ben siskind, I   July 16

M, Σ mouse pair

$$\begin{tikzcd} M \arrow[r, "S"] \arrow[rr, bend right] & P \arrow[r, "T"] & Q \end{tikzcd}$$

$$X(S,T) \longrightarrow W(S,T)$$

th. Σ has very strong hull condensation
(condensation for weak hulls)

pf. idea: $\Phi : S^\frown b \xrightarrow[\text{weak hull}]{} T$
$\uparrow$ by Σ

S by Σ, $c = \Sigma(S)$

NTS: $b = c$.

compare $S^\frown b$, $S^\frown c$. /

a pseudo hull embedding

$$\Phi : S \longrightarrow T$$

$$\mathcal{M}^S_\alpha \longrightarrow \mathcal{M}^T_{u(\alpha)}$$

$$E^S_\alpha \longmapsto E^T_{u(\alpha)}$$

$$\Phi = (u, v, \{t^0_\xi\}_{\xi < lh(S)}, \{t^1_\xi\}_{\xi+1 < lh(S)})$$

(1) $v : lh(S) \longrightarrow lh(I)$

$v(\xi) = \sup\{u(\eta)+1 : \eta < \xi\}$

if $\eta \leq_S \xi$, then $v(\eta) \leq_T v(\xi)$.

$v(\xi) \leq_T u(\xi)$.

(2) $t^0_\xi : M^S_\xi \longrightarrow M^T_{v(\xi)}$

the following diagram commutes

$\eta \leq_S \xi$:

$$\begin{array}{ccc} M^S_\eta & \xrightarrow{t^0_\eta} & M^T_{v(\eta)} \\ S\downarrow & & T\downarrow \\ M^S_\xi & \xrightarrow{t^0_\xi} & M^T_{v(\xi)} \end{array}$$

$t^1_\xi = \hat{i}^T_{v(\xi), u(\xi)} \circ t^0_\xi$.

$E^T_{u(\xi)} = t^1_\xi(E^S_\xi)$; for $\eta > \xi$, $t^0_\eta \restriction lh(E_\xi)+1 =$ $t^1_\xi \restriction lh(E_\xi)+1$.

(3) $\eta+1 < lh(S)$, $\beta = S\text{-pred}(\eta+1)$,

$\beta^* = T\text{-pred}(u(\eta)+1)$, then $\beta^* \in [v(\beta), u(\beta)]_T$.

$$M^S_\eta \xrightarrow{t^1_\eta} M^T_{u(\eta)}$$

$$E^S_\eta \longmapsto E^T_{u(\eta)}$$

$$\begin{tikzcd}
M^S_\beta \arrow[r, "t^0_\beta"] \arrow[d, "E^S_\eta"'] & M^T_{v(\beta)} \arrow[r, "T"] & M^T_{\beta^*} \arrow[r, dashed] \arrow[d, "E^T_{u(\eta)}"] & M^T_{u(\beta)} \\
M^S_{\eta+1} \arrow[rr, dashed, "t^0_{\eta+1}"'] & & M^T_{\underbrace{u(\eta)+1}_{=\, v(\eta+1)}} &
\end{tikzcd}$$

$t^0_{\eta+1}$ given by shift

$$\Phi^{W(S,T)} : S \longrightarrow W(S,T)$$

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<u>one step embedding normalization</u>

- $W(S, S', F)$
- $\Phi^{W(S, S', F)}$
- $\pi^{W(S, S', F)}$

S, S' are trees with last models,
F on last model of S'.

let α least s.t. F is on $\mathcal{M}^{S'}_\alpha$ – seq.

let β least s.t. $\beta = \alpha \vee \operatorname{crit}(F) < \lambda(E^{S'}_\beta)$.

supp. $S \restriction \beta+1 = S' \restriction \beta+1$,

$\lambda(E^S_\beta) \geq \lambda(E^{S'}_\beta)$.

