john steel july 16

HOD pair capturing

ref. :

(1) NITCIS
(2) Local hod comparison
(3) notes on hod pair capturing

context : AD^+.

A. mouse pairs. (M, Σ) :

(1) M is a pure extender premouse (λ- indexing)

or

(2) M is a lpm

(3) Σ is a complete iteration strategy for M that

(a) normalizes well :

$$\begin{tikzcd} M \arrow[r, "\mathcal{T}"] \arrow[drr, "W = W(\mathcal{T}, \mathcal{U})"'] & N \arrow[r, "\mathcal{U}"] & P \arrow[d, "\pi"] \\ & & R \end{tikzcd}$$

$(\mathcal{T}, \mathcal{U})$ by Σ iff $W(\mathcal{T}, \mathcal{U})$ by Σ and

$$\Sigma^{\pi}_{W,R} = \Sigma_{(\mathcal{T},\mathcal{U}),P}$$

(b) Σ has strong hull condensation :

if $\Phi : \mathcal{T} \longrightarrow \mathcal{U}$ is a pseudo hull embedding + $\mathcal{U}$ is by Σ, then $\mathcal{T}$ is by Σ.

an <u>lbr hod pair</u> is a mouse pair (M, Σ) s.t. M is a lbm.

—

if $\pi : (M, \Sigma) \longrightarrow (Q, \Psi)$, π is elementary iff π is $\underset{=}{k(M)}$- elementary for M to Q,

degree of soundness
of M

and $\Sigma = \Psi^{\pi}$ [where $\Psi^{\pi}(\mathcal{I}) = \Psi(\pi \mathcal{I})$ is the pullback strategy].

$\Sigma_s(t) = \Sigma(s^\frown t)$, s, t stacks of normal trees

for $\mathcal{I}$ on M, $(M_\alpha^{\mathcal{I}}, \Sigma_\alpha^{\mathcal{I}}) = \alpha^{\underline{th}}$ pair.

<u>facts</u>: (1) $\underset{\substack{\beta\gamma \\ \shortparallel}}{\hat{\imath}}$ is elementary (in the category of pairs)

largest fragment of an emb. from $\beta^{\underline{th}}$ model to $\gamma^{\underline{th}}$ model

(2)

$$(M_\alpha^{\mathcal{I}}, \Sigma_\alpha^{\mathcal{I}}) \mid lh(E_\alpha^{\mathcal{I}}) =$$

$$(M_\beta^{\mathcal{I}}, \Sigma_\beta^{\mathcal{I}}) \mid lh(E_\beta^{\mathcal{I}}), \quad \alpha \leq \beta.$$

(3) dodd-jensen (iteration maps are pointwise minimal)

(4) comparison of pairs.

$$\begin{tikzcd}
(P, \Sigma) \arrow[dr, dashed] & \\
& (R, \Omega) \\
(Q, \Psi) \arrow[ur] &
\end{tikzcd}$$

gives a mouse order $\leq^*$ on pairs.

(5) direct limit generated by (P, Σ),

$\mathcal{F}(P, \Sigma)$ = system of all ctbl. iterates.

so $M_\infty(P, \Sigma)$ = direct lim of $\mathcal{F}(P, \Sigma)$

each $M_\infty(P, \Sigma) \in$ HOD.

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(6) Σ is OD(π), wh $\pi : P \longrightarrow M_\infty(P, \Sigma)$ is the iteration map by Σ.

(7) Solidity, universality, Condensation.

(8) Zeman thm. 9.3.2. (trang-steel)
(condensation)

dodd solidity

$\square_\kappa$ iff κ is not supercompact.

(9) the hod mouse constructions of a Γ-woodin universe $(N^*, \Sigma^*, \delta^*)$ doesn't break down.

(10) Σ has very strong hull condensation.

(11) strategy fully normalizes well

$$\begin{tikzcd} M \arrow[r, "\mathcal{T}"] \arrow[rr, bend right, "X(\mathcal{T},\mathcal{U})"'] & N \arrow[r, "\mathcal{U}"] & P \end{tikzcd}$$

(12) also true for inf. stacks; in part., have a unique normal tree $M \longrightarrow M_\infty(M,\Sigma)$ s.t. every ctble. weak hull is by Σ.

(13) Σ^{rel} (= Σ on trees $\mathcal{T}$ that have extenders not dropping on main branch) is $|M_\infty(P,\Sigma)|$-suslin (and not κ-suslin for any $\kappa < |M_\infty(P,\Sigma)|$ by kunen–martin).

