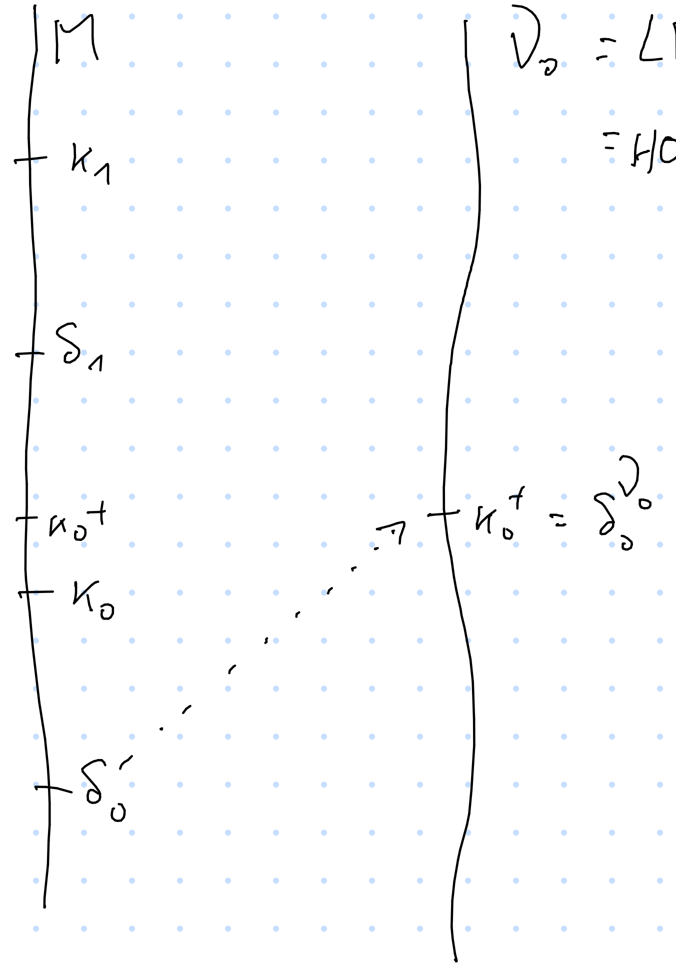
$M = M_{swsw}$

$\mathcal{V}_0 = L[\mathcal{M}_\infty, g \mapsto g^*]$
$= HOD^{M^{Col(\omega,<\kappa_0)}}_{\mathcal{E}}$



$\mathcal{E}$ in $\mathcal{V}_0$

In M, $\mathcal{M}_\infty$ is fully iterable at the bottom Woodin cardinal

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$\mathcal{V}_0 = \mathcal{P}$ - construction in M over $(\mathcal{M}_\infty | \kappa_0^{+\mathcal{M}_\infty}; \in, * \restriction \delta_0^{\mathcal{M}_\infty})$.

More precisely

If $P|\nu$ is constructed, $\nu > \kappa_0^{+\mathcal{M}_\infty}$, $E_\nu^M \neq \emptyset$ with crit $> \kappa_0$, then $E_\nu^P = E_\nu^M \restriction (P|\nu)$.

If $E_\nu^M \neq \emptyset$, crit $= \kappa_0$, then

$$E_\nu^P = \{ (\mathcal{T}, b) \mid \mathcal{T} \in P | \lambda_{E_\nu^M} \text{ on } \mathcal{M}_\infty | \delta_0^{\mathcal{M}_\infty}, \; b = \Sigma_{\mathcal{M}_\infty}(\mathcal{T}) \}$$

$$\mathcal{M}_\infty | \delta_0^{\mathcal{M}_\infty} \longrightarrow (\mathcal{M}_\infty | \delta_0^{\mathcal{M}_\infty})^{Ult(M,F)}$$

$$\pi_{E_\nu^M} \upharpoonright \mathcal{M}_\infty | \delta_0^{\mathcal{M}_\infty}$$

$$M|\nu = \mathcal{P}|\nu[M|\kappa_0^{+M}] \quad , \nu > \kappa_0^{+\mathcal{M}_\infty}$$

$$M\|\nu = \mathcal{P}\|\nu[M|\kappa_0^{+M}]$$

Lemma $V_0 = \bigcap$ all $<\kappa_0$ grounds of M.