<u>case 1.</u> (dropping case)

F is appl. to $\lhd\ M^S_\beta \mid lh(E^S_\beta)$

$W = S' \restriction \alpha+1 ^\frown \langle F \rangle$.

$\Phi = id : S \restriction \beta+1 \to W$

<u>case 2.</u> (non-dropping) otherwise.

$W = S' \restriction \alpha+1 ^\frown \langle F \rangle ^\frown i^{M^S_\beta}_F \,/\!/\, S^{> \operatorname{crit}(F)}$

Φ

$$u(\xi) = \begin{cases} \xi & \text{if } \xi < \beta \\ \alpha+1+(\xi-\beta) & \text{if } \xi \geq \beta . \end{cases}$$

$u : [\beta, lh(S-1)) \xrightarrow[\text{onto}]{} [\alpha+1, lh(W-1))$.

$E^W_{u(\xi)} = t'_\xi(E^S_\xi)$.

$$M^S_\beta \xrightarrow{t^1_\beta} M^W_{\alpha+1}$$

$$t^1_\beta = i^{M^S_\beta}_F, \quad E^W_{\alpha+1} = i^{M^S_\beta}_F(E^S_\beta)$$

$$t^0_{\beta+1} = t^1_{\beta+1}.$$

$$t^0_\xi = id \quad \text{for } \xi \leq \beta.$$

<u>def.</u> $\vec{S} = (S_\xi, F_\eta, \Phi^{\zeta,\xi} : \eta+1, \xi < lh(\vec{S}), \zeta \leq_{\vec{S}} \xi)$

is an <u>m-tree</u> (meta tree) iff

(0) $lh(\vec{S})$ is an ordinal

$\leq_{\vec{S}}$ is a tree order on $lh(\vec{S})$

(1) for $\zeta \leq_{\vec{S}} \eta \leq_{\vec{S}} \xi$,

(a) S_ξ is a normal tree on M in a last model

(b) if $\eta+1 < lh(\vec{S})$, F_η is on the seq. of last model of S_η

(c) $\Phi^{\eta,\xi} : S_\eta \longrightarrow S_\xi$

(2) (normality)

for $\eta+1 < lh(\vec{S})$,

(a) for $\xi < \eta$ $lh(F_\xi) < lh(F_\eta)$

(b) for $\xi = \vec{S}-\text{pred}(\eta+1)$,

(i) ξ least s.t. $\text{crit}(F_\eta) < \lambda(F_\xi)$

(ii) $S_{\eta+1} = W(S_\xi, S_\eta, F_\eta)$

(iii) $\Phi^{\xi,\eta+1} = \Phi^{W(S_\xi, S_\eta, F_\eta)}$

~~def~~ <u>def.</u> $\vec{S}$ <u>drops at</u> $\eta+1$ iff we're in the dropping case of $W(S_\xi, S_\eta, F_\eta)$.

(3) for $\lambda < lh(\vec{S})$, $b = [0,\lambda)_{\vec{S}}$,

$b \subset \lambda$ cofinal

$S_\lambda = \text{dir lim}\{S_\eta, \Phi^{\eta,\xi} : \eta \leq \xi \in b\}$.

$$\vec{W}(S,T) = (W_\xi, F_\xi, \Phi^{\eta,\xi} : \eta \leq_T \xi)$$

- $W_0 = S$ $\qquad \pi_\xi : M_\xi^T \to$ lam model of W_ξ
- $F_\xi = \pi_\xi(E_\xi^T)$
- $\xi = T-\text{pred}(\eta+1) \Rightarrow \text{crit}(F_\eta) < \lambda(F_\xi)$.

$$W_{\eta+1} = W(W_\xi, W_\eta, F_\xi).$$

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analogies:

models M $\longleftrightarrow$ normal trees

ultrapowers $\longleftrightarrow$ one step embedding normalization

iterated trees $\longleftrightarrow$ m-trees

elementary embedding $\longleftrightarrow$ pseudo hull embedding

Sishind 2

m-tree.

$$\vec{S} = (S_\xi,\ F_\xi,\ \Phi^{\eta,\xi} : \eta \leq_{\vec{S}} \xi)$$

example: $W(\mathcal{T}, \mathcal{U})$.