[prf.: $\mathcal{T}$ by Σ iff there is a weak hull embedding of $\mathcal{T}$ into $\mathcal{U}$ (cf. (12)).]

<u>defn.</u> HPC is the statement: for every suslin-co-suslin $A \subset \mathbb{R}$, there is a lbr hod pair (P,Σ) s.t. $A \leq_w \mathrm{Code}(\Sigma)$

LEC: same, but (P, Σ) is a pure extender pair.

remark: LEC $\Rightarrow$ HPC.

LEC $\Leftrightarrow$ $\forall x, y$ ($x \in OD_y$ iff

$x \in$ iterate mouse over y).

NLE is the statement:
there is no cth. ω_1-iterable pure extender model with a long extender on its sequence.

<u>conjecture</u>: (1) AD^+ + NLE $\Rightarrow$ LEC.
(2) AD^+ + NLE $\Rightarrow$ HPC.

<u>partial results</u>:

<u>def.</u> $\underset{\sim}{\Delta}^h_{max} = \{A \subset \mathbb{R} : \exists$ lbr hod pair (P, Σ)
s.t. $A \leq_w \mathrm{Code}(\Sigma)\}$.

$\underset{\sim}{\Delta}^p_{max} = \{$ .... pure ext pair .... $\}$

$\underset{\sim}{\Delta}^p_{max} \subset \underset{\sim}{\Delta}^h_{max} \subset S_\infty \cap \breve{S}_\infty$

$\shortparallel$

Suslin sets

$= ?$

assume $S_\infty \cap \breve{S}_\infty \not\subseteq \underset{\sim}{\Delta}^h_{max}$.

want : strategy for pm in a long extender.

have $\underset{\sim}{\Delta}^h_{max} \subset \Gamma_0 \subsetneqq \Gamma_1 \subsetneqq \Gamma_2$

Γ_1 ind. like pt. class with scale property

let A_i = complete Γ_i set. assume

$A_i^\# <_w A_{i+1}$

let $(N^*, \Sigma^*, \delta^*)$ be a Γ_2 – woodin, capturing A_0, A_1, A_2.

Σ^* on knows stacks in $N^* | \delta^*$.

let $\mathbb{C}$ = hod mouse constr. of $N^* | \delta^*$.

$\mathbb{C}$ consists of pairs $(M^{\mathbb{C}}_{\nu,k}, \Omega^{\mathbb{C}}_{\nu,k})$, $\nu < \delta^*$, $k < \omega$.

come from Σ^*

let $H = M^{\mathbb{C}}_{\delta^*,0}$, $\Omega = \Omega^{\mathbb{C}}_{\delta^*,0}$.

(H, Ω) can be thought of as an lbr pair in V ($\models AD^+$).

wat to use $\Omega^{C}_{j,k} \in \underset{\sim}{\Delta}^{h}_{max}$, all j,k, to get some $(M_{\eta,0}, \Omega_{\eta,0})$ to which we can add a long extender.

<u>defn.</u> $\kappa_\alpha = \alpha^{\underline{th}}$ strong or limit of strongs in H.

we'll show:

(1) $H \models$ "κ_0 is a limit of woodin cardinals"

(2) $L(\underset{\sim}{\Delta}^{h}_{max}, \mathbb{R}) \models AD_{\mathbb{R}} + \theta$ reg.

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assume AD^+ thruout.

HPC = hod pair capturing

$\forall A \in S_\infty \cap \breve{S}_\infty$ $\exists$ lbr hod pair (P, Σ) s.t.

$A \leq_w Code(\Sigma)$.

assume $\neg$HPC : show there is an iterate pure ext. pm with a long extender.

let $\underset{\sim}{\Delta}^h_{max} = \{A \subset \mathbb{R} : \exists$ lbr hod pair (P, Σ)
$A \leq_w code(\Sigma)\}$

we'll show $L(\overset{\underset{\sim}{\Delta}^h_{max}}{\overset{\shortparallel}{\underset{\sim}{\Delta}_{max}}}, \mathbb{R}) \cap \mathcal{P}(\mathbb{R}) = \underset{\sim}{\Delta}_{max}$

and $L(\underset{\sim}{\Delta}_{max}, \mathbb{R}) \models AD_\mathbb{R} + \Theta$ regular

let $\underset{\sim}{\Delta}_{max} \subsetneq \Gamma_0 \subsetneq \Gamma_1 \subsetneq \Gamma_2$ ind. like, scale prop.

