap M$ is ctble. then A is Lebesgue measurable and has the property of Baire.

Example: There is a tree S on $\omega \times \omega \times \omega_1$ s.t. $p[S] = \{ (x,y) : \exists f \, \forall n \, (x\restriction n, y \restriction n, f \restriction n) \in S \}$ is a universal $\underset{\sim}{\Sigma}^1_2$ set, i.e. for any $A \in \underset{\sim}{\Sigma}^1_2$ there is $y_A \in \mathbb{R}$ with $A = \{ x : (x, y_A) \in p[S] \}$.
Moreover, if S is constructed as usual, $S \in L$. S is called "the" Shoenfield tree.

We hence get, for example:
Suppose that $\omega_1^{L[y]} < \omega_1$ f.a. $y \in \mathbb{R}$. Then every $\underset{\sim}{\Sigma}^1_2$ set is Lebesgue measurable and has the property of Baire. [Given $A \in \underset{\sim}{\Sigma}^1_2$, set $M = L[y_A]$. Then $x \in A \iff$

$$(x, y_A) \in p[S] \iff$$

$$M[x] \models (x, y_A) \in p[S] \qquad (!).]$$

We aim to lift the argument from the example to higher levels of the projective hierarchy.

<u>Woodin's</u> <u>lemma</u>. Let κ be λ-strong. Let T, T' be trees s.t. $p[T] = {}^{\omega}\omega \setminus p[T']$ in every $V[G]$, G size $< \kappa$ generic / V. Let H be $Col(\omega, 2^{2^{\kappa}})$-generic / V. Then in $V[H]$ the following holds true:

There are U, U', trees, s.t. $p[U] = \exists^{\mathbb{R}} p[T]$ and $p[U] = {}^{\omega}\omega \setminus p[U']$ in every size $< \lambda^+$ generic extension.

<u>Corollary to Woodin's lemma</u>: Let $\kappa_0 < \kappa_1 < \ldots$ with limit λ witness the # 12 hypothesis. Let G be $Col(\omega, \lambda)$-gen. / V. Then in $V[G]$ PM and PC hold.

Prf. of Cor.: The Shoenfield tree on $\omega^2 \times \lambda^+$ can easily be used to get trees $S_1, T_1 \in L$ on $\omega^2 \times \lambda^+$ s.t. in all size $< \lambda^+$ extensions:

$$p[T_1] = \text{univ. } \underset{\sim}{\Pi}^1_1 \text{ set, and}$$

$$p[T_1] = ({}^\omega\omega)^2 \setminus p[S_1].$$

Working in $V[G]$, we may pick $(G_i : i<\omega)$ s.t. G_0 is $Col(\omega, 2^{2^{\aleph_0}})$-gen. / V, and G_{i+1} is $Col(\omega, 2^{2^{\kappa_{i+1}}})$-gen. / $V[G_0, \dots, G_i]$. Applying Woodin's lemma ω many times we get trees $S_i, T_i \in V[G_0, \dots, G_{i-2}]$ on $\omega^2 \times \lambda^+$ s.t. in all size $< \lambda^+$ extensions (of $V[G_0, \dots, G_{i-2}]$, in particular in $V[G]$):

$$p[T_i] = \text{univ. } \underset{\sim}{\Pi}^1_i \text{ set, and}$$

$$p[T_i] = ({}^\omega\omega)^2 \setminus p[S_i],$$

for $i \geq 2$. Now let $A \in \underset{\sim}{\Pi}^1_i$ in $V[G]$, $A \in \Pi^1_i(y)$, say. Let $\dot{y}$ be a name for y in V; w.l.o.g., $\dot{y} \in H_{\lambda^+}$. We may hence take an ultrapower of $V[G_0, \dots, G_{i-2}]$, using an extender E with c.p. κ_{i-1},

$$\pi : V[G_0, \ldots, G_{i-2}] \longrightarrow_E M[G_0, \ldots, G_{i-2}] = \tilde{M},$$

s.t. y is $\mathbb{P}$-gen. / $\tilde{M}$ for some $\mathbb{P}$ with $\tilde{M} \models \overline{\overline{\mathbb{P}}} < \lambda^+$.

Essentially, we may assume $x \in A \leftrightarrow (x,y) \in p[T_i]$.
Our result now follows from the next claim together with the Cor. to Solovay's lemma.

<u>Claim</u>: $x \in A \Leftrightarrow \tilde{M}[y][x] \models (x,y) \in p[\pi(T_i)]$.

[Notice $V[G]$ knows $\tilde{M}[y]$ has only ctbly. many reals!]

Pf.: Let $x \in A$, $\forall n \; (x \upharpoonright n, y \upharpoonright n, f \upharpoonright n) \in T_i$, say. Then $(x \upharpoonright n, y \upharpoonright n, \pi(f \upharpoonright n)) \in \pi(T_i) \; \forall n$, so $(x,y) \in p[\pi(T_i)]$.
On the other hand, supp. $x \notin A$. Then $(x,y) \in p[S_i]$, so $(x,y) \in p[\pi(S_i)]$ as above.
We have

$$V[G_0, \ldots, G_{i-2}] \models p[S_i] \cap p[T_i] = \emptyset$$

in all size $< \lambda^+$ extensions,

So using π,

$$\tilde{M} \models p[\pi(S_i)] \cap p[\pi(T_i)] = \emptyset$$

in all size $< \pi(\lambda^+)$ extensions.

However, by the absoluteness of well-foundedness, this implies

$$V[G] \models p[\pi(S_i)] \cap p[\pi(T_i)] = \emptyset.$$

In particular, $(x, y) \notin p[\pi(T_i)]$. $\dashv$

Proof of Woodin's lemma: Will be shown by amalgamating various $\mathcal{U}$'s working for particular forcings. Let $\overline{\overline{P}} < \lambda^+$ in $V[H]$, say $P \in H_{\lambda^+}$ there. Let

$$\pi : V \longrightarrow M$$

at κ s.t. still $P \in M[H]$.

We easily get $\mathcal{U}$ s.t. $p[\mathcal{U}] = \exists^{\mathbb{R}} p[T]$.

Moreover, by an argument as above, we'll have that $p[\mathcal{U}] = p[\pi(\mathcal{U})]$.

We now want to get, in $V[H]$, a tree $\mathcal{U}'$ s.t. for any K $\mathbb{P}$-gen. / $M[H]$,

$$M[H][K] \models p[\pi(\mathcal{U})] = {}^{\omega}\omega \setminus p[\mathcal{U}'] .$$

[As any real in any size $< \lambda^+$ extension of $V[H]$ is in some such $M[H][K]$ we can then just take the "union" of all $\mathcal{U}'$'s.]

We have the long extender E given by π, i.e.

$$X \in E_a \iff a \in \pi(X)$$

for $a \in [\pi(\kappa)]^{<\omega}$, $X \in \mathcal{P}([\kappa]^{\bar{\bar{a}}})$. Set

$\nu_a := \pi(E_a)$, being a measure on M.

Notice $(s, a) \in \pi(\mathcal{U}) \iff a \in \pi(\mathcal{U}_s) \iff$

$$\mathcal{U}_s \in E_a \iff \pi(\mathcal{U}_s) \in \nu_a .$$

In $V[H]$, enumerate the ν_a's as $(\sigma_i : i < \omega)$ s.t. every ν_a occurs inf. often. For any $i<\omega$ there is $\pi_i : M \longrightarrow_{\sigma_i} \mathrm{Ult}(M; \sigma_i)$, and if $k, i < \omega$ are s.t. σ_k projects to σ_i then there is

$\pi_{ik} : \mathrm{Ult}(M; \sigma_i) \longrightarrow \mathrm{Ult}(M; \sigma_k)$.

We may then define, in $V[H]$:

$$(s, (\alpha_0, \dots, \alpha_{n-1})) \in \mathcal{U}' \quad \text{iff}$$

$$\forall i < k < n \quad (\pi(\mathcal{U}_{s\restriction \# i}) \in \sigma_i \;\wedge$$

$$\pi(\mathcal{U}_{s\restriction \# k}) \in \sigma_k \;\wedge$$

$$\sigma_k \text{ projects to } \sigma_i$$

$$\longrightarrow \pi_{ik}(\alpha_i) > \alpha_k),$$

where $\# i$ = the length of a s.t. $\sigma_i = \nu_a$.