$\mathcal{T}$ normal tree

$\langle \mathcal{T} \upharpoonright \xi+1,\ E_\xi^{\mathcal{T}} : \xi < lh(\mathcal{T}) \rangle$

$\eta = T\text{-pred}(\xi+1)$

$$\mathcal{T} \upharpoonright \xi+2 = W(T\upharpoonright \eta+1,\ T\upharpoonright \xi+1,\ E_\xi^{\mathcal{T}}) = \mathcal{T}\upharpoonright\xi \frown E_\xi^{\mathcal{T}}$$

models M $\longleftrightarrow$ trees S

ultrapowers by F $\longleftrightarrow$ one-step embedding norm $W(S, F)$

trees $\longleftrightarrow$ m-trees

elementary embedding $\longleftrightarrow$ pseudo hull embeddings

shift lemma. $\Psi : S' \longrightarrow T'$

$F \longmapsto G$

there is a unique Γ

(M, Σ) a mouse pair

S, T, S', T'

by Σ.

(a)

(b)
$$\begin{tikzcd} S \arrow[r, "\Psi"] \arrow[d] & T \arrow[d] \\ W(S,S',F) \arrow[r, "\Gamma"'] & W(T,T',F) \end{tikzcd}$$

$\Phi : \mathcal{U} \to \mathcal{V} \subset \mathcal{V}^*$, $\quad lh(\mathcal{U}) = \gamma+1$,
$\quad lh(\mathcal{V}) = \delta+1$

$$\begin{tikzcd} & & M^{\mathcal{V}}_{\delta} \\ \mathcal{M}^{\mathcal{U}}_{\gamma} \arrow[rr, "t^0_\gamma"] \arrow[urr, dashed] & & \mathcal{M}^{\mathcal{V}}_{v(\gamma)} \arrow[u] \end{tikzcd}$$

$\mathcal{U}$ $\quad\quad$ $\mathcal{V}$

Φ is <u>cofinal</u> iff
$v(\gamma) \leq_{\mathcal{V}} \delta$

in this case, we let
$u(\gamma) = \delta$ and
$t'_\gamma = \hat{\imath}^{\mathcal{V}}_{v(\gamma), u(\gamma)} \circ t^0_\gamma$.

Let $\alpha =$ least s.t. F is on the $M^{S'}_\alpha$-seq.
α^* $\quad$ F^* $\quad$ $M^{T'}_{\alpha^*}$-seq.

$\beta =$ least s.t. $\beta = \alpha$ or $crit(F) < \lambda(E^{S'}_\beta)$
β^* $\quad$ α^* $\quad$ F^* $\quad$ $E^{T'}_{\beta^*}$

Supp. $\Psi \restriction \beta+1 = \Psi' \restriction \beta+1$ (so $W(S, S', F)$ exists.)

(mean: $S \restriction \beta+1 = S' \restriction \beta+1$,

$u \restriction \beta = u' \restriction \beta$,

$T \restriction \nu(\beta) = T' \restriction \nu'(\beta)$

$\Rightarrow$ $t^0_\xi = t^{0\prime}_\xi$, $\xi \le \beta$

$t^1_\xi = t^{1\prime}_\xi$, $\xi < \beta$

[notation: $\Psi = (u, v, \langle t^0_\xi \rangle, \langle t^1_\xi \rangle)$

$\Psi' = (u', v', \langle t^{0\prime}_\xi \rangle, \langle t^{1\prime}_\xi \rangle)$]

supp. also $T \restriction \beta^* + 1 = T' \restriction \beta+1$.

$\lambda(E^S_\beta) \ge \lambda(E^{S'}_\beta)$.

<u>then</u> (1) (a) $\alpha^* \in [\nu'(\alpha), u'(\alpha)]_T$,

(b) $\beta^* \in [\nu(\beta), u(\beta)]_T$,

(c) $W(T, T', F^*)$ exists.

