Goldberg, The ultrapower axiom

UA = ultrapower axiom

thm. (UA) Supp. κ is a cardinal, the following are equivalent.

(1) κ is strongly compact.

(2) κ is supercompact or a measurable limit of supercompact.

def. M, N inner models, an embedding $j: M \to N$ is an ultrapower if there is some M-ultrafilter U s.t. $N = \mathrm{Ult}(M; U)$, $j = j_U^M$. j is an internal ultrapower embedding if we can take $U \in M$.

UA: if $j_0 : V \to M_0$, $j_1 : V \to M_1$, are ultrapower embeddings, then there are internal ultrapower embeddings $k_0 : M_0 \to N$, $k_1 : M_1 \to N$ s.t.

$$\begin{tikzcd} & N & \\ M_0 \arrow[ur, "k_0"] \arrow[rr, phantom, "///"] & & M_1 \arrow[ul, "k_1"'] \\ & V \arrow[ul, "j_0"] \arrow[ur, "j_1"'] & \end{tikzcd}$$

plan.

part I. show the least strongly compact is supercompact.

part II. show the 2^{nd} strongly compact is supercompact.

part I: we'll fix regular δ, consider $\mathcal{U}_\delta$ "least" on δ.

part II: define "irreducible" ultrafilter.

def. if α is an ordinal, an ultrafilter $\mathcal{U}$ on α is uniform if for all $\beta < \alpha$, $\alpha \setminus \beta \in \mathcal{U}$.

fix δ regular, carrying a ctbly. complete uniform ultrafilter.

def. a ctbly. complete u.f. $\mathcal{U}$ on δ is zero order iff

(1) $[id] = \sup j_\mathcal{U}{}''\delta$ ($\mathcal{U}$ weakly normal)

(2) $\sup j_\mathcal{U}{}''\delta$ carries no countably complete uniform u.f. in $M_\mathcal{U}$

($\Leftrightarrow \mathrm{cf}^{M_\mathcal{U}}(\sup j_\mathcal{U}{}''\delta)$ carries no uniform u.f.)

<u>th.</u> (ketonen) δ carries a zero order ultrafilter.

<u>universal property of zero order ultrafilters (UA):</u>

supp. $\mathcal{U}$ is zero order on δ. supp. $j: V \to M$ is an ultrapower emb. s.t. $\sup j''\delta$ carries no cntbly. complete uniform ultrafilter in M. then there is an internal ultrapower $k: M_{\mathcal{U}} \to M$ s.t.

$$\begin{tikzcd} M_{\mathcal{U}} \arrow[r, "k"] & M \\ V \arrow[u, "j_{\mathcal{U}}"] \arrow[ur, "j"'] \arrow[ur, phantom, "///"] & \end{tikzcd}$$

and $k(\sup j_{\mathcal{U}}''\delta) = \sup j''\delta$.

<u>proof.</u>

$$\begin{tikzcd} & N & \\ M_{\mathcal{U}} \arrow[ur, "i_0"] & /// & M \arrow[ul, "i_1"'] \\ & V \arrow[ul, "j_{\mathcal{U}}"] \arrow[ur, "j"'] & \end{tikzcd}$$

<u>claim.</u> $i_0''M_{\mathcal{U}} \subset i_1''M$

pf.: since $M_{\mathcal{U}} = \mathrm{Hull}^{M_{\mathcal{U}}}(j_{\mathcal{U}}''V \cup \{[id]_{\mathcal{U}}\})$

suff. to show $i_0[j_{\mathcal{U}}''V] \subset i_1''M$.

$i_0(\sup j_0''\delta) \in i_1''M$.

to see $i_0(\sup j''\delta) \in i_1''M$:

$$
\begin{aligned}
i_0(\sup j''\delta) &= \sup i_0 \circ j_U''\delta \\
&= \sup i_1 \circ j''\delta \\
&= i_1(\sup j''\delta).
\end{aligned}
$$

define $k : M_U \longrightarrow M$ by $k = i_1^{-1} \circ i_0$.

take $a \in M$ s.t. $M = \mathrm{Hull}^M(j''V \cup \{a\})$.