<u>Claim.</u> If K is $\mathbb{P}$-gen./$M[H]$ then

$$M[H][K] \models p[\pi(\mathcal{U})] = {}^{\omega}\omega \setminus p[\mathcal{U}'] .$$

Prf.: Assume first $x \in p[\pi(\mathcal{U})] \cap p[\mathcal{U}']$, say $(x, f) \in [\pi(\mathcal{U})]$ and $(x, \vec{\alpha}) \in [\mathcal{U}']$. For all n, $\pi(\mathcal{U}_{x\restriction n}) \in \nu_{f\restriction n}$, hence using $\vec{\alpha}$ we see that

$$\operatorname{dir\,lim}_n \; \mathrm{Ult}(M; \nu_{f\restriction n})$$

is ill-founded. However, this direct limit can be embedded into $\mathrm{Ult}(M; \pi(E))$ which is well-founded. Contradiction!

Now let $x \notin p[\pi(\mathcal{U})]$. Define, for $i < \omega$,

$$f_i(\vec{\gamma}) := |(x \restriction i, \vec{\gamma})|_{\pi(\mathcal{U})_x} .$$

For any i, π_i canonically extends to $\tilde{\pi}_i$ and σ_i can be fattened to $\tilde{\sigma}_i$ (a measure on $M[H][K]$); for appropriate k, i π_{ik} extends to $\tilde{\pi}_{ik}$.

We may then set $\alpha_k := [f_{\#k}]_{\tilde{\sigma}_k}$.

It is straightforward to check that $(x, \vec{\alpha}) \in [\mathcal{U}']$: if $\pi(\mathcal{U}_{s \restriction \# i}) \in \sigma_i$, $\pi(\mathcal{U}_{s \restriction \# k}) \in \sigma_k$, σ_k projects to σ_i, then $\pi_{ik}(\alpha_i) = \tilde{\pi}_{ik}(\alpha_i) =$

$$= \tilde{\pi}_{ik}([\vec{\gamma} \mapsto |(x \restriction \# i, \vec{\gamma})|_{\pi(\mathcal{U})_x}]_{\tilde{\sigma}_i}) =$$

$$= [\vec{\varepsilon} \mapsto |(x \restriction \# i, \vec{\varepsilon} \restriction \# i)|_{\pi(\mathcal{U})_x}]_{\tilde{\sigma}_k} >$$

$$> [\vec{\varepsilon} \mapsto |(x \restriction \# k, \vec{\varepsilon})|_{\pi(\mathcal{U})_x}]_{\tilde{\sigma}_k} =$$

$$= \alpha_k .$$

This proves Woodin's lemma. $\dashv$

Unfortunately, it is open whether PU holds in $V[G]$ in general — for $V[G]$ as in the statement of the Cor. to Woodin's lemma. However, PU does hold if V is an appropriate "core model":

We shall from now on assume that $V = L[E]$ is the minimal fully iterable fine structural inner model satisfying the #12 hypothesis. We're left with having to prove PU in $V[G]$.

We'll make heavy use of the fact that $L[E]$ is the core model (in a precise sense) of all of its (set-) generic extensions. We'll hence also write $L[E] = K$.

For the rest of the pf. of thm. 1 let us fix $A \subset \mathbb{R}^2$, $A \in \underset{\sim}{\Pi}^1_n$, some $n < \omega$, say $A \in \Pi^1_n(z_0)$, $z_0 \in \mathbb{R}$. Let T_n, S_n be as given by (the pf. of) the Cor. to Woodin's lemma, $T_n, S_n \in V[G_0, \dots, G_{n-2}]$. Actually, by the construction of T_n, S_n we may assume that $T_n \restriction \kappa_{n-1}$, $S_n \restriction \kappa_{n-1}$ are s.t. in any size $< \kappa_{n-1}$ extension of $V[G_0, \dots, G_{n-2}]$:

$$p[T_n \restriction \kappa_{n-1}] = \text{univ. } \underset{\sim}{\Pi}^1_n \text{ set } (\subset \mathbb{R}^3), \text{ and}$$

$$p[T_n \restriction \kappa_{n-1}] = \mathbb{R}^3 \setminus p[S_n \restriction \kappa_{n-1}] .$$

Let x range over reals. We aim to find, uniformly in x, $F(x)$ in a projective way s.t.

$$\exists y \, (x, y) \in A \rightarrow$$
$$\exists y \, ((x,y) \in A \wedge y = F(x)), \text{ i.e.}$$

$$\exists y \, (x, y, z_0) \in p[T_n] \Rightarrow (x, F(x), z_0) \in p[T_n] .$$

<u>Definition</u>. We call a premouse $\mathcal{M}$ (z_0, x)-good iff (a) $\mathcal{M} \triangleright \mathcal{J}^K_{\kappa^{+K}_{n-1}}$,

(b) the phalanx $((K, \mathcal{M}), \kappa^{+K}_{n-1})$ is iterable,

(c) $\mathcal{M}$ has a top extender, $F^{\mathcal{M}}$, with critical point κ_{n-1},

(d) if $\pi : \mathcal{M} \rightarrow_{F^{\mathcal{M}}} \tilde{\mathcal{M}}$ and

$\tilde{\pi} : \mathcal{M}[G_0, \dots, G_{n-2}] \rightarrow \tilde{\mathcal{M}}[G_0, \dots, G_{n-2}]$

is the canonical extension then

$\exists y \; (x, y, z_0) \in p[\tilde{\pi}(T_n \upharpoonright \kappa_{n-1})]$

[notice $T \upharpoonright \kappa_{n-1} \in \mathcal{J}^{K}_{\kappa^+_{n-1}} \in \mathcal{M}$], AND

(e) $\mathcal{M}$ is least (in the mouse order) with

(a) thru (d).

Fact 1. The set of all real codes for (z_0, x)-good premice is projective. [(b) is, by Hauser.]

Fact 2. Any two (z_0, x)-good $\mathcal{M}, \mathcal{M}'$ simply coiterate above κ^+_{n-1}.

By arguments pretty similar to those on pp. 8 f. we get

Claim. $\exists y \; (x, y, z_0) \in p[T_n]$ (i.e., $\exists y \; (x, y) \in A$) iff there is a (z_0, x)-good $\mathcal{M}$.

Prf. of Claim: If $\exists y \; (x, y, z_0) \in p[T_n]$

then there is an initial segment $\mathcal{M}$ of K with top extender $F^{\mathcal{M}}$ at κ_{n-1} s.t. (x, y, z_0) is size $< \tilde{\pi}(\kappa_{n-1})$ - generic over $\tilde{\mathcal{M}}[G_0, \ldots, G_{n-2}]$ (with notation as in (d) above).

Suppose $(x, y, z_0) \notin p[\tilde{\pi}(T_n \upharpoonright \kappa_{n-1})]$. Then by applying $\tilde{\pi}$ to the properties of $T_n \upharpoonright \kappa_{n-1}$ and $S_n \upharpoonright \kappa_{n-1}$ we get $(x, y, z_0) \in p[\tilde{\pi}(S_n \upharpoonright \kappa_{n-1})] \subset p[\tilde{\pi}(S_n)]$.*) But $(x, y, z_0) \in p[T_n]$ implies $(x, y, z_0) \in p[\tilde{\pi}(T_n)]$. Hence

$$p[\tilde{\pi}(S_n)] \cap p[\tilde{\pi}(T_n)] \neq \emptyset$$

which implies

$$p[S_n] \cap p[T_n] \neq \emptyset .$$

Contradiction!

Now let $\mathcal{M}$ be (z_0, x)-good, with top extender $F^{\mathcal{M}}$. Let $(x, y, z_0) \in p[\tilde{\pi}(T_n \upharpoonright \kappa_{n-1})] \subset p[\tilde{\pi}(T_n)]$.*)

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*) We can take the ultrapower of $V[G_0, \ldots, G_{n-1}]$ by $F^{\mathcal{M}}$; we here write $\tilde{\pi}$ for this ultrapower map, too.

If $(x, y, z_0) \notin p[T_n]$ then $(x, y, z_0) \in p[S_n]$, so $(x, y, z_0) \in p[\tilde{\pi}(S_n)]$, and we get the same contradiction as above.

⊣

<u>Definition.</u> We write $(y, \vec{\alpha})^{\mathcal{M}}$ for the leftmost witness of $\exists y\ (x, y, z_0) \in p[\tilde{\pi}(T_n \upharpoonright \kappa_{n-1})]$.