$A_0^\# \leq_w A_1^\# \leq_w A_2^\#$

let $(N^*, \Sigma^*, \delta^*)$ capture A_2.

can arrange $\forall \eta < \delta^*$ there is a term τ_η s.t. for g $Col(\omega, \eta)$-gen. $/ N^*$,

τ_η^g is a $\leq \delta^*$-u. baire code for $\Sigma^* \upharpoonright$ trees on $N^*|\eta$ in $N^*|\delta^*[g]$.

let $\mathbb{C}$ = the hmc of N^*.

hmc = hod mouse construction

gives $(M^{\mathbb{C}}_{\nu,k}, \Omega_{\nu,k})$, $\nu \leq \delta^*$, $k < \omega$, lbr pairs.

$H = M_{\delta^*,0}$, $\Omega = \Omega_{\delta^*,0}$.

(H, Ω) is an lbr pair in V.

each $\Omega_{\nu,k} \in \underset{\sim}{\Delta}_{max}$.

<u>fact</u>. $\forall \eta < \delta^*$ there is a term τ_η s.t. $\forall g$ $Col(\omega,\eta)$-generics, $\Omega_{\eta,0} = (A_0)_{\tau^g_\eta}$: correctness

$\uparrow$ a real

of $N^*[g]$ and $\Omega_{\eta,0} = (A_0)_x$, some $x \in V$.

<u>defn</u>. $\kappa_\alpha = \alpha^{th}$ strong or lim of strongs in H.

so $\kappa_0, \kappa_{\alpha+1}$ are strong in N^*.

<u>~~vague~~ goal</u>: for club many $\eta < \delta^*$,

$$\mathcal{P}(\eta)^H \subset L(T_{\Gamma_1}, N^*|\eta).$$

T_{Γ_1} : tree of Γ_1 scale on A_1

this would do it.

"universality argument"

easy to see this if η is from a club s.t.
$H|\eta = M_{\eta,0}$ and η a cutpoint in H.

gives woodin's proof that $\exists$ non tame mouse if
"goal" is false.

<u>thm I.</u> $H \models \kappa_0$ is a limit of woodins.

proof: let $\nu < \kappa_0$ be a cutpoint of H, a cardinal in H, and

$$(R, \Phi) = (M_{\nu,0}, \Omega_{\nu,0})$$
$$\shortparallel$$
$$(H|\nu, \Omega_{H|\nu})$$

supp. there are no woodins $> \nu$ in H.

let δ = ~~least~~ some Γ_1 woodin of $N^* > \nu$.

= woodin in $L(T_{\Gamma_1}, N^*|\delta)$

can assume $H|\delta = M_{\delta,0}$.

let Q = the Q-str. showing δ is not woodin in H. show $Q \in L(T_{\Gamma_1}, N^*|\delta)$ to get a $\lightning$.

let $\sigma = \mathbb{R}^{N^*[g]}$, g $\mathrm{Col}(\omega, \delta)$ - gen.

$\Psi = \Omega_q$.

(*) show Q is OD in $L(A_1 \cap \sigma, \sigma)$. note

$\mathrm{Code}(\Psi) = (A_0)_x$, some x in σ.

we may assume $\mathrm{code}(\Phi) = (A_0)_y$, $y \in N^*[h]$,

h on $\mathrm{Col}(\omega, \iota)$, $h \in N^*[g]$.

<u>enough</u> then: $Q \in L(T_{\Pi_1}, N^*|\delta, h)$.

~~shows~~ (*) more prec.:

show Q is OD in $L(A_1 \cap \sigma, \sigma)^{\models AD}$ from $A_1 \cap \sigma$,

$N^*|\delta$, h.

so we can form $M_\infty(Q, \Psi)$ in $L(A_1 \cap \sigma, \sigma)$.

have $M_\infty(Q, \Psi)$ is OD here

(arg. th for the rank in the mouse order).

have $\pi : Q \longrightarrow M_\infty(Q, \Psi)$.