$\dashv$

<u>Corollary.</u> there's a unique zero order u.f. on δ.

Pf: say U_0, U_1 zero order.

$$
\begin{tikzcd}
M_{U_0} \arrow[loop left, "k_1 \circ k_0"] \arrow[r, shift right, "h_0"'] & M_{U_1} \arrow[l, shift right, "h_1"'] \\
V \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

$h_0(\sup j_{U_0}''\delta) = \sup j_{U_1}''\delta$.

$k_1 \circ h_0(\sup j_{U_0}''\delta) = \sup j_{U_0}''\delta$.

def. (UA) U_δ = the unique zero order u.f. on δ.

cor. (UA) supp. $i : M_{U_\delta} \longrightarrow M$ is an ultrapower embedding. TFAE.

(1) i is internal.

(2) i is cont. at $\sup j_U{}''\delta$.

$$\begin{tikzcd} M_{U_\delta} \arrow[r, "i"] & M \\ V \arrow[u, "j_{U_\delta}"] \arrow[ur, "j"'] & \end{tikzcd}$$

defn $j : V \longrightarrow M$ as $j = i \circ j_{U_\delta}$

note that $\sup j''\delta = i(\sup j_{U_\delta}{}''\delta)$, so $\sup j''\delta$ carries no uniform u.f. in M.

so let $k : M_{U_\delta} \longrightarrow M$ come fr the universal property. So k is internal.

$k \circ j_{U_\delta} = i \circ j_{U_\delta}$ and $k(\sup j_{U_\delta}{}''\delta) = \sup j''\delta$
$= i(\sup j_{U_\delta}{}''\delta)$.

Since k and j agree on $j_{U_\delta}{}''V \cup \{\sup j_{U_\delta}{}''\delta\}$.

prop. (UA) supp. $\gamma < \mathrm{cf}^{M_{U_\delta}}(\sup j_{U_\delta}{}''\delta)$.
then for any ctbly. complete M_{U_δ}-u.f. W on γ, $W \in M_{U_\delta}$.

proof. let $i = {}^{\mathcal{U}_\delta} j_W$. then $i : M_{\mathcal{U}_\delta} \to M$ and i is cont. at $\sup j_{\mathcal{U}_\delta}{}'' \delta$. so i is $M_{\mathcal{U}_\delta}$-internal. follows that $W \in M_{\mathcal{U}_\delta}$, since W is the u.f. derived from j using $[id]_W$. $\dashv$

fact. (UA) Supp. $\gamma < cf^{M_{\mathcal{U}_\delta}}(\sup j_{\mathcal{U}_\delta}{}'' \delta)$. assume $crit(\mathcal{U}_\delta) = \kappa_{\mathcal{U}_\delta}$ is γ-strongly compact. then $\mathcal{P}(\gamma) \subset M_{\mathcal{U}_\delta}$.

defn. supp. Y is a set and κ is a cardinal. a family $\mathcal{F} \subset \mathcal{P}(Y)$ is κ-independent iff f.a. $A \subset \mathcal{F}$, the filter generated by $A \cup \{Y \setminus X : X \in \mathcal{F} \setminus A\}$ is κ-complete.

thm. (hausdorff) if κ is inacc. and $\gamma^{<\kappa} = \gamma$, $\gamma \in Card$, then there is an κ-independent family $\mathcal{F} \subset \mathcal{P}(\gamma)$ s.t. $\overline{\overline{\mathcal{F}}} = 2^\gamma$.

proof: WMA that $cf(\gamma) \geq \kappa$, where $\kappa = \kappa_{U_\delta}$. γ is a cardinal.

WMA $(\gamma^{<\kappa})^{M_{U_\delta}} = \gamma$.

note that in M_{U_δ}, κ is γ-strongly compact. so γ carries a κ-complete u.f. in M_{U_δ}. every γ' s.t. γ' is M_{U_δ}-regular and $\gamma' \in [\kappa, \gamma]$ carries a κ-complete u.f. in M_{U_δ}.