<u>Definition.</u> We call a (z_0, x)-good $\mathcal{M}$ stable iff there is no simple iterate $\mathcal{M}^*$ above κ_{n-1}^+ of $\mathcal{M}$ with $(y, \vec{\alpha})^{\mathcal{M}^*} <_{lex} (y, \vec{\alpha})^{\mathcal{M}}$.

<u>Claim.</u> If there is a (z_0, x)-good $\mathcal{M}$ then there is a stable (z_0, x)-good $\mathcal{M}$.

This finishes the prf. of thm 1, as we may now set $F(x) =$ the y s.t. there is a stable (z_0, x)-good $\mathcal{M}$ and there is $\vec{\alpha}$ s.t. $(y, \vec{\alpha}) = (y, \vec{\alpha})^{\mathcal{M}}$.

Prf. of the claim. Let $\mathcal{M}$ be (z_0, x)-good.

We may simply ~~iterate~~ iterate $\mathcal{M}$ to get $\mathcal{M}_1$ s.t. f.a. simple iterates $\mathcal{M}_1^*$ of $\mathcal{M}_1$ we have: if $(y, \vec{\alpha}) = (y, \vec{\alpha})^{\mathcal{M}_1}$, $(y^*, \vec{\alpha}^*) = (y, \vec{\alpha})^{\mathcal{M}_1^*}$ then $y(0) \leq y^*(0)$ (i.e., $y(0) = y^*(0)$). Again, we may simply iterate $\mathcal{M}_1$ to get $\mathcal{M}_2$ s.t. f.a. simple iterates $\mathcal{M}_2^*$ of $\mathcal{M}_2$ we have: if $(y, \vec{\alpha}) = (y, \vec{\alpha})^{\mathcal{M}_2}$, $(y^*, \vec{\alpha}^*) = (y, \vec{\alpha})^{\mathcal{M}_2^*}$ then $\alpha_0 \leq \alpha_0^*$ (i.e. $\alpha_0 = \alpha_0^*$). We may continue in this fashion to get $\mathcal{M}_\omega$, the direct limit of all the $\mathcal{M}_i$'s, which is stable and (z_0, x)-good.

We have been using that simple iterates of (z_0, x)-good $\mathcal{M}$'s are themselves (z_0, x)-good.

$\dashv$ (theorem 1)

We remark that core model theory has been used for finding an upper bound.

Let lightface PU :≡ every lightface projective subset of $\mathbb{R} \times \mathbb{R}$ admits a lightface projective uniformization.

Classical : PD ⇒ lightface PU.

<u>Question</u> : (open !)

Does PM ∧ PC ∧ lightface PU ⇒ PD ?

Let $V \models$ the # 12 hypothesis as witnessed by $\kappa_0 < \kappa_1 < \ldots$ with sup λ. Let $\mathbb{P} \in V$ with $\overline{\overline{\text{r.o.}(\mathbb{P})}} < \lambda^+$, and let G be $\mathbb{P}$-generic over V. Let G' be $Col(\omega, \lambda)$-generic over $V[G]$. As $\mathbb{P} \times Col(\omega,\lambda) \cong Col(\omega,\lambda)$, (G, G') is essentially $Col(\omega,\lambda)$-generic over V. Moreover, $V[G]$ is Σ^1_ω-correct in $V[G][G']$ by Woodin's Lemma. Hence if $V[G] \models$ lightface PU then $V[G][G'] \models$ lightface PU. But the latter is impossible, in general:
lightface PU trivially implies Basis(Π^1_3, OD) ; applied to "$x \notin K$" the weak homogeneity of

$Col(\omega, \lambda)$ easily gives a contradiction, if $V =$ what we called K above.

Hence we need new techniques to prove

$PM \wedge PC \wedge$ lightface $PU \not\Rightarrow PD$.

On the other hand, I have shown, using coding techniques, that

$PM \wedge PC \wedge PU \wedge$
"there is a lightface $\Delta_3(J_2(\mathbb{R}))$ - well-ordering of $\mathbb{R}$"
$\not\Rightarrow PD$.

In the model witnessing this we'll have, in particular, that any lightface projective subset of $\mathbb{R} \times \mathbb{R}$ admits a uniformizing function which is lightface $\Sigma_3(J_2(\mathbb{R}))$.
[Indexing : $J_1(\mathbb{R}) = V_{\omega+1}$; hence $\mathcal{P}(\mathbb{R}) \cap J_2(\mathbb{R})$ = the projective sets of reals, etc.]

<u>Question</u> : (open !)

Does $PM \wedge PC \wedge \forall n \; Unif(\Pi^1_{2n+1}, \Pi^1_{2n+1})$
$\Rightarrow PD$?

[Here, $\mathrm{Unif}(\Gamma, \Gamma')$ means that every $A \in \Gamma \cap \mathcal{P}(\mathbb{R}^2)$ admits a uniformizing function in $\Gamma' \cap \mathcal{P}(\mathbb{R})$.]

<u>Thm.</u> 3 (Steel, 1993) $PM \wedge PC \wedge PU \wedge \mathrm{Unif}(\Pi^1_3, \Pi^1_3) \Rightarrow \underset{\sim}{\Pi}^1_2$ determinacy.

We'll prove this in the form of a transfer theorem, i.e., we'll show that the hypothesis implies the existence of certain inner models with a Woodin cardinal, from which we may derive the conclusion.

The proof of thm. 3 will heavily use the following lemma. Recall that a transitive model M is called Σ^1_n correct iff for all $\phi(\vec{v}) \in \Sigma^1_n$ and for all $\vec{a} \in \mathbb{R} \cap M$

$$M \models \phi(\vec{a}) \iff V \models \phi(\vec{a})$$

Every inner model is Σ^1_2 correct by Shoenfield's absoluteness lemma.

Steel's correctness lemma. Let κ, Ω be measurable cardinals, $\kappa < \Omega$. Suppose that there is no inner model with a Woodin cardinal, and let K denote Steel's core model of height Ω. Then there is a tree $T \in K$ s.t. $p[T]$ = a universal Π^1_2 set of reals; in particular, K is Σ^1_3 correct.

Also, the following is the key to proving Thm's 2 and 3:

Woodin's 2nd lemma. Suppose PM + PC + PU. Then for every $n < \omega$ there is a transitive model M_n with $M_n \cap OR = \omega_1^V$, $M_n \models ZFC$, s.t. whenever $g \in V$ is $\mathbb{P}$-generic over M_n for some p.o. $\mathbb{P} \in M_n$ then $M_n[g]$ is Σ^1_n correct. Moreover, if $x \in \mathbb{R}$ is arbitrary then we may arrange $x \in M_n$.

We'll omit the proof of this lemma.

<u>Thm. 4</u> (Woodin; 1981) PM + PC + PU implies that every real has a sharp, i.e., $\underset{\sim}{\Pi}^1_1$ determinacy holds.

Proof: Deny. Pick $x \in \mathbb{R}$ s.t. $x^\#$ doesn't exist. Set $M = M_3$ where M_3 is as in Woodin's 2nd lemma with $x \in M_3$. Inside M, there is a cardinal κ s.t. $\kappa^+ = \kappa^{+L[x]}$ (Jensen's covering lemma + absoluteness of $x^\#$). Let $g \in V$ be $Col(\omega, \kappa)$-generic over M. Notice $\kappa^{+M} = \omega_1^{M[g]}$. Inside $M[g]$, the following holds true.

$$(*) \quad \forall \alpha < \omega_1 \ \exists \beta \ L_\beta[x] \models \overline{\overline{\alpha}} \leq \kappa .$$

We can write $(*)$ as a formula being Π^1_3 in x, y where $y \in \mathbb{R} \cap M[g]$ is a code for κ. By correctness, $(*)$ hence has to hold in V. But ω_1^V can't be a successor cardinal in $L_{\omega_1}[x] \subset M$. Contradiction!

Proof of thm. 3: As $\mathbb{R}$ is closed under sharps, by a theorem of Woodin it suffices to prove that f. a. $x \in \mathbb{R}$ there is an inner model containing x which has a Woodin cardinal being countable in V.

Deny. Fix $x \in \mathbb{R}$, a counterexample. We shall prove that $\mathrm{Unif}(\Sigma^1_3(x), \Sigma^1_3(x))$ holds. This will give a contradiction as follows.