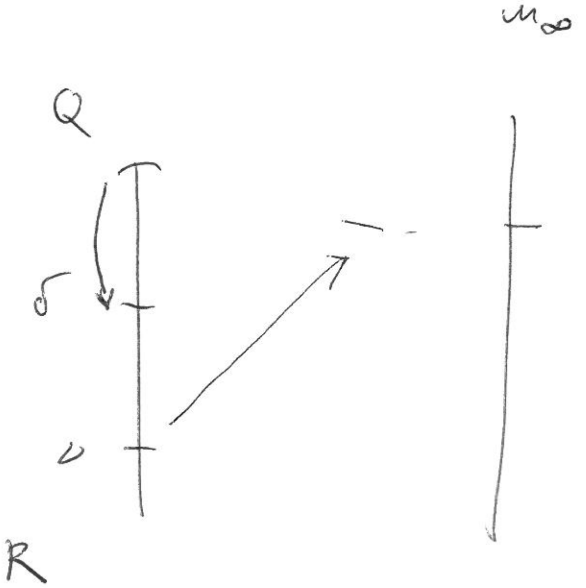
$\pi = i_{\mathcal{U}}$, for $\mathcal{U}$ a normal tree on Q,

all ctbly weak hulls by Ψ.

want to compute $\mathcal{U}$ from $Q|\delta = H|\delta$

(which is 1^{st} order over $N^*|\delta$)

using (R, Φ),



$\mathcal{U} = \mathcal{U}_0 {}^\frown \mathcal{U}_1$.

$\mathcal{U}_0$ = maximal part on R.

$\mathcal{U}_1$ on $M_\infty^{\mathcal{U}_0}$ above $i^{\mathcal{U}_0}(\iota)$

a tree on $M_\infty^{\mathcal{U}_0} | i^{\mathcal{U}_0}(\delta)$.

then Q = unique extension of $H|\delta$ proj. to δ and s.t. $i^{\mathcal{U}}$ extends to an embedding of it into $M_\infty(Q, \Psi)$.

<u>lem.</u> let $\sigma = \mathbb{R} \cap N^*[g]$, g $Col(\omega, <\kappa_0)$-gen.

then there is h s.t. $\sigma = \mathbb{R} \cap H[h]$, h on $Col(\omega, <\kappa_0)$, and

$$(\mathrm{Hom}^*_h)^H = \Delta_{max}(\sigma)$$

notation: for $\sigma = \mathbb{R} \cap N^*[k]$,
k size $< \delta^*$-generic.

$$A_i(\sigma) = A_i \cap \sigma$$

$$\Gamma_i(\sigma) = \{ (A_i(\sigma))_x : x \in \sigma \}$$

note: have $j: L(A_i(\sigma), \sigma) \prec L(A_i, \mathbb{R})$, an el. embedding.

$$\underset{\sim}{\Delta}_{max}(\sigma) = \Delta_m(\sigma) = \{ (A_0)_x \cap \sigma : x \in \sigma \wedge (A_0)_x \in \underset{\sim}{\Delta}_{max} \}.$$

since: for $\eta < \kappa_0$,
$\Omega_{H|\eta}$ "$\in$" $(\mathrm{Hom}^*_h)^H$. see §7 of NITCIS.

moreover, they are cofinal in $(\mathrm{Hom}^*_h)^H$.

but also: every $(Q, \psi) \in \Delta_m(\sigma)$ gets out-iterated by some $(H|\eta, \Omega_{H|\eta})$, where $\eta < \kappa_0$.

also: $\forall A \in (\mathrm{Hom}^*_h)^H \; \exists B \in (\mathrm{Hom}^*_h)^H$
s.t. B is not OD for A in $L(\mathrm{Hom}^{*H}_h, \sigma)$.

uses: the strategies $\Omega_{H|\eta}$ for η a cardinal of H, $< \kappa_0$, are all fullness preserving in $L(\mathrm{Hom}^*_h H, \sigma)$.

<u>Corollary</u>. $L(\Delta_m(\sigma), \sigma) \cap \mathcal{P}(\sigma) = \Delta_m(\sigma)$,

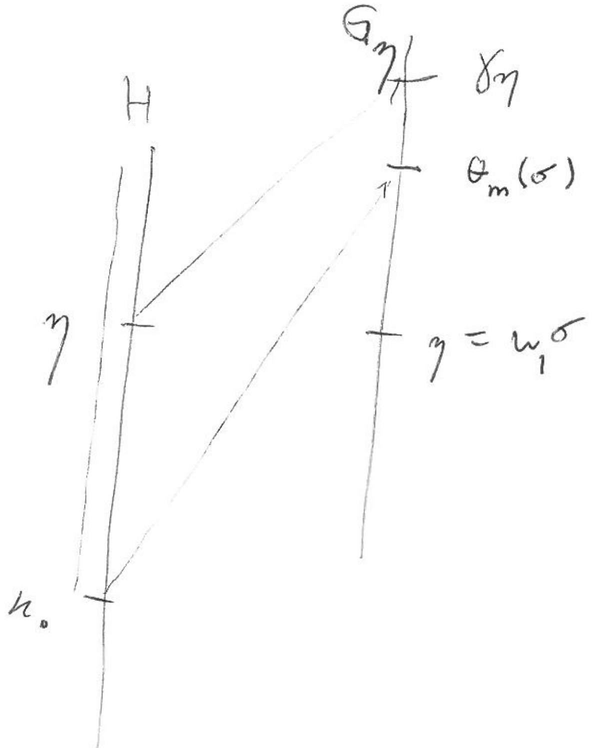
$L(\Delta_m(\sigma), \sigma) \models AD_{\mathbb{R}}$.