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in M_{U_δ}, let $\mathcal{F} \subset \mathcal{P}(\gamma)$ be a κ-independent family s.t. $|\mathcal{F}|^{M_{U_\delta}} = \gamma$.

supp. $\sigma, \tau \in \mathcal{P}_\kappa(\mathcal{F})$, $\sigma \cap \tau = \emptyset$.

need $\bigcap \sigma \cap \bigcap \{\gamma \setminus X : X \in \tau\} \neq \emptyset$.

claim. $\mathcal{P}(\mathcal{F}) \subset M_{U_\delta}$.

pf.: supp. $A \subset \mathcal{F}$. the filter G generated by $A \cup \{\gamma \setminus X : X \in A\}$ is κ-complete

So it extends to a κ-complete u.f. $W \supset G$. but $W \cap M_{U_\delta} \in M_{U_\delta}$.

so $A \in M_U$. $A = (W \cap M_{U_\delta}) \cap \mathcal{F}$.

Since $|\mathcal{F}|^{M_{U_\delta}} = \gamma$, we get $\mathcal{P}(\gamma) \subset M_{U_\delta}$.

<u>Proposition.</u> (UA) if $\gamma < cf^{M_{U_\delta}}(\sup j_{U_\delta}{}''\delta)$ and κ_{U_δ} is γ-strongly compact, then $\mathcal{P}(\gamma) \subset M_{U_\delta}$.

<u>Cor.:</u> if κ_{U_δ} is δ-strongly compact, then $cf^{M_{U_\delta}}(\sup j_{U_\delta}{}''\delta) = \delta$.

proof: deny. then $\mathcal{P}(\delta) \subset M_{U_\delta}$. but every M_{U_δ}-u.f. on δ is in M_{U_δ}.

So $U_\delta \in M_{U_\delta}$ $\lightning$.

<u>def.</u> Supp. M is a inner model, γ is a cardinal, and γ' is a M-cardinal. M has the (γ, γ') <u>covering property</u> iff every $A \subset M$, $\overline{\overline{A}} \leq \gamma$ can be covered by

a set $A' \in M$ s.t. $|A'|^M \leq \gamma'$.

th. (Ketonen) supp. γ is reg., $j : V \longrightarrow M$ is a ultrapower. TFAE.

(1) M has the (γ, γ') covering property.

(2) $\mathrm{cf}^M(\sup j''\gamma) < \gamma'$.

proposit. if κ_{U_δ} is δ-strongly compact, then for any $\gamma < \delta$, $(M_{U_\delta})^\gamma \subset M_{U_\delta}$.

proof: suff. to show $j_{U_\delta}''\gamma \in M_{U_\delta}$.

assume γ is regular.

consider ~~supp.~~ $\mathrm{cf}^{M_{U_\delta}}(\sup j_{U_\delta}''\gamma) < \delta$.

by Ketonen, have $A' \in M_{U_\delta}$ s.t. $j_{U_\delta}''\gamma \subset A'$, $|A'|^{M_{U_\delta}} < \delta$.

by a previous proposition, $\mathcal{P}(|A'|^{M_{U_\delta}}) \subset M_{U_\delta}$. $\Rightarrow$ $\mathcal{P}(A') \subset M_{U_\delta}$. $\dashv$

thm. (UA) supp. δ is a succ. cardinal, or an inacc. cardinal. then κ_{U_δ} is δ-strongly compact.

thm. (UA) if δ is a successor, then

$$(M_{U_\delta})^\delta \subset M_{U_\delta}.$$

if δ is inaccessible, $(M_{U_\delta})^{<\delta} \subset M_{U_\delta}$ + M_{U_δ} has (δ,δ)-covering.

cor. (UA) the least strongly compact is supercompact.

pf.: call it κ. fix succ. cardinal $\delta \geq \kappa$. consider U_δ. since U_δ witnesses κ_{U_δ} is δ-supercompact, we need to know that $\kappa_{U_\delta} \leq \kappa$.

th. (Ketonen) if U is a zero order on δ and every regular cardinal in the interval $[\iota, \delta]$ carries a countably complete u.f., then M_U has the $(\delta, < j_U(\iota))$-covering property. $\Rightarrow$ $\mathrm{crit}(U) \leq \iota$.

since every regular cardinal in $[\kappa, \delta]$ carries an κ-complete u.f., can conclude $\kappa_{\mathcal{U}\delta} \leq \kappa$.

def. A u.f. $\mathcal{U}$ is γ-zero order if

(0) $\mathcal{U}$ is ν-complete

(1) In $M_{\mathcal{U}}$, $\sup j_{\mathcal{U}}''\delta$ carries a $j_{\mathcal{U}}(\nu)$ complete u.f.