For $\Gamma = \Sigma^1_3(x)$ or $\Pi^1_3(x)$, $\mathrm{Unif}(\Gamma, \Gamma)$ implies that Γ has the reduction property, hence $\neg\Gamma$ has the separation property. But $\Sigma^1_3(x)$ cannot both have the reduction as well as the separation property. [Let $A \subset \mathbb{R}^2$ be universal for $\Sigma^1_3(x) \cap \mathcal{P}(\mathbb{R})$, $A \in \Sigma^1_3(x)$. Set $P(\bar{x}) \iff ((\bar{x})_0, \bar{x}) \in A$, $Q(\bar{x}) \iff ((\bar{x})_1, \bar{x}) \in A$. Let P^*, Q^* in $\Sigma^1_3(x)$ reduce P, Q, and let R in $\Delta^1_3(x)$ separate P^* from Q^*. Let y, z s.t. $R(\bar{x}) \iff (y, \bar{x}) \in A$, $\neg R(\bar{x}) \iff (z, \bar{x}) \in A$. It is then easy to see that $(z, y) \in R \iff (z, y) \notin R$. Contradiction!]

We're now going to verify $Unif(\Sigma^1_3(x), \Sigma^1_3(x))$. Set $M = M_{50}$ where M_{50} is as in Woodin's 2nd lemma with $x \in M_{50}$. Clearly, it suffices to show $Unif(\Sigma^1_3(x), \Sigma^1_3(x))$ holds in M. Similar to how thm. 4 was shown above we can check that M is (globally!) closed under $\dagger$'s (daggers). Consider $(\mathbb{R}^M)^\dagger$ *). We may assume that $(\mathbb{R}^M)^\dagger$ doesn't have an inner model containing x which has a Woodin cardinal. Let Ω denote the top measurable of $(\mathbb{R}^M)^\dagger$. For any $y \in \mathbb{R}^M$, we may now build Steel's $K_{(x,y)}$, the core model over (x,y), inside $(\mathbb{R}^M)^\dagger$. By (the proof of) Steel's Correctness lemma, $K_{(x,y)}^{(\mathbb{R}^M)^\dagger}$ is Σ^1_3 correct (from the point of view of M, or of V).

Let us work inside M. Let $A \subset \mathbb{R}^2$ be $\Sigma^1_3(x)$, $(y,z) \in A \iff \phi(x,y,z)$, say, where ϕ is Σ^1_3. We have $\forall y \left(\exists z \, (y,z) \in A \rightarrow \exists z \in K \, (y,z) \in A \right)$

---

*) If $(\mathbb{R}^M)^\dagger \not\models AC$, we might want to work with a different dagger containing all of $\mathbb{R}^M$.

by Σ^1_3 correctness of K; we here write $K = K^{(\mathbb{R}^M)^+}_{(x,y)}$. However, this K has a good $\Delta^1_3(x,y)$-well ordering $<_K$, i.e. one s.t.

$$(u,v) \in R \iff \{(u)_i : i<\omega\} \text{ is the set of all } <_K\text{-predecessors of } v$$

is a $\Delta^1_3(x,y)$ relation. So if $\exists z\,(y,z) \in A$ holds then we may canonically pick a witness z as follows. Let $\phi(x,a,b) \iff \exists c\, \bar{\Phi}(x,a,b,c)$, $\bar{\Phi}$ Π^1_2.

$$\exists u \exists v \,[\; z = (v)_0 \;\wedge\; (u,v) \in R \;\wedge\; \bar{\Phi}(x,y,(v)_0,(v)_1) \;\wedge\; \forall i<\omega\; \neg\bar{\Phi}(x,y,((u)_i)_0,((u)_i)_1)\;].$$

As Σ^1_2 is closed under $\forall^\omega$, this is Σ^1_3.

However, all this works uniformly in y. This proves $\mathrm{Unif}(\Sigma^1_3(x), \Sigma^1_3(x))$ in M.

The proof of Thm. 3 is complete.

We now aim to indicate how one might be able to prove Steel's correctness lemma. The argument to follow is due to Steel and myself.

Let A be a (universal) Π^1_1 set. Let s range over ${}^{<\omega}\omega$. We may associate $s \mapsto <_s$, an ordering of $lh(s)$, with $s \subset s' \Rightarrow <_s \subset <_{s'}$ in such a way that $x \in A \leftrightarrow <_x$ is a well-ordering; here, $<_x = \bigcup_n <_{x \restriction n}$.

The Shoenfield tree S may be defined as

$$(s, \vec{\alpha}) \in S \iff \forall n < n' < lh(s) \, (\alpha_n < \alpha_{n'} \leftrightarrow n <_s n').$$

Clearly, $p[S] = A$.

We can easily rearrange S so as to have $p[S] = B$ = a (universal) Σ^1_2 set. We shall also have $x \in B \iff \exists y <_{x,y}$ is a well-ordering.

Now κ is a measurable cardinal, witnessed by $\mathcal{U}$, say. We may iterate V by $\mathcal{U}$, obtaining

$$M_0 = V \xrightarrow[\mathcal{U}]{\pi_0} M_1 \xrightarrow[\pi_0(\mathcal{U})]{\pi_1} M_2 \rightarrow \ldots \rightarrow M_i \xrightarrow{\pi_i} M_{i+1} \rightarrow \ldots,$$

$i \in OR$, taking direct limits at limit ordinals.

Let $\kappa_i = c.p.(\pi_i)$; so $\kappa_0 = \kappa$.

For $n \leq n'$ and $\phi : n \xrightarrow{\sim} n'$ order preserving we may define $\pi_\phi : M_{n+1} \longrightarrow M_{n'+1}$ by setting

$$\pi_n \circ \dots \circ \pi_0 (f)(\kappa_0, \dots, \kappa_n) \overset{\pi_\phi}{\longmapsto} \pi_{n'} \circ \dots \circ \pi_0 (f)(\kappa_{\phi(0)}, \dots, \kappa_{\phi(n)}).$$

Let $s \subset s'$. Then

$$(lh(s) ; <_s) \underset{f_s}{\overset{\sim}{\longleftrightarrow}} (lh(s) ; \in) \text{ "}\subset\text{" } (lh(s') ; \in) \underset{f_{s'}^{-1}}{\overset{\sim}{\longleftrightarrow}} (lh(s') ; <_{s'})$$

induces $\phi : lh(s) \xrightarrow{\sim} lh(s')$ order preserving, which in turn gives π_ϕ. We write $\pi_{s,s'}$ for π_ϕ.

In much the same way, if $s \subset s'$, $t \subset t'$, $lh(s) = lh(t)$, $lh(s') = lh(t')$ then we get a map $\pi_{(s,t),(s',t')}$.

It is easy to verify that $<_{x,y}$ is a well-ordering iff

$$\lim \operatorname{dir} \left(M_n \, , \, \pi_{(x\restriction n, y\restriction n),(x\restriction n', y\restriction n')} : n \leq n' < \omega \right)$$

is well-founded.

[$\Leftarrow$ trivial; $\Rightarrow$: embed the direct limit into $M_{otp(<_{x,y})}$.]

Let $(r_n : n<\omega)$ be a nice enumeration of $(s : s\in {}^{<\omega}\omega)$.
We define T_2, the Martin–Solovay tree, by

$$(s,\vec{\alpha}) \in T_2 \text{ iff } \forall n<n'<lh(s)\ [\, r_n \subsetneq r_{n'} \rightarrow \pi_{(s\restriction lh(r_n),\, r_n),\,(s\restriction lh(r_{n'}),\, r_{n'})}(\alpha_n) > \alpha_{n'} \,].$$

If x is a real built on the 1st coordinate then T_2 continuously proves $x\notin B$: namely, we have that $x\in p[T_2]$ iff $x\notin B$.
[$\Rightarrow$ easy; $\Leftarrow$: if $\forall y$ $<_{x,y}$ is ill-founded then S_x is well-founded. Let

$$\alpha_n = |(r_n, \vec{\kappa})|^{M_n}_{\pi_{n-1}\circ\dots\circ\pi_0(S_x)}$$

where $\vec{\kappa} = (\kappa_{f^{-1}_{(x\restriction lh(r_n),\, r_n)}(0)}, \dots, \kappa_{f^{-1}_{(x\restriction lh(r_n),\, r_n)}(lh(r_n)-1)})$

with $f_{(-,-)}$ being as on the previous page.
Notice that then $\pi_{(s\restriction lh(r_n),\, r_n),\,(s\restriction lh(r_{n'}),\, r_{n'})}(\alpha_n) =$

$$|(r_n, (\kappa_{f^{-1}_{(x\restriction lh(r_{n'}),\, r_{n'})}(0)}, \dots))|^{M_{n'}}_{\pi_{n'-1}\circ\dots\circ\pi_0(S_x)} >$$

$$|(r_{n'}, (\underline{\qquad}, \kappa_{f^{-1}_{(x\restriction lh(r_{n'}),\, r_{n'})}(lh(r_n))}, \dots))|^{M_{n'}}_{\pi_{n'-1}\circ\dots\circ\pi_0(S_x)} = \alpha_{n'}]$$

Let us now assume that $0^{\P}$ doesn't exist, and let K denote the core model.