Let $\eta \geq \kappa_0$, η inacc. in N^*, a limit of Γ_1 woodins in N^*;

$\sigma = \mathbb{R} \cap N^*[g]$, g $\mathrm{Col}(\omega, <\eta)$

$\mathrm{HOD}^{L(\Delta_m(\sigma), \sigma)} | \Theta_m(\sigma) = \lim \mathcal{I}^*(\eta) = G_\eta$

↑ system of H.



$L(A_1(\sigma), \sigma)$ knows $G_\eta | \gamma_\eta$??

<u>Steel</u> July 20

AD^+. $\underset{\sim}{\Delta}_{max} = \{ A \subset \mathbb{R} : A \leq_w \text{Code}(\Sigma),$

som ~~lp~~ lbr hod pair

$(P, \Sigma) \}$

$$\underset{\sim}{\Delta}_{max} \subsetneq \Gamma_0 \subsetneq \Gamma_1 \subsetneq \Gamma_2$$

$$A_0 \qquad A_1 \qquad A_2$$

$(N^*, \Sigma^*, \delta^*)$ a coarse Γ_2 woodin,

$(H, \Omega) = (M^{\mathbb{C}}_{\delta^*, 0}, \Omega_{\delta^*, 0})$ of N^*

goal: one can expand som $H|\gamma$ to $(H|\gamma, E)$,

long, the structure iterate.

letting $\kappa_\alpha = \alpha^{th}$ strong or limit of strongs

$H \models \kappa_0$ is a limit of woodins.

letting $\sigma = \mathbb{R} \cap N^*[g]$, g on $\text{Col}(\omega, < \kappa_0)$.

then

$$L(\underset{\sim}{\Delta}_m(\sigma), \sigma)$$

is a derived model

$A_i(\sigma) = A_i \cap \sigma$,

$\sigma \in P_{\omega_1}(\mathbb{R})$

$\Gamma(\sigma) = \{ A \cap \sigma :$

$A = A_x$,

som $x \in \sigma \}$

$L(\text{Hom}^*_g, \mathbb{R}^*_g) \models AD_{\mathbb{R}}$.

$L(\underset{\sim}{\Delta}_m(\sigma), \sigma) \cap P(\sigma) = \underset{\sim}{\Delta}_m(\sigma)$.

θ regular : look at $HOD^{L(\underset{\sim}{\Delta}_m(\sigma),\sigma)} | \theta_m(\sigma)$

$= \lim \mathcal{I}(\sigma)$.

$\mathcal{I}^*(\kappa_0)$ = system of all iterates of H via trees in $N^*|\kappa_0$.

$\mathcal{I}^*(\kappa_0)$ is directed ! ← class in $\mathcal{I}(\sigma)$

rmk : for any hod pair (P, Σ) non-dropping iterates $(Q, \Psi), (R, \Lambda)$, the two compare by iterating least extender disagreement (by Σ positional !).

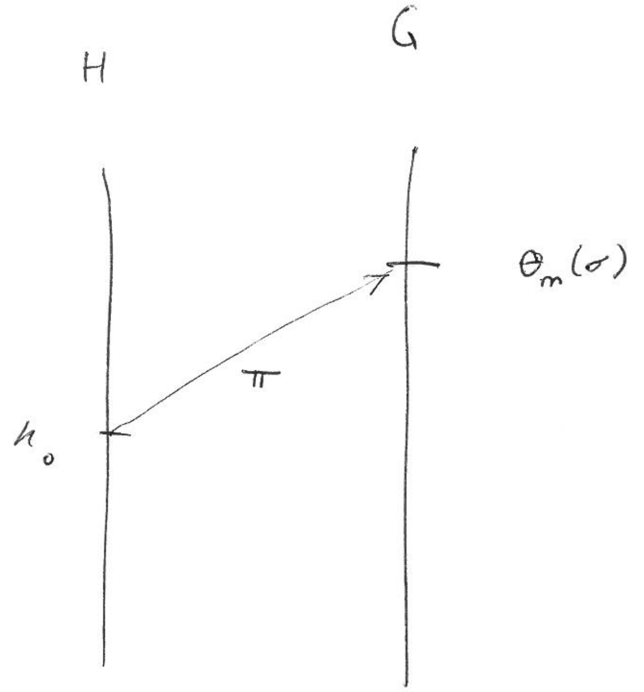
$\mathcal{I}(\sigma)$ = dir system of all $(P, \Sigma) \in \sigma \times \underset{\sim}{\Delta}_m(\sigma)$ that are fullness pres.