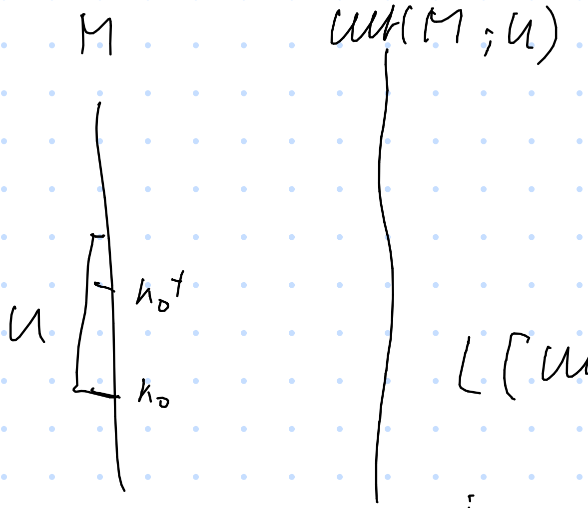
Proof. "$\subseteq$" exercise.

"$\supseteq$" Let U be the least measure on κ_0 in M.

$$j : M \to Ult(M;U)$$

Fix A, a set of ordinals, $A \in \bigcap <\kappa_0$ grounds of M.

$j(A) \in \bigcap <j(\kappa_0)$ grounds of $Ult(M;U)$



$\mathcal{M}_\infty$ of κ_0 in $Ult(M;U)$

$$\| $$

$$Ult(\mathcal{M}_\infty ; \pi_{M,\mathcal{M}_\infty}(U))$$

$$L[Ult(\mathcal{M}_\infty, \pi_{M,\mathcal{M}_\infty}(U)), * \upharpoonright \delta_0^{\mathcal{M}_\infty}]$$

is a $<j(\kappa_0)$ ground of $Ult(M;U)$

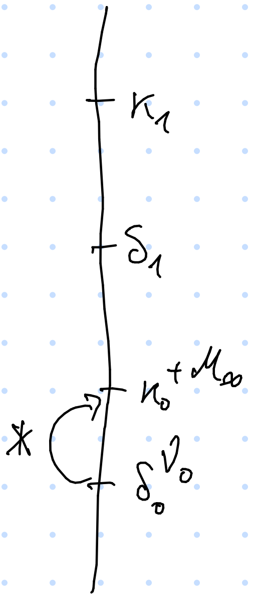
$\Rightarrow j(A) \in$

$j \restriction \kappa_0^+ \in \mathcal{V}.$

$\mathcal{T} \in A \iff j(\mathcal{T}) \in j(A)$

$\Rightarrow A \in \mathcal{V}_0.$ $\dashv$

$M \supseteq \mathcal{V}_0$



<u>Goal</u>. Define another M_∞-system based on $\mathcal{V}_0$.

1^{st} issue: Iterability of $\mathcal{V}_0$ in V.

$$\begin{tikzcd}
L[M, \pi_{M,M_\infty}] \arrow[r, "\tilde{\pi}_{M,M_\infty}"] \arrow[d, "\mathcal{T}_0"] & \mathcal{V}_0 \subseteq M \arrow[d] & M \arrow[d, "\mathcal{T}^0"] \\
L[M', \pi'] \arrow[r, "\tilde{\pi}'"] \arrow[d, "\mathcal{T}_1"] & \mathcal{V}_0^{M'} \subseteq M' \arrow[d, "\tilde{\pi}'\mathcal{T}_1"'] & M' \arrow[d, "\tilde{\pi}'\mathcal{T}_1"] \\
L[M'', \pi''] \arrow[r] & \mathcal{V}_0^{M''} & M''
\end{tikzcd}$$

Let $\eta < \kappa_1$ be a relative strong cutpoint of $\mathcal{V}_0$