We aim to prove that there is $T \in K$ with $p[T] = p[T_2]$.

To commence, $\pi_0 : V \longrightarrow M_1$ induces $\pi_0 \restriction K : K \to K^{M_1}$.

By $\neg 0^{\P}$, K^{M_1} is an iterate of K; moreover, the iteration giving K^{M_1} can be absorbed by an iteration $\mathcal{I}$ belonging to K, i.e., there is $\sigma_0 : K \to K_1$ given by $\mathcal{I} \in K$ together with $k : K^{M_1} \longrightarrow K_1$ s.t. $\sigma_0 = k \circ \pi_0$.

Let $\tilde{K} = \mathrm{ult}(K^{M_1} ; \pi_0(\mathcal{I}))$ *). Then

$$K^{M_1} \models \text{"} \tilde{K} = \mathrm{ult}(K^{M_1} ; \pi_0(\mathcal{I})) \models \phi(\vec{x}) \text{,"} \quad \text{so using } k \Rightarrow$$

$$K_1 \models \text{"} \underbrace{\mathrm{ult}(K_1 ; \overbrace{k \circ \pi_0}^{} (\mathcal{I}))}_{} \models \phi(k(\vec{x})) \text{."}$$

where $k \circ \pi_0 = \sigma_0$, and $\mathrm{ult}(K_1 ; \sigma_0(\mathcal{I})) =: K_2$.

Hence $k : \tilde{K} \longrightarrow K_2$ is elementary, which gives

---

*) by which we mean the iterate of K^{M_1} by $\pi_0(\mathcal{I})$. Notice $\pi_0(\mathcal{I}) \in K^{M_1}$!

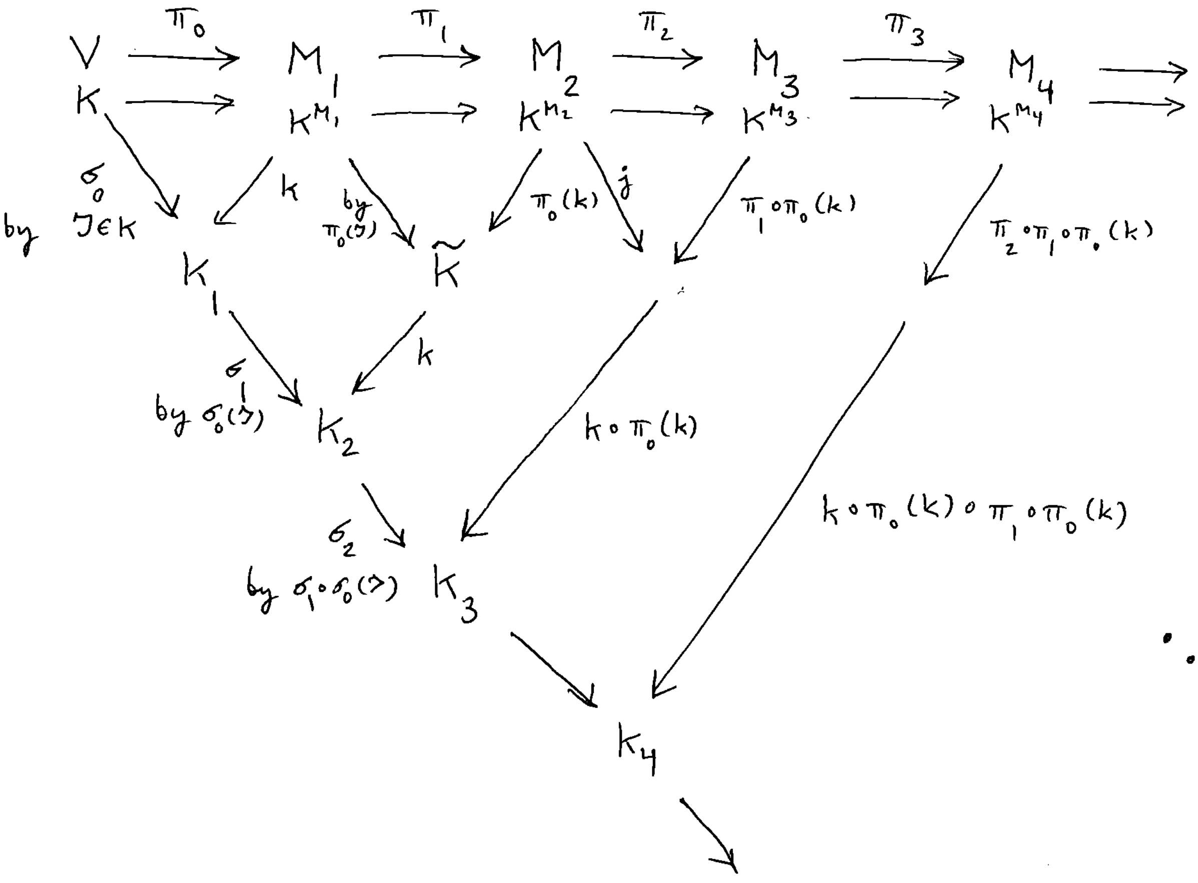
$k \circ \pi_0(k) : K^{M_2} \longrightarrow K_2$. Similarily,

$K^{M_2} \models$ " $\mathrm{Ult}(K^{M_2}; \pi_1 \circ \pi_0(\mathcal{J})) \models \phi(\vec{x})$, " $\overset{\text{using } k \circ \pi_0(k)}{\Longrightarrow}$

$K_2 \models$ " $\mathrm{Ult}(K_2; \underbrace{k \circ \pi_0(k)(\pi_1 \circ \pi_0(\mathcal{J}))}_{= \; \sigma_1 \circ \sigma_0(\mathcal{J})}) \models \phi(\vec{x})$. "

where $\sigma_1 : K_1 \to K_2$ is given by $\sigma_0(\mathcal{J})$ $(\in K_1)$.

In this fashion we obtain the following:



$\ldots$,

where the pattern is that

$k \circ \pi_0(k) \circ \pi_1\pi_0(k) \circ \ldots \circ \pi_{i-2}\pi_{i-3}\cdots\pi_0(k) : K^{M_i} \longrightarrow K_i$;

call this map k_i (so $k = k_1$). We have

commutativity in the form $k_{i+1} \circ \pi_i = \sigma_i \circ k_i$,

shown inductively : for example, $k_3 \circ \pi_2(x) =$

$k \circ \pi_0(k) \circ \underbrace{\pi_1\pi_0(k)}_{\substack{\text{map given by } \pi_1\pi_0(J), \\ \text{call it } j}} \circ \pi_2 \; (x) = k_2 \circ j\,(x) = k_2(j)\big(k_2(x)\big)$;

but $k_2(j) =$ the map given by $\sigma_1\sigma_0(J)$, as $k_2\,\pi_1\,\pi_0 =$

$\sigma_1\,k_1\,\pi_0 = \sigma_1\sigma_0$, so $k_2(j)(k_2(x)) = \sigma_2\,(k_2(x))$.

But we need more : we need such diagrams for all $s, t \in {}^{<\omega}\omega$ and their associated "shift maps" $\pi_{(s\restriction n,\, t\restriction n),\,(s\restriction n',\, t\restriction n')}$, s.t. if $s \subset s'$, $t \subset t'$ then the diagram for one extends the one for the other.