let $G_{\kappa_0} = \lim \mathcal{I}^*(\kappa_0)$, so

$HOD^{L(\underset{\sim}{\Delta}_m(\sigma),\sigma)} \trianglelefteq^* G_{\kappa_0} = G$.

let $\pi : H \longrightarrow G$ be the iteration map.

then $\pi(\kappa_0) = \theta_m(\sigma)$.



$G(\Delta_m(\sigma))$ is a "symmetric Vopěnka" extension of G.

$Vop^{<\omega}$. conditions : $A \subset {}^\omega\gamma_1 \times {}^\omega\gamma_2 \times \dots \times {}^\omega\gamma_n$,

$\gamma_1 \lneq \gamma_2 \lneq \dots \lneq \gamma_n < \theta$; $supp(A) = n$,

A OD.

$A \leq B$ iff $supp(A) \supset supp(B)$,

$\forall s \in A,\ s \restriction supp(B) \in B$.

gis $f : \omega \longrightarrow \bigcup_{\gamma < \theta} {}^\omega\gamma$

$G(\Delta_m(\sigma), \sigma)$ is a symm. extension.

$AD_{\mathbb{R}}$ every set of reals is OD from a seq. of homogeneity measures, hence from an ~~$\mathcal{E}$~~ ω-seq. of ordinals, so adding all ω_γ, $\gamma < \theta$, to background model gives all of it.

so $G(\underset{\sim}{\Delta}_m(\sigma), \sigma) \models \theta$ reg. $+ AD_{\mathbb{R}} + \mathcal{P}(\mathbb{R}) = \Delta_m(\sigma)$, θ strong.

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let now $\kappa_0 \leq \eta < \delta^*$, η inacc. limit of Γ woodins in N^*.

let g on $Col(\omega, <\eta)$, $\mathcal{E} = \mathbb{R} \cap N^*[g]$,

$L(\underset{\sim}{\Delta}_m(\mathcal{E}), \mathcal{E}) \models AD_{\mathbb{R}} + \theta$ reg.

have $\mathcal{E}^*(\eta) =$ dir lim system of iterates of H via trees in $N^*|\eta$.

let $G_\eta =$ dir lim

$\pi : H \longrightarrow G_\eta$

$\pi(\kappa_0) = \theta_m(\tau)$.

$G(\underset{\sim}{\Delta}_m(\tau), \tau) \models AD_{\mathbb{R}} + \theta$ reg.

for $\kappa_0 \leq \eta < \gamma < \delta^*$,

η, γ inacc. limit of Π_1-woodins.

$\pi_{\eta\gamma} : G_\eta \longrightarrow G_\gamma$ is the it. map by Ω_{G_η}.
let $\psi_\eta : H \to G_\eta$ it. map by Ω_{G_η}, $\gamma_\eta = \psi_\eta(\eta)$.

G-def.bility conjecture:

let $\kappa_0 \leq \eta < \xi$, η, ξ are inacc. limits of Π_1 woodins in N^*. let g be N^*-generic over $Col(\omega, <\xi)$, $\tau = \mathbb{R} \cap N^*[g]$,
$\sigma = \mathbb{R} \cap N^*[g \restriction (\omega \times \eta)]$.

then:

1. $G_\eta \mid \gamma_\eta$ is def.ble over $L(A_1(\sigma), \sigma)$ from $A_1(\sigma)$, uniformly in σ.

2. (a) $\pi_{\eta,\xi} \restriction (G_\eta \mid \gamma_\eta)$ is def.ble on $L(A_1(\tau), \tau)$ from $A_1(\tau)$, $N|\eta$, and σ (uniformly in τ).