(2) $[id]_{\mathcal{U}} = \sup j_{\mathcal{U}}''\delta$.

def. if $\mathcal{U}, W$ are ctbly. complete u.f.,

$\mathcal{U} \leq_{RF} W$ iff there is an internal ultrapower $k : M_{\mathcal{U}} \to M_W$ s.t. $k \circ j_{\mathcal{U}} = j_W$.

an u.f. W is <u>irreducible</u> if for all $\mathcal{U} \leq_{RF} W$, either $\mathcal{U} \equiv W$, or $\mathcal{U}$ is principal.

an u.f. W is $<\lambda$-irreducible if f.a. $\gamma < \lambda$, f.a. $\mathcal{U}$ on γ, if $\mathcal{U} \leq_{RF} W$, then $\mathcal{U}$ is principal.

the 2^{nd} strongly compact.

<u>thm.</u> (UA) Supp. δ is a succ. cardinal, U_δ exists, and W is $<\delta$-irreducible. then $(M_W)^\delta \subset M_W$.

<u>cor.</u> (UA) if U is irreducible (strongly) uniform on λ + λ is a successor or a strong limit singular, then U is λ-supercompact.

<u>prop.</u> Supp. κ is the least cardinal above ν that is δ-strongly compact (δ a reg. cardinal). then there is a irreducible u.f. U on δ s.t. $\kappa_U = \kappa$.

<u>thm.</u> (UA) if W is $<\delta$ irreducible, δ is a succ. and U_δ exists, then W is δ-supercompact.

proof:

$$\begin{tikzcd} & N & \\ M_W \arrow[ur, "i_0"] & & M_{U_\delta} \arrow[ul, "i_1"'] \\ & V \arrow[ul] \arrow[ur] & \end{tikzcd}$$

want: $\text{crit}(i_1) > \delta$.

given this, $OR^\delta \subset N \subset M_W$, so W is δ-supercompact.

<u>def.</u> if j_0, j_1 are ultrapowers of V, $j_0 : V \longrightarrow M_0$, $j_1 : V \longrightarrow M_1$, then a pair (i_0, i_1), $i_0 : M_0 \to N$, $i_1 : M_1 \longrightarrow N$ is a <u>can. comparison</u> of (j_0, j_1) if f.a. comparisons (i_0', i_1') of (j_0, j_1) to N'

$$\begin{tikzcd}
 & N' & \\
 & N \arrow[u, "h"] & \\
M_0 \arrow[ur, "i_0"] \arrow[uur, "i_0'"] & & M_1 \arrow[ul, "i_1"'] \arrow[uul, "i_1'"'] \\
 & V \arrow[ul, "j_0"] \arrow[ur, "j_1"'] &
\end{tikzcd}$$

there is $h : N \longrightarrow N'$, an internal ultrapower s.t. $i_0' = h \circ i_0$, $i_1' = h \circ i_1$.

<u>th.</u> (UA) any pair of ultrapowers admits a canonical comparison.

Pf.:

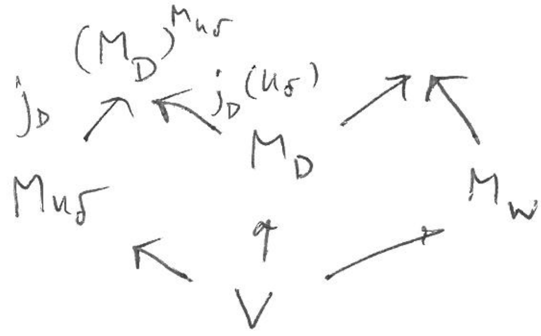
$$\begin{tikzcd}
(M_D)^{M_W} \arrow[r, bend left, "k'"] & N & \\
 & & \\
D & & \\
M_0 \arrow[uuu, dashed, "D"] \arrow[uuur, "k"] & & M_1 \arrow[uuul, "(U_{\delta'})^{M_W}"'] \\
 & V \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$M_1 \ni \sup j_W{}''\delta$.