Notice that the sequence of k_i's is an iteration of K by the long extender derived from σ_0 ("$\in$" K). This allows us to define maps

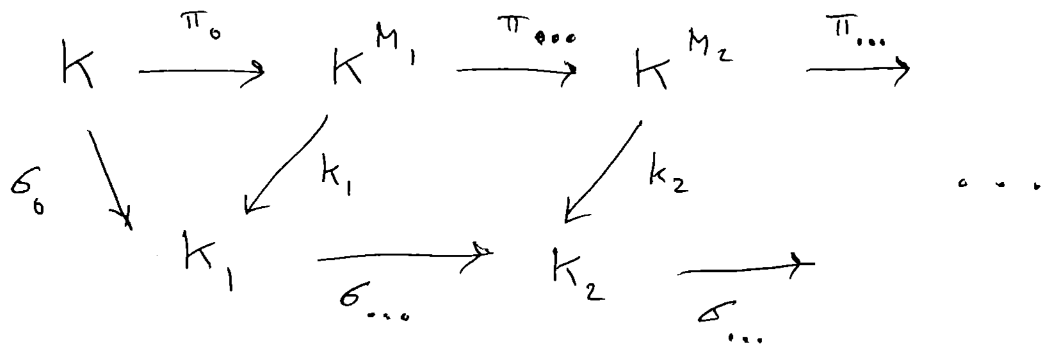
$$\sigma_{(s,t),(s',t')} : K_{lh(s)} \longrightarrow K_{lh(s')}$$

in a similar way as the $\pi_{(s,t),(s',t')}$'s were defined above. [Example: we may define $K_1 \to K_2$ by $\sigma_0(f)(a) \mapsto \sigma_1 \sigma_0(f)(\sigma_0(a))$, as

$$K_1 \models \phi(\sigma_0(f)(a)) \Leftrightarrow a \in \underset{\substack{\shortparallel \\ \text{map given by } \mathcal{J}}}{\sigma_0}(\{u : K \models \phi(f(u))\})$$

$$\Leftrightarrow \sigma_0(a) \in \overset{\substack{\text{map given by } \sigma_0(\mathcal{J}) \\ \shortparallel}}{\sigma_1}(\{u : K_1 \models \phi(\sigma_0(f)(u))\}) \Leftrightarrow$$

$K_2 \models \phi(\sigma_1 \sigma_0(f)(\sigma_0(a)))$; etc.] Further, we have our maps $k_i : K^{M_i} \to K_i$ s.t. the following diagram commutes.



We have defined a system

$$(K_i, \sigma_{(s,t),(s',t')}) \text{ "}\in\text{" } K,$$

with associated diagrams as above. We now define a tree $T \in K$, by

$(s, \vec{\beta}) \in T$ iff $\forall n < n' < lh(s)\ [r_n \subsetneq r_{n'} \rightarrow$

$\sigma_{(s \restriction lh(r_n), r_n), (s \restriction lh(r_{n'}), r_{n'})}(\beta_n) > \beta_{n'}\ ]$.

We claim $p[T] = p[T_2]$, so that T is a sort of Martin-Solovay tree in K.

"$\supset$": Let $(x, \vec{\alpha}) \in [T_2]$. Set $\beta_n = k_n(\alpha_n)$ for $n < \omega$. Then if $r_n \subsetneq r_{n'}$:

$\sigma_{(s \restriction lh(r_n), r_n), (s \restriction lh(r_{n'}), r_{n'})}(\beta_n) =$

$\sigma_{(s \restriction lh(r_n), r_n), (s \restriction lh(r_{n'}), r_{n'})} \circ k_n(\alpha_n) =$

$k_{n'} \circ \pi_{(s \restriction lh(r_n), r_n), (s \restriction lh(r_{n'}), r_{n'})}(\alpha_n) >$

$k_{n'}(\alpha_{n'}) = \beta_{n'}$.

"$\subset$": If $x \notin p[T_2]$ then $\exists y <_{x,y}$ is a well-ordering. Then one can check that

$$\lim\operatorname{dir}(K_i, \sigma_{(x \restriction n, y \restriction n), (x \restriction n', y \restriction n')})$$

can be embedded into $K_{otp(<_{x,y})}$, and is hence well-founded. But this implies $x \notin p[T]$.

We believe, but haven't checked carefully, that the full correctness lemma can be shown in this fashion.

Conjecture: (open!) Let $n<\omega$. Suppose that V is closed under $M_n^\#$, but there is no inner model with $n+1$ Woodin cardinals. Then K is Σ^1_{n+3} correct.

Proof of thm. 2:

Main Lemma (Schindler) Assume that there is no inner model with a Woodin cardinal, and K exists (i.e., there is a model K s.t. etc.). Suppose that $\mathcal{J}^K_{\omega_1^V} \models$ "there are finitely many strong cardinals." Then $\mathcal{J}^K_{\omega_1^V}$ is projective in the codes (i.e. $\{x \in \mathbb{R} : x \text{ codes } \mathcal{M} \trianglelefteq \mathcal{J}^K_{\omega_1^V}\}$ is a projective set. In fact, if $\mathcal{J}^K_{\omega_1^V} \models$ "there are exactly n strongs," then $\mathcal{J}^K_{\omega_1^V}$ is $\underset{\sim}{\Delta}^1_{n+5}$ in the codes.

Rmk.: This generalizes a result of Hjorth who had shown this in the special case $n=0$. If $\mathcal{J}^K_{\omega_1^V} \models$ "$\exists$ exactly n strongs" then:

ω_1^V inacc. in $K \Rightarrow \mathcal{J}^K_{\omega_1^V}$ is (lightface) Δ^1_{n+5};

ω_1^V is a succ. in $K \Rightarrow \mathcal{J}^K_{\omega_1^V}$ is $\utilde{\Delta}^1_{n+3}$.

Further generalizations (by myself) show for ex. that if ω_1^V is not an inacc. limit of strongs then the set of reals coding initial segments of $\mathcal{J}^K_{\omega_1^V}$ is in $L_{\omega_1^V}(\mathbb{R})$.

Now assume Ω subtle + PM + PC + PU + the #12 hypothesis fails. We aim to derive a contradiction. We can build K (of height Ω), and by the failure of the #12 hypothesis,

$\mathcal{J}^K_{\omega_1^V} \models$ "there are exactly n strong cardinals,"

for some $n<\omega$. By virtue of the Main Lemma, ω_1^V can't be a successor cardinal in

K, as otherwise it is easy to construct a projective sequence of pairwise distinct reals of length ω_1, contradicting PM and a theorem of Shelah. So ω_1^V is inaccessible in K.

We now let $\phi_n(x)$ denote the following predicate: x codes a premouse $\mathcal{M}$ such that there is a universal weasel $W \triangleright \mathcal{M}$ with

(a) $\mathcal{M} \cap OR$ is a successor cardinal in W,

(b) if $E_\nu^W \neq \emptyset$ with $\kappa = c.p.(E_\nu^W) < \mathcal{M} \cap OR$ and $\nu > \mathcal{M} \cap OR$ then $\mathcal{J}_\kappa^{\mathcal{M}} \models$ "there are $<$ n strong cardinals," [$\mathcal{M}$ is an n-cutpoint in W]

(c) if W' is a class sized iterate of W with iteration map $i: W \to W'$ then W' has the definability property at all $\kappa < i(\mathcal{M} \cap OR)$ s.t. $\mathcal{J}_\kappa^{W'} \models$ "there are $< n$ strong cardinals" and $\kappa >$ the generators of all extenders used along the branch from W to W'.

the main technical result towards showing the above Main Lemma is that $\{x : \phi_n(x)\}$ is a Π^1_{n+3} set of reals. This in turn requires the construction of a new type of models, $K^c_n(\mathcal{M})$.

In fact, if $\mathcal{J}^K_{\omega_1^V} \models$ "there are $\leq n$ strongs" and ω_1^V is inaccessible in K then one can show that x codes an initial segment of K iff x codes $\bar{\mathcal{M}}$ and there is y coding $\mathcal{M} \rhd \bar{\mathcal{M}}$ with $\phi_n(y)$ s.t. for all z coding $\mathcal{N}$ with $\phi_n(z)$ we have: IF $\mathcal{M}, \mathcal{N}$ coiterate to $\mathcal{M}^*$, $\mathcal{N}^*$ s.t. $\mathcal{M}^* \trianglelefteq \mathcal{N}^*$ THEN $\mathcal{M}^*$ is an n-cutpoint in $\mathcal{N}^*$, there is no drop along the main branches on either side, and if

$$i : \mathcal{M} \to \mathcal{M}^*,$$
$$j : \mathcal{N} \to \mathcal{N}^*$$

denote the maps coming from the coiteration then $i''\mathcal{M} \subset j''\mathcal{N}$.