[more notation: $\Phi = \Phi^{W(S, S', F)} = (\bar{u}, \bar{v}, \langle \bar{t}^0_\xi \rangle, \langle \bar{t}^1_\xi \rangle)$.

$\overset{*}{\Phi} = \Phi^{W(T, T', F^*)} = (\overset{*}{u}, \overset{*}{v}, \langle \overset{*}{t}{}^0_\xi \rangle, \langle \overset{*}{t}{}^1_\xi \rangle)$.]

$W = W(S, S', F)$, $\overset{*}{W} = W(T, T', F^*)$

(2) there is a unique cardinal Γ s.t.

(1) (a) $\Gamma \restriction \alpha+1 = \Psi' \restriction \alpha+1$.

(b) $\hat{u}(\alpha) = \alpha^*$.

[even more notation: $\Gamma = \langle \hat{u}, \hat{v}, \ldots \rangle$.]

(c) the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "\Psi"] \arrow[d, "\phi"'] & T \arrow[d] \\ W \arrow[r, "\Gamma"'] & W^* \end{tikzcd}$$

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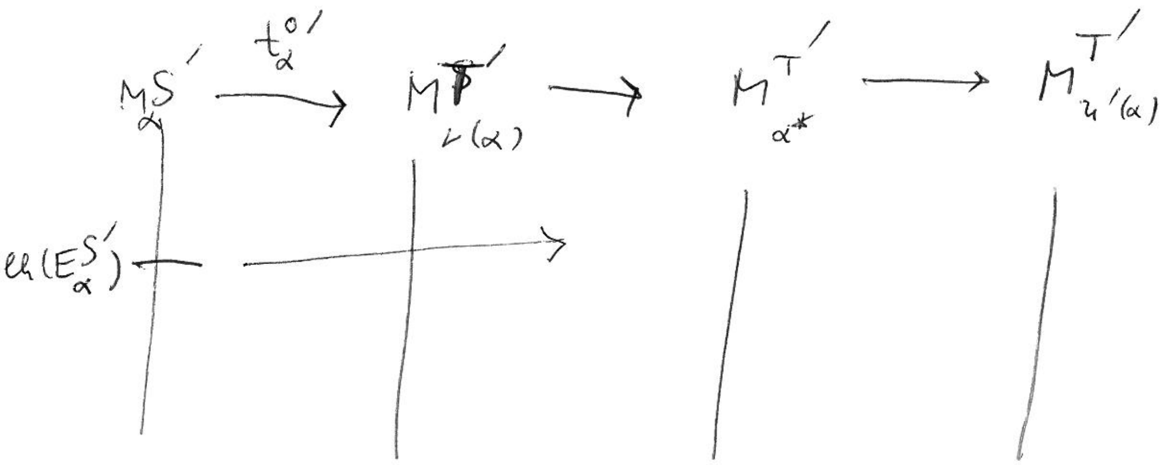

$t_\alpha^{1'} \restriction lh(E_\alpha^{S'})+1 =$ last

t^1 – embedding $\restriction lh\ t_\alpha^{S'}+1$

$t_\alpha^{1'}(F) = t_j^{1'}(F) = F^*$

$\hat{\imath}^{I}_{\iota(\beta),\beta^*} \circ t_\beta^0(\lambda(E_\beta^S)) \geq \hat{\imath}^{T}_{\iota(\beta),\beta^*} \circ t_\beta^0(\lambda(E_\beta^{S'}))$

$\shortparallel$

version $\quad (\lambda(E_\beta^{S'}) = \lambda(E_{\beta^*}^{I'}))$

$\underline{\Gamma\restriction\alpha+2}$

$\Gamma\restriction\alpha+1 = \Psi\restriction\alpha+1$.

