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Γ a class of posets

$FA^{+\omega_1}(\Gamma)$: $\forall Q \in \Gamma$ $\forall$ $\aleph_1$ sized coll. of dense sets, $\mathcal{D}$, + every seq $(\dot{S}_i : i < \omega_1)$ of Q-names for stat. subsets of ω_1, there is a filter g, $g \cap D \neq \emptyset$ $\forall D \in \mathcal{D}$, and $\dot{S}^g = \{\alpha < \omega_1 : \exists p \in g \; p \Vdash \check{\alpha} \in \dot{S}\}$ stat, f.a. $\dot{S} = \dot{S}_i$.

Lem. Let Γ be nice. Assume $FA^{+\omega_1}(\Gamma)$. and $\mathbb{P}$ [illegible] : f.a. $\mathbb{P}$-names $\dot{Q}$ for an elt. of $\dot{\Gamma}$ there ex. $\mathbb{P} * \dot{Q}$ name $\dot{\mathbb{R}}$ s.t.

① $\mathbb{P} * \dot{Q} * \dot{\mathbb{R}} \in \Gamma$, $\Vdash_{\mathbb{P} * \dot{Q}} \dot{\mathbb{R}}$ is stat. set pres.

② whenever $j : V \to N$ is a gen. elementary embedding with $crit(j) = \omega_2$, $H_\theta^V \in wfp(N)$, $j \restriction H_\theta^V \in N$ + $|H_\theta^V|^N = \omega_1$ +

and there ex. a V-gen. filter ~~K~~

$G * H * K \in N$ s.t. all stat. subsets

of ω_1 in $V[G*H*K]$ are stat. in N

then $j[G]$ has a lower bound in $j(\mathbb{P})$.

<u>then</u> $V^{\mathbb{P}} \models FA^{+\omega_1}(\dot{\Gamma})$.

<u>defn.</u> (foreman - todorcevic)

IA = ~~set~~ class of W of size $\aleph_1$ s.t. there is a <u>filtration of W</u> (i.e., a cont. $\subset$ incr. $\vec{N} = (N_i : i < \omega_1)$ with union W ~~&~~ s.t. every $(N_i : i < \alpha) \in W$, $\alpha < \omega_1$.

IC = class of W of size $\aleph_1$ s.t. $W \cap [W]^\omega$ contains a club

IS = ... $W \cap [W]^\omega$ is stationary.

IU = class of W of size $\aleph_1$ s.t. $W \cap [W]^\omega$ is $\subset$-cofinal in $[W]^\omega$

$IA \subset IC \subset IS \subset IU$.

strict under PFA

strict under MM, PFA^+ (not PFA),

strict under MM (not under $PFA^{+\omega_1}$).

<u>fact</u>. ae $2^{\aleph_1} = \aleph_2$. TFAE.

(1) approachability property fails at ω_2, i.e., $\omega_2 \notin I[\omega_2]$.

(2) the inclusion $IA \subset IU$ is strict in $\mathcal{P}_{\omega_2}(H_{\omega_2})$, i.e., there are stat. many $W \in IU \setminus IA$ in $\mathcal{P}_{\omega_2}(H_{\omega_2})$.

(i.e., at least one of the 3 inclusions is strict).

<u>stationary reflection.</u>

$RP_{IC} \equiv$ for every reg. $\theta \geq \omega_2$ + every stat. $S \subset [\theta]^{\omega}$, there is some $W \in IC$ s.t. $S \cap [W]^{\omega}$ is stat.

clearly,

$$\begin{array}{ccccccc} RP_{IA} & \Rightarrow & RP_{IC} & \Rightarrow & RP_{IS} & \Rightarrow & RP_{IU}. \\ & & \Downarrow & & \Uparrow & & \\ & & & RP_{internal} & & & \end{array}$$

def. $RP_{internal}$ = $\forall \theta \geq \omega_2$ $\forall$ stationary $S \subset [\theta]^\omega$ there is W s.t. $|W| = \aleph_1$ and $S \cap W \cap [W]^\omega$ is stationary.

(fuchino – usuba) $RP_{internal}$ is equivalent to: for every cble. $M \prec (H_\theta ; \in, \Delta)$ there are $\subset$-cofinally many $W \in [H_\theta]^{\aleph_1}$ s.t. letting $M(W) = \text{Hull}(M \cup \{W\})$ we have $M(W) \cap W = M$ (similar to doebler–schindler).

th. 1. assume $V \models PFA^{+\omega_1}$; there is a $<\omega_2$ strategically closed poset $\mathbb{P}$ s.t. in $V^{\mathbb{P}}$:

(a) PFA holds

(b) $PFA^{+\omega_1}_{H^V_{\omega_2} \notin IC}$ holds, i.e., $FA^{+\omega_1}(\Gamma)$, where Γ is the class of proper posets that collapse ω_2 + force $H^V_{\omega_2} \notin IC$.