[this is Σ^1_{n+5} !]

Let $\Psi_n(x)$ denote the predicate ~~[crossed out]~~ just described (on the bottom half of p. 38). Let Ξ abbreviate the statement:

$$\forall x \forall y \left(\Psi_{n+1}(x) \wedge \Psi_{n+1}(y) \wedge x \text{ codes } \mathcal{M}_x,\ y \text{ codes } \mathcal{M}_y \rightarrow \mathcal{M}_x \trianglerighteq \mathcal{M}_y \text{ or } \mathcal{M}_x \trianglelefteq \mathcal{M}_y \right) \wedge$$

if $\mathcal{M} = \bigcup \{ \mathcal{M}_x : \Psi_n(x) \wedge x \text{ codes } \mathcal{M}_x \}$ then $\mathcal{M}$ is a (transitive) model of "ZFC + $\exists$ exactly n strong cardinals" of height ω_1.

By what we've said so far, $V \models \Xi$ and Ξ is Π^1_{n+7}. Now let $M = M_{n+50}$ where M_{n+50} is as in Woodin's 2nd lemma. M is globally closed under #'s, so that we can build K^M. By the main property of M and a thm. of Hauser, K^M has at least $n+30$ strong cardinals, $\kappa_0 < \ldots < \kappa_{n+29}$. Let $M \models$ "λ is a singular cardinal $> \kappa_{n+29}$." Pick $g \in V$ $Col(\omega, \lambda)$-generic over M. Then $K^{M[g]} = K^M$ and $M[g] \models$ " my ω_1 is a successor cardinal in K, and $\mathcal{J}^K_{\omega_1} \models$ 'there are

$>$ n strong cardinals.' "

We have $M[g] \models \Xi$. Let $\mathcal{M}$ (of height $\omega_1^{M[g]} = \lambda^{+M}$) be as given by $M[g] \models \Xi$. It is easy to check that if κ_n is the $n+1^{st}$ strong cardinal of $K^{M[g]}$ then $\mathcal{J}_{\kappa_n}^{K^{M[g]}} \lhd \mathcal{M}$; moreover, in $M[g]$ there is a universal $W \rhd \mathcal{M}$ having the definability property at $\kappa_0, \dots, \kappa_{n-1}$, by the choice of $\mathcal{M}$.

There is $\sigma : K^{M[g]} \to W$ coming from the coiteration of $K^{M[g]}, W$. As $K^{M[g]}$ (or a very Soundness witness for a long enough segment thereof) has the definability property at $\kappa_0, \dots, \kappa_n$, $\sigma \upharpoonright \kappa_n + 1 = \mathrm{id}$. On the other hand, κ_n is strong in $\mathcal{J}_{\omega_1^{M[g]}}^{K^{M[g]}}$ but not in $\mathcal{J}_{\omega_1^{M[g]}}^{W}$, whereas

$\mathcal{J}_{\omega_1^{M[g]}}^{K^{M[g]}} \models$ "κ_n is strong" $\Longrightarrow$

$\mathcal{J}_{\sigma(\omega_1^{M[g]})}^{W} \models$ "κ_n is strong" $\Longrightarrow$

$\mathcal{J}_{\omega_1^{M[g]}}^{W} \models$ "κ_n is strong."

Contradiction!

We'll now turn towards problems inspired by the question on p. 19.

Let $\underset{\sim}{\Sigma}^1_n M :\equiv$ every $\underset{\sim}{\Sigma}^1_n$ set of reals is Lebesgue measurable, $\underset{\sim}{\Sigma}^1_n C :\equiv$ every $\underset{\sim}{\Sigma}^1_n$ set of reals has the property of Baire, for $n < \omega$.

<u>Question</u>: Let $n<\omega$. Is it consistent to have $\underset{\sim}{\Sigma}^1_n M$ + $\underset{\sim}{\Sigma}^1_n C$ + there is a (lightface) projective well-ordering of the reals?

<u>Rmk.</u>: $n=1$: $L \models \underset{\sim}{\Sigma}^1_1 M, \underset{\sim}{\Sigma}^1_1 C$, $\exists \Delta^1_2$-w.o. of $\mathbb{R}$ (by Gödel). $n=2$: $\underset{\sim}{\Sigma}^1_2 M, \underset{\sim}{\Sigma}^1_2 C$, Δ^1_3-w.o. in a gen. ext. of L with an inaccessible (by David). $n=3,4$: models by S. Friedman.

<u>Thm.</u> 5 (S. Friedman / Schindler, 1998)
Let $n<\omega$. It is consistent to have $\underset{\sim}{\Sigma}^1_n M$ + $\underset{\sim}{\Sigma}^1_n C$ + there is a lightface Δ^1_{n+2}-well ordering of $\mathbb{R}$.
More precisely, let $L[E]$ be the minimal fine structural fully iterable inner model with n strong cardinals and an inaccessible above.

then there is a set generic extension of $L[E]$ in which

(a) every $\underset{\sim}{\Sigma}^1_{n+2}$ set of reals is universally Baire,

(b) $\underset{\sim}{\Sigma}^1_{n+3}$ M + $\underset{\sim}{\Sigma}^1_{n+3}$ C, and

(c) there is a lightface Δ^1_{n+5}-well ordering of the reals.

Rmk.: It is open whether $n+5$ can be replaced by $n+4$ in (c). By an observation of Hauser, (a) (for $n \geq 1$) implies the ex. of an inner model with 1 strong cardinal.

We'll omit the proof of thm. 5 and prove instead:

Thm. 6 (S. Friedman / Schindler, 1998)
Let $n < \omega$. Let $L[E]$ be the minimal fine structural fully iterable inner model with n strong cardinals (closed under #'s if $n = 0$). Then there is a real a, set generic over $L[E]$, s.t. in $L[E][a]$ the following hold true.

(a) every $\underset{\sim}{\Sigma}^1_{n+2}$ set of reals is universally Baire (so $\underset{\sim}{\Sigma}^1_{n+2}$ M + $\underset{\sim}{\Sigma}^1_{n+2}$ C),

(b) there is a $\Delta^1_{n+3}(a)$ – well ordering of $\mathbb{R}$, and

(c) a is a Π^1_{n+4} singleton (so there is a Δ^1_{n+5} – w.o. of $\mathbb{R}$).

Warm up: Assume $n > 0$. Let $\kappa_1 < \dots < \kappa_n$ be the strong cardinals of $L[E]$. Let g be $Col(\omega, \kappa_n^{++L[E]})$ – generic over $L[E]$, and let $a \in \mathbb{R}$ code g. We claim that (a) + (b) hold.

As to (a), $\underset{\sim}{\Sigma}^1_{n+2}$ M + $\underset{\sim}{\Sigma}^1_{n+2}$ C follow from the argument showing the cor. to Woodin's lemma (pp. 7 – 9). In fact, the trees themselves, coming from Woodin's lemma, witness universal Baireness of $\underset{\sim}{\Sigma}^1_{n+2}$ sets.

(b): Let $x \in \mathbb{R} \cap L[E][a]$. As the forcing has the κ_n^{+++} – c.c., x has a name $\dot{x} \in (H_{\kappa_n^{+++}})^{L[E]} = J_{\kappa_n^{+++L[E]}}[E]$. This in turn implies $x \in J_{\kappa_n^{+++L[E]}}[E]$. Notice $\kappa_n^{++L[E]} = \omega_1^{L[E][a]}$; let us write this as ω_1.

We may well-order the reals of $L[E][a]$ as follows:

$$x < y \text{ iff } \exists \text{ (long enough) } \mathcal{M} \lhd J_{\omega_1}[E] \text{ s.t. } x <_{\mathcal{M}[a]} y,$$

where $<_{\mathcal{M}[a]}$ denotes the order of constructibility of $\mathcal{M}[a]$. It hence suffices to show that the set of real codes for initial segments of $J_{\omega_1}[E]$ is $\underset{\sim}{\Sigma}^1_{n+3}$.