In $\delta' = cf^{M_W}(\sup j_W``\delta)$

in N, $\sup j^{M_W}_{U_{\delta'}}``\delta'$ carries ~~..~~ no ctbly. complete u.f.

<u>fact.</u> there is an ultrafilter D on some $\gamma < \delta$ and an internal ultrapower $k : (M_D)^{M_{U_\delta}} \to N$ s.t. $k' \circ j_D = k$.



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def. if M, N are ω models, $j : M \to N$ el. embedding, then

(1) j is an ultrapower embedding if $\exists$ M-u.f. U s.t. $j = j_U^M$.

(2) j is internal if U can be chosen in M.

def. supp. $j_0 : V \to M_0$, $j_1 : V \to M_1$ are ultrapowers. say $(j_0, j_1) : (M_0, M_1) \to N$ is a comparison of (i_0, i_1) if

(1) $j_0 : M_0 \to N$, $j_1 : M_1 \to N$ are internal ultrapowers

(2) $j_0 \circ i_0 = j_1 \circ i_1$.

ultrapower axiom. (UA) any pair of ultrapowers admits comparison.

main thm. (UA) supp. κ is a cardinal. TFAE.

(1) κ is strongly compact.

(2) κ is supercompact or a measurable limit of supercompacts.

<u>def.</u> if $\mathcal{U}$, W are uf's, then $\mathcal{U} \leq_{RF} W$ iff there is an internal $k : M_{\mathcal{U}} \to M_W$ s.t. $k \circ j_{\mathcal{U}} = j_W$.

an u.f. W is <u>irreducible</u> if f.a. $\mathcal{U} \leq_{RF} W$, either $\mathcal{U}$ is principal or $\mathcal{U} \equiv W$.

same u.power emb.

λ a cardinal, W is <u>$<\lambda$-irreducible</u> if f.a. $\mathcal{U}$ on $\gamma < \lambda$, if $\mathcal{U} \leq_{RF} W$, then $\mathcal{U}$ is principal.

<u>irreducibility th.</u> (UA) Supp. δ is a succ. cardinal that carries a uniform u.f. Supp. W is a u.f. TFAE:

(1) W is $<\delta$-irreducible.

(2) W is δ-supercompact.

<u>plan:</u> ① ired. th. $\Longrightarrow$ main th.

② sketch ired. th.

① : <u>lem</u> (UA) if W_0 is an u.f. $\leq_{RF} \restriction W$ has the ascending chain condition.

$$\begin{tikzcd} M_{W_0} & & \\ V \arrow[u, "W_0"] \arrow[r, "u_0"'] & M_1 \arrow[ul, "W_1"'] \arrow[r, "u_1"'] & M_2 \arrow[l, "W_2"'] \arrow[r] & \cdots \end{tikzcd}$$

proof: (irred. $\Rightarrow$ main)

Supp. δ is a succ., κ δ-strongly supercompact. We can show that κ is δ-supercompact.

Supp. W_0 is κ-complete.

$$\mathrm{cf}^{M_{W_0}}(\sup j_{W_0}{}''\delta) < j_{W_0}(\kappa).$$

if W_1 is not $< j_{u_0}(\delta)$-irreducible.

$$\begin{tikzcd} M_{W_0} & & & & \\ V \arrow[u, "W_0"] \arrow[r, "u_0"'] \arrow[rrrr, bend right, "i"'] & M_1 \arrow[ul, "W_1"'] \arrow[r] & M_2 \arrow[r] & \cdots \arrow[r, "u_{n-1}"'] & M_n \arrow[ullll] \end{tikzcd}$$

W_n is $< i(\delta)$-irreducible in M_n.

by irred. thm., W_n where κ_{W_n} is $j(\delta)$-supercompact in M_n.

claim. $\kappa_{W_n} \in [\kappa, j(\kappa)]$.

pf.: $\kappa_{W_n} \geq \kappa$, since $(j_{W_n})^{W_n} \circ i = j_{W_0}$ and W_0 has $\kappa_{W_0} = \kappa$.

we'll show $(j_{W_n})^{M_n}(i(\kappa)) > i(\delta)$, so $\kappa_{W_n} \leq i(\kappa)$.