   (b) if η is a limit of strongs in δ^*, then $\pi_{\eta,\xi} = \psi_{\sigma,\tau}$ on γ_η

$\psi_{\sigma,\tau} : L(A_1(\sigma),\sigma) \to L(A_1(\tau),\tau)$ given

by $A_1(\tau)^{\#}_{\cap\sigma} \longrightarrow A_1(\tau)^{\#}$.

rmk.: these are true with

γ_η repl. by $\Theta_m(\sigma)$.

lea. let N^* be as above, and assume Θ-dag. conjecture; then there is an iteration strategy for a mouse with a long extender.

pf.:

claim. let η be a strong limit of strong cardinals in H. let $\xi > \eta$ be a inacc. lim of Γ_1-woodins. let g be $Col(\omega, <\xi)$-generic over N^*, $\tau = \mathbb{R} \cap N^*[g]$.

then f.a. $B \subset \eta$, $B \in H$, B is OD in $L(A_1(\tau),\tau)$ from $N^*|\eta$ as a parameter.

pf. : let $i_0 : H|\eta \to G_\eta / \gamma_\eta$ be the it. map by $\Omega_{H|\eta}$. i_0 is copied.

then $i_0 \in$ first admissible set on $N^*|\eta, \Sigma^* \restriction N^*|\eta$. from $\Phi^*(\eta)$.

now, working in $L(A_1(\tau), \tau)$, pick any h that is $Col(\omega, <\eta)$-gen. over $N^*|\eta$; let $\sigma = \mathbb{R} \cap N^*|\eta[h]$. may assume h is N^*-generic.

let $B \subset \eta$, $B \in H$.

$\pi_{\eta,\xi} \circ i_0(B) \subset \delta < \gamma_F$.

so $\pi_{\eta,\xi} \circ i_0(B)$ is OD in $L(A_1(\tau), \tau)$ for $A_1(\tau)$, by part (1). but

$\pi_{\eta,\xi} \restriction i_0(\eta) = \psi_{\sigma,\tau} \restriction i_0(\eta)$ by part (2) and $\psi_{\sigma,\tau}$ is def.ble from $A_1(\tau)^{(\#)}$ and σ.

so $i_0(B)$ is OD in $L(A_1(\tau), \tau)$ from σ_h for comeager many h.

So $i_0(B)$ is OD in $L(A_1(\tau), \tau)$.

So B is OD in $L(A_1(\tau), \tau)$

$\vdash N^*/\eta$.

Steel July 23

(joint with Sargsyan)

assume AD^+.

$\underset{\sim}{\Delta}^h_{max} = \underset{\sim}{\Delta}_{max} = \{A \subset \mathbb{R} : A \leq_W \text{code}(\Sigma)$, some lbr hod pair $(P, \Sigma)\}$.

assume $S_\infty \cap \breve{S}_\infty \neq \underset{\sim}{\Delta}_{max}$.

so $\supsetneq$

have shown $L(\underset{\sim}{\Delta}_{max}, \mathbb{R}) \models$ "$AD_{\mathbb{R}} + \theta$ reg.,
$\mathcal{P}(\mathbb{R}) = \underset{\sim}{\Delta}_{max}$."

let $\underset{\sim}{\Delta}_{max} \subsetneq \Gamma_0 \subsetneq \Gamma_1 \subsetneq \Gamma_2$

$A_0 \quad A_1 \quad A_2$

$(N^*, \Sigma^*, \delta^*)$ a coarse Γ_2 woodin,

$H = (M^{\mathbb{C}}_{\delta^*, 0}, \Omega^{\mathbb{C}}_{\delta^*, 0})$, $\mathbb{C}$ hod mouse constr. of N^*.

showed: $H \models \kappa_0$ is a limit of woodins

($\kappa_\alpha = \alpha^{th}$ strong or lim of strongs in H.)

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for η inacc. lim of Γ_0 woodins,

g on $\mathrm{Col}(\omega, <\eta)$, $\quad \sigma = \mathbb{R} \cap N^*[g]$.

$\bar{I}^*(\eta) =$ dir lim. system of all iterates of (H, Ω) via trees in N^*/η.

$\tilde{\sigma}H^*(\sigma) =$ hod system of $L(\Delta_m(\sigma), \sigma)$

$G_\eta = \lim \bar{I}^*_\eta$, $\quad \mathrm{HOD}^{L(\Delta_m(\sigma), \sigma)} = \lim \bar{I}^*(\sigma)$.