Let $b \in \mathbb{R}$ code $J_{\kappa_n^{++L[E]}}[E]$, $b \in L[E][a]$, and write $\mathcal{N} = J_{\kappa_n^{++L[E]}}[E]$. We have that

$\mathcal{N} \trianglelefteq \mathcal{M} \lhd J_{\omega_1}[E]$ iff

$\exists \tilde{\mathcal{M}} \trianglerighteq \mathcal{M}$, $\tilde{\mathcal{M}}$ ctble., $\mathcal{M} \trianglerighteq \mathcal{N}$, s.t.

$\rho_\omega(\tilde{\mathcal{M}}) = \mathcal{N} \cap \mathrm{OR}$ and

$\exists$ universal $W \rhd \tilde{\mathcal{M}}$ having the definability property at $\kappa_1, \ldots, \kappa_{n-1}$.

[Proof: $\Rightarrow$ clear. $\Leftarrow$: Coiterate W, $K = L[E]$. As K has the definability property at $\kappa_1, \ldots, \kappa_n$, the coiteration is above κ_n on the K-side, and above $\kappa_n + 1$ on the W-side. But then the W-side can't move $\tilde{\mathcal{M}}$ at all. A standard argument gives $\tilde{\mathcal{M}} \triangleleft K$.]

By the method which shows Φ_n on p. 37 is Π^1_{n+3}, one can show that the set of real codes for such $\mathcal{M}$ is Π^1_{n+2} (b).

However, (c) can't hold in $L[E][a]$, as it is a homogeneous extension of the ground model (cf. pp. 19 f.).

<u>Proof of thm. 6</u>: (Sketch)

Set $\lambda = \kappa_n^{++L[E]}$. Inside $L[E]$, we define a "nice" sequence of λ^+-Suslin trees, $(T_k : k < \omega)$. We first force with $\prod_k T_k$, adding cofinal branches B_k thru T_k (the forcing has the λ^+-c.c.). We then force with $Col(\omega, \lambda)$, adding G. We'll again write $\omega_1 = \omega_1^{L[E][\vec{B}][G]} = \lambda^{+L[E]} = \kappa_n^{+++L[E]}$.

Any ~~two~~ two distinct cofinal branches thru $({}^{<\omega}2, \subset) \in L$

give a pair of a.d. subsets of ω (reals). Let $(a_k : k<\omega) \in L$ be given by the first (along $<_L$) ω many branches. Write

$$x^{dec} = \{ k<\omega : x \cap a_k \text{ is finite} \}$$

for reals x.

Pick a real $g \subset \omega$, $g \in L[E][\vec{B}][G]$ coding $J_\lambda[E]$. We want to force over $L[E][\vec{B}][g]$ to obtain a real a s.t. $g = a^{dec}$, and a is a Π^1_{n+3} singleton inside $L[E][a]$. We shall also have (a) and (b) in $L[E][a]$ by arguments pretty much as before.

Let $(a_i : i<\omega_1) \in L[E][g]$ be ~~the~~ the sequence of pairwise a.d. reals given by the first (along $<_{L[E][g]}$) ω_1 many branches thru $({}^{<\omega}2, \subset)$. Notice the 2 def.ns of $(a_k : k<\omega)$ given coincide.

The forcing $\mathbb{R}$ consists of $p = (\ell(p), r(p))$ where $\ell(p) : k \to 2$, some $k<\omega$, and $r(p) \subset \omega_1$ finite.

We set $q \le p$ iff $\ell(q) \supset \ell(p)$, $r(q) \supset r(p)$,

AND

$$k < \mathrm{dom}(\ell(p)) \wedge k \in g \implies$$

$$\{m \in \mathrm{dom}(\ell(q)) \setminus \mathrm{dom}(\ell(p)) : \ell(q)(m) = 1\} \cap a_k = \emptyset,$$

$$k < \mathrm{dom}(\ell(p)) \wedge \ell(p)(k) = 1 \wedge \alpha \in r(p) \cap B_{2k} \Rightarrow$$

$$\{m \in \mathrm{dom}(\ell(q)) \setminus \mathrm{dom}(\ell(p)) : \ell(q)(m) = 1\} \cap a_{\alpha+\omega+2k} = \emptyset,$$

and

$$k < \mathrm{dom}(\ell(p)) \wedge \ell(p)(k) = 0 \wedge \alpha \in r(p) \cap B_{2k+1} \implies$$

$$\{m \in \mathrm{dom}(\ell(q)) \setminus \mathrm{dom}(\ell(p)) : \ell(q)(m) = 1\} \cap a_{\alpha+\omega+2k+1} = \emptyset.$$

Let $a \subset \omega$ be given by a generic. By the 1st two lines above, $a^{dec} = g$.

Set $D_k = \{\alpha : a \cap a_{\alpha+\omega+k}$ is finite$\}$. We also have (by the last 4 lines):

$$k \in a \Rightarrow B_{2k} = D_{2k} \wedge D_{2k+1} = \emptyset, \text{ and}$$

$$k \notin a \Rightarrow B_{2k+1} = D_{2k+1} \wedge D_{2k} = \emptyset.$$

It is crucial that moreover we'll have that

$k \in a \Rightarrow T_{2k+1}$ is Aronszajn in $L[E][a]$, and

$k \notin a \Rightarrow T_{2k}$ is Aronszajn in $L[E][a]$.

these properties of a make it a Π^1_{n+4} singleton ("David's trick"). We let $\phi(x) :\equiv$

x^{dec} codes $J_\lambda[E]$, and

for all $\mathcal{N}$, $J_\lambda[E] \triangleleft \mathcal{N} \triangleleft J_{\omega_1}[E]$, with

(a) λ is the 2^{nd} largest cardinal of $\mathcal{N}$,

(b) $\mathcal{N}[x] \models ZF^-$,

we have that, if $(T_n^{\mathcal{N}} : n < \omega)$ and $(a_i^{\mathcal{N},x} : i < \omega_1^{\mathcal{N}[x]})$ are defined inside $\mathcal{N}$, $\mathcal{N}[x]$, as (T_n), (a_i) was defined above in $L[E]$, $L[E][g]$, and if we set $B_k^{\mathcal{N},x} = \{\alpha : x \cap a^{\mathcal{N},x}_{\alpha+\omega+k}$ is finite$\}$, then

(a)' $k \in x \Rightarrow B_{2k}^{\mathcal{N},x}$ is a cof. branch thru $T_{2k}^{\mathcal{N}}$, and

(b)' $k \notin x \Rightarrow B_{2k+1}^{\mathcal{N},x}$ is a cof. branch thru $T_{2k+1}^{\mathcal{N}}$.

As on p. 44 f., $\phi(x)$ can be checked to be Π^1_{n+4}. We're left with having to verify

$\phi(x) \Leftrightarrow x = a$.

Let $x \neq a$, let w.l.o.g. $\ell \in x \setminus a$. In particular, $T_{2\ell}$ is an Aronszajn tree in $L[E][a]$. We may pick $\sigma : \mathcal{W}[x] \to J_{\omega_2}[E][x]$ with $\mathcal{W}$ ctble., c.p.$(\sigma) > \lambda$. There is a condensation lemma telling us that $\mathcal{W} \triangleleft L[E]$. But then $\mathcal{W}[x] \models$ "$T_{2k}^{\mathcal{W}}$ is not Aronszajn," by (a)′, so $J_{\omega_2}[E][x] \models$ "T_{2k} is not Aronszajn." Contradiction!

To prove that $\phi(a)$ holds one first observes that $(T_k^{\mathcal{W}} : k < \omega) = (T_k \cap \mathcal{W} : k < \omega)$ and $(a_i^{\mathcal{W}, a} : i < \omega_1^{\mathcal{W}[a]}) = (a_i : i < \omega_1^{\mathcal{W}[a]})$ for any $\mathcal{W}$ as in $\phi(x)$. In particular, $B_k^{\mathcal{W}, a} = B_k \cap \mathcal{W}$ for $k < \omega$. But then (a)′, (b)′ will be obvious.

<u>Question</u>: (open!) Let $3 \leq n < k < \omega$. What is the consistency strength of $\underset{\sim}{\Sigma}^1_n M + \underset{\sim}{\Sigma}^1_n C +$ there is a (lightface) Δ^1_k-well-ordering of the reals?

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The previous notes also contain unpublished material from Woodin, Steel, and Steel + Schindler.