$j_{W_n}^{M_n}(i(\kappa)) = j_{W_0}(\kappa)$.

$$j_{W_n}^{M_n}(i(\kappa)) = j_{W_0}(\kappa) > cf^{M_{W_0}}(\sup j_{W_0}{}''\delta)$$

$$= cf^{M_{W_0}}(\sup j_{W_n}^{M_n} \circ i{}''\delta)$$

$$= cf^{M_{W_0}}(\sup j_{W_n}^{M_n}{}'' i(\delta))$$

$$\leq cf^{M_n}(\sup j_{W_n}^{M_n}{}'' i(\delta)) = i(\delta).$$

case 1. $\kappa_{W_n} = j(\kappa)$

by elt., since $i(\kappa)$ is $i(\delta)$-supercompact in M_n, κ is δ-supercompact.

case 2. $\kappa_{W_n} \in [\kappa, i(\kappa))$.

if $\alpha < \kappa$, then $i(\alpha) = \alpha < \kappa$, so in M_n, there is a supercompact in $(i(\alpha), i(\kappa))$, so $\exists$ one in (α, κ).

(2) irred. thm.

- canonical comparisons.

def. if $j_0 : V \to M_0$, $j_1 : V \to M_1$ are ultrapowers and $(i_0, i_1) : (M_0, M_1) \to N$ is a comparison, then (i_0, i_1) is canonical if f.a. $(i_0', i_1') : (M_0, M_1) \to N'$ there is a internal $h : N \to N'$ s.t.
$h \circ i_0 = i_0'$, $h \circ i_1 = i_1'$.

proposition (UA) any pair of ultrapowers of V has a can. comparison.

lem (UA). Supp. $(i_0, i_1) : (M_0, M_1) \to N$ is a comparison of $j_0 : V \to M_0$, $j_1 : V \to M_1$. TFAE.

(1) (i_0, i_1) is canonical.

(2) $N = \mathrm{Hull}^N(i_0{}'' M_0 \cup i_1{}'' M_1)$.

lem. supp. $(i_0, i_1) : (M_0, M_1) \to N$ is a can. comparison of (j_0, j_1). supp. $k : N \to N'$ is a u.p. embedding.

TFAE.

(1) k is internal

(2) k is def. the on M_0 and on M_1.

pf.:

$$\begin{tikzcd}
 & N' & \\
 & N \arrow[u, "k"] & \\
M_0 \arrow[ur, "i_0"] \arrow[uur, bend left, "i_0'"] & & M_1 \arrow[ul, "i_1"'] \arrow[uul, bend right, "i_1'"'] \\
 & V \arrow[ul, "j_0"] \arrow[ur, "j_1"'] &
\end{tikzcd}$$

$i_0' = k \circ i_0$, $i_1' = k \circ i_1$.

$(i_0', i_1') : (M_0, M_1) \to N'$ is a comparison, so $\exists$ internal $h : N \longrightarrow N'$ s.t. $h \circ i_0 = i_0'$, $h \circ i_1 = i_1'$.

$$\left.\begin{aligned} h \restriction i_0{}'' M_0 &= k \restriction i_0{}'' M_0 \\ h \restriction i_1{}'' M_1 &= k \restriction i_1{}'' M_1 \end{aligned}\right\} \Rightarrow h = k \Rightarrow$$

k internal to N.

<u>def.</u> δ is a reg. cardinal. $\mathcal{U}_\delta$ is the unique up to iso. s.t. for any ultrapower $j : V \to M$ in $\sup j''\delta$ carries no uniform u.f., there is $k : M_{\mathcal{U}_\delta} \longrightarrow M$ internal s.t. $k \circ j_{\mathcal{U}_\delta} = j$ & $k(\sup j_{\mathcal{U}_\delta}{}''\delta) = \sup j''\delta$.