$\hat{u}(\alpha) = \alpha^*$

$\hat{t}^1_\alpha = \hat{\iota}^{W^*}_{\iota(\alpha),\alpha^*} \circ \overset{*}{t}{}^0_\alpha$.

$$\begin{tikzcd} M^W_\alpha \arrow[r, "\hat{t}^1_\alpha"] & M^{W^*}_{\alpha^*} \end{tikzcd}$$

$$F \longmapsto F^*$$

$\hat{t}^0_{\alpha+1} : M^W_{\alpha+1} \longrightarrow M^{\overset{*}{W}}_{\alpha^*+1}$.

$$\begin{tikzcd} M^S_\beta \arrow[r, "t^0_\beta"] \arrow[d, "F"'] & M^T_{\iota(\beta)} \arrow[r] & M^T_{\beta^*} \arrow[r, "\text{crits} > \text{crit}(F^*)"] \arrow[d, "F^* = \overset{*}{t}{}^1_\beta"] & M^J_{\xi+1} \arrow[r] \arrow[d, "\overset{*}{t}{}^0_{\xi+1}"] & M^J_{u(\beta)} \arrow[d, "\overset{*}{t}{}^0_{u(\beta)}"] \\ M^W_{\alpha+1} \arrow[rr, dashed, "\text{shift}", "\hat{t}^0_{\alpha+1}"'] & & M^{W^*}_{\alpha^*+1} \arrow[r, "\overset{*}{t}{}^1_\xi(E^J_\xi)"'] & {} \arrow[r] & M^{W^*}_{u^*\circ u(\beta)} \end{tikzcd}$$

$\iota^{M^S_\beta}_F = \bar{t}^1_\beta$

$$\bar{u}(\xi) = \begin{cases} \xi, & \xi < \beta \\ \alpha+1+(\xi-\beta), & \xi \geq \beta \end{cases}.$$

$\bar{u} : [\beta, lh(S)) \xrightarrow[\text{onto}]{} [\alpha+1, lh(W))$.

$t^0_{\bar{u}(\xi)}$, $t^1_{\bar{u}(\xi)}$.

$$\begin{tikzcd}
M^S_\xi \arrow[rrr, "t^1_\xi"] \arrow[ddd, "\bar{t}^1_\xi"'] & & & M^T_{u(\xi)} \arrow[ddd, "\overset{*}{t}{}^1_{u(\xi)}"] \\
& E^S_\xi \arrow[r, mapsto] \arrow[d, mapsto] & E^{\mathcal{T}}_{u(\xi)} \arrow[d, mapsto] & \\
& E^W_{\bar{u}(\xi)} \arrow[r, mapsto] & E^W_{\overset{*}{u}\circ u(\xi)} & \\
M^W_{\bar{u}(\xi)} \arrow[rrr, "\hat{t}^1_{u(\xi)}"'] & & & M^{W^*}_{\overset{*}{u}\circ u(\xi)}
\end{tikzcd}$$

$\eta = \cancel{\xi}\ S\text{-pred}(\xi+1)$ $\quad$ $\eta^* = T\text{-pred}(u(\xi)+1)$

$\bar{\eta} = W\text{-pred}(\bar{u}(\xi)+1)$ $\quad$ $\tilde{\eta} = W^*\text{-pred}(u^*\circ u(\xi)+1)$

$$\begin{tikzcd}
E^S_\xi \arrow[rrr, "t^0_{\xi+1}"] \arrow[ddd, "\bar{t}^0_{\xi+1}"'] & & & E^{\mathcal{T}}_{u(\xi)} \arrow[ddd, "t^0_{_}"] \\
& M^S_\eta \arrow[ul] \arrow[r] \arrow[d] & M^T_{\eta^*} \arrow[ur] \arrow[d] & \\
& M^W_{\bar{\eta}} \arrow[r] \arrow[dl] & M^{W^*}_{\tilde{\eta}} \arrow[dr] & \\
E^W_{\bar{u}(\xi)} \arrow[rrr, dashed, "\text{shift}", "\hat{t}^0_{\bar{u}(\xi)+1}"'] & & & E^{W^*}_{\overset{*}{u}\circ u(\xi)}
\end{tikzcd}$$