(c) RP_{IC} fails.

th. 2. $PFA^{+\omega_1}_{H^V_{\omega_2} \notin IC} \Longrightarrow RP_{internal}$.

proof of th. 1.

$V \models PFA \Rightarrow 2^{\aleph_1} = \aleph_2$.

fix a bijection $\Phi : \omega_2 \longrightarrow H_{\omega_2}$.

$\mathbb{P}_{nrIC}$ = the set of closed, bounded subsets S of ω_2 s.t. whenever $\gamma \in S^2_1 = \omega_2 \cap cf(\omega_1)$ and $\Phi'' \gamma \in IC$, then $S \cap \gamma$ is nonstationary.

order by end-extension.

<u>facts</u>. (1) $\mathbb{P}$ is σ-closed.

(2) $\mathbb{P}$ is $< \omega_2$ strategically closed.

(3) $\mathbb{P}$ forces $\neg RP_{IC}$.

left to prove:

(a) preserves PFA.

(b) preserves $PFA^{+\omega_1}_{H^V_{\omega_2} \notin IC}$.

we prove (b); (a) is similar.

by the lea on p. 1 :

supp. $\mathbb{P} \Vdash$ "$\dot{\mathbb{Q}}$ is proper + forces $H_{\omega_2}^{V^{\mathbb{P}}} \notin IC$ "

then $\mathbb{P} * \dot{\mathbb{Q}}$ is proper + forces $H_{\omega_2}^V \notin IC$.

let $\dot{\mathbb{R}}$ be the $\mathbb{P} * \dot{\mathbb{Q}}$ name for the poset.

① of lea satisfied.

② asse $j : V \longrightarrow N$, $j \upharpoonright H_\theta^V \in N$,
$|H_\theta^V|^N = \aleph_1$, and N has the same
$G * H$ that is V-generic +
$V[G * H]$, N agree about stationarity of
subsets of ω_1. need to check that
$j''G$ has a lower bound.
$\;\;=$
G

it suff. to show $N \models \underbrace{j(\Phi)''\omega_2^V}_{= H_{\omega_2}^V} \notin IC$

but $H_{\omega_2}^V \notin IC^{V[G*H]}$, this is
upward absolute to N b/c $V[G*H]$
+ N agrees about stat. subsets of ω_1.
so lea applies. $\dashv$

<u>thm 2.</u> $PFA^{+\omega_1}_{H^V_{\omega_2} \notin IC} \Longrightarrow RP_{\text{internal}}$.

proof of th. 2 : assume $PFA^{+\omega_1}_{H^V_{\omega_2} \notin IC}$.

fix a reg. $\theta \geq \omega_2$.

let $Q = Add(\omega) * Col(\omega_1, H^V_\theta)$

by gitik-velickovic, $\Vdash_{Add(\omega)} [\omega_2]^\omega \setminus V$ is stationary

($[\omega_2]^\omega \cap V$ is also stat. by properness)

it follows easily that

$$\Vdash_Q H^V_{\omega_2} \in IS \setminus IC .$$

so Q is a member of the class of the forcing axiom. by ~~the~~ a lea of woodin, then in stat. many $W \prec (H_{(2^\theta)^+}; \in, -)$ s.t. $\overline{W} = \aleph_1 \subset W$ there ex. a (W, Q)-gen. filter g s.t. f.a. $\dot{T} \in W$ that names a stat. subset of ω_1, $\dot{T}^g$ is stationary.

Supp. R is any stat. subset of $[\theta]^\omega$ s.t. $R \in W$.

(we'll show $R \cap W \cap [W]^\omega$ is stationary.)

let $(\dot{N}_i : i < \omega_1)$ be a $\mathbb{Q}$ name for a filtration of H_θ^V. let $\dot{S}_R = \{i : \dot{N}_i \in \check{R}\}$.

by properness of $\mathbb{Q}$, $\dot{S}_R$ is forced to be stationary. it's also in W.

so $\dot{S}_R^{\,g}$ is stat. in ω_1.

notice: $\Vdash$ if $i \in \dot{S}_R$, then $\dot{N}_i \in \check{R} \subset V$.

it follows that in the filtration $((\dot{N}_i)^g : i < \omega_1)$ of W, where $i \in \dot{S}_R^{\,g}$, then $N_i^{\,g} \in W \cap R$. $\dashv$

$RP_{IS} \Longrightarrow RP_{Iu}$.

Open: is it reversible?