<u>th.</u> (UA) if δ is a succ. cardinal, then

$$\mathrm{OR}^\delta \subset M_{\mathcal{U}_\delta}.$$

<u>lem.</u> (UA) for any reg. δ, $\mathcal{U}_\delta$ is irreducible.

<u>irred. thm.</u> (UA) supp. δ is a succ. cardinal that carries a uniform u.f. and W is a u.f. TFAE.

(1) W is $<\delta$ irreducible.

(2) W is δ-supercompact.

proof:

$$\begin{tikzcd}
 & N & \\
M_{U_\delta} \arrow[ur, dotted, "k"] & & M_W \arrow[ul, "U_{\delta'}^{M_W} = U'"'] \\
 & V \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

let $\delta' = \mathrm{cf}^{M_W}(\sup j_W{}''\delta)$.
let $U' = (U_{\delta'})^{M_W}$

<u>claim.</u> $\sup j_{U'}^{M_W} \circ j_W{}''\delta$ carries no u.f. in N.

pf.: $\mathrm{cf}^N(\sup j_{U'}^{M_W} \circ j_W{}''\delta) = \mathrm{cf}^N(\sup j_{U'}^{M_W}{}''\delta')$.

in N, $\sup j_{U'}{}''\delta'$ carries no u.f. since $U' = (U_{\delta'})^{M_W}$. but then $\mathrm{cf}^N(\sup j_{U'}{}''\delta')$ carries no u.f. so $\sup j_{U'} \circ j_W{}''\delta$ carries no u.f. $\dashv$

therefore by universal property of U_δ, $\exists k : M_{U_\delta} \to N$ s.t. $k \circ j_{U_\delta} = j_{U'}^{M_W} \circ j_W$.

<u>key fact.</u> there is an u.f. D on $\gamma < \delta$ and an internal ultrapower $k' : (M_0)^{M_{u_\delta}} \longrightarrow N$ s.t. $k = k' \circ j_0^{M_{u_\delta}}$ + $\mathrm{crit}(k') > \delta$.

<u>goal.</u> $\mathrm{crit}(k) > \delta$.

$OR^\delta = OR^\delta \cap M_{u_\delta} \subset N \subset M_W$.

<u>new goal:</u> D is principal.

then $k = k'$, so $\mathrm{crit}(k) > \delta$.

suff. to see $D \leq_{RF} W$, since W is $<\delta$-irred.

<u>note:</u> $(M_D)^{M_{u_\delta}} = j_D(M_{u_\delta}) = M^{M_D}_{j_D(u_\delta)}$.

$$\begin{tikzcd}
M_D^{M_{u_\delta}} \arrow[rr, "k'"] & & N \arrow[r, "\ell'"] & N' \\
& a \arrow[ur, "h"] & & \\
M_D \arrow[uu, "j_D(u_\delta)"] \arrow[ur, "\ell"] \arrow[rr, "j"] & & M_W \arrow[uu, "u' = (u_{\delta'})^{M_W}"'] \arrow[ul, bend right] &
\end{tikzcd}$$

if $j = \mathrm{id}$, then ✓

<u>new goal:</u> $h \neq \mathrm{id}$.

assume $h = \mathrm{id}$.

$\boxed{\text{key fact.} \quad u' = j_D(u_\delta).}$

in part., $\delta' = j_D(\delta)$

$\mathcal{P}(\delta') \cap M_W = \mathcal{P}(j_D(\delta)) \cap M_D$.

let $\ell' : N \longrightarrow N'$ be an u.p. of N by $U' = j_D(U_\delta)$. so ℓ' is def. then on M_D and M_W.

$$\ell \restriction OR = j^{M_W}_{U'} \restriction OR = j_D(j_{U_\delta}) \restriction OR$$

it follows that $j_D(U_\delta)$ is principal. but we assumed δ carries a unifor u.f., so U_δ is not principal. $\dashv$