$\pi : H \longrightarrow G_\eta$

$\pi(\kappa_0) = \theta_m(\sigma)$

$G = G_\eta(\theta_m(\sigma))$

Let $\gamma_\eta = \pi(\eta)$.

G – def. conjecture.

(1) $G_\eta \mid \gamma_\eta$ is def. in $L(A_1(\sigma), \sigma)$ from $A_1(\sigma)$ uniformly in σ.

(2) $\psi_{\sigma,\tau} \restriction \gamma_\eta = \pi_{\eta,\xi} \restriction \gamma_\eta$

$\pi_{\eta\xi} : G_\eta \to G_\xi$

$\psi_{\sigma,\tau} : L(A_1(\sigma), \sigma) \to L(A_1(\tau), \tau)$.

Saw: G-det. conj. $\Longrightarrow$

some $M^{\mathbb{C}}_{\nu,k}$ can be expanded to an iterable

mouse with a long extender.

(saw: for club many $\eta < \delta^*$,

$\mathcal{P}(\eta) \cap H \subset L(T_{\Gamma_1}, N^*|\eta)$.

~~[illegible]~~ Say $M = L_\alpha(N^*, T_{\Gamma_1}) \models ZFC^-$.

$M_\eta = \mathrm{Hull}^M(\eta)$

$\pi_\eta : M_\eta \longrightarrow M$, $\sigma_{\eta\xi} : M_\eta \rightarrow M_\xi$

$\sigma_{\eta\xi} : M_\eta \rightarrow M_\xi$, $\sigma_{\eta\xi} = \pi_\xi^{-1} \circ \pi_\eta$

$\vec{M}$ in $L(N^*, T_{\Gamma_1})$.

$\left.\begin{array}{l}\mathcal{P}(\eta) \cap H \subset M_\eta \text{ for club many } \eta \\ \mathrm{crit}(\pi_\eta) = \eta.\end{array}\right\}$ call C the club.

in N^*, we have $j : V \longrightarrow Q$ witnessing

δ^* is woodin w.r.t. $(\vec{M}, \vec{\sigma})$, $\mathrm{crit}(j) \in C$.

$$\begin{array}{ccl} & & j(M_\eta) \\ & \nearrow & \\ & & M_\xi^Q = M_\xi^\nu,\quad \xi \in C. \\ M_\eta & & M_\eta \end{array}$$

can show the minimal lay extender of $\sigma_{\eta,\xi}$ can be added to the seq. of H^Q.

(see Neeman–Steel, "equiconsistencies at subcompacts.")

rmk: $G_\eta \restriction \theta_m(\sigma)$ is defnable as in (1).

lem. $G_\eta \restriction \theta_m(\sigma)^{+G_\eta}$ is def.ble in $L(A_1(\sigma),\sigma)$ from $A_1(\sigma)$.

proof: assume $AD_{\mathbb{R}}$. (Solovay measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$.) have $\sigma = \mathbb{R} \cap N^*[g]$, g on $Col(\omega, <\eta)$.

claim. ~~AD~~ TFAE.

(a) $(G|\theta_m, \Omega_G) \trianglelefteq (M,\Sigma) \trianglelefteq (G|\theta_m^+, \Omega_G)$ and $\rho_\omega(M) = \theta_m$.

(b) $L(A_1(\sigma),\sigma) \models \Vdash^{Col(\omega,\mathbb{R})}$ (M,Σ) is the least hod pair extending $(G|\theta_m, \Omega_{G|\theta_m})$ and $\rho_\omega(M) = \theta_m$.

reason: let $T^* = \prod_{\tau \in \mathcal{P}_{\omega_1}(\mathbb{R})} T_{\Gamma_1}$ / Solovay measure

$S^* = \prod_{\tau \in \mathcal{P}_{\omega_1}(\mathbb{R})} T_{\check{\Gamma}_1}$ / Sol. measure.

(T^*, S^*) are abs. complementing in $L(A_1(\sigma), \sigma)[h]$,

h $Col(\omega, \mathbb{R})$.

T^*, S^* are correct in $N^*[g][h]$ for Γ_1, $\check{\Gamma}_1$.

<u>alternative argmt</u>:

Say M is good iff

(1) $G | \theta_m(\sigma) \trianglelefteq M$, $\rho_\omega(M) = \theta_m(\sigma)$, and

(2) $\forall^* X \in \mathcal{P}_{\omega_1}(M)$, let

$\underset{X}{\pi} : \underset{X}{N} \to M$, $\kappa = \sup X \cap \theta_m(\sigma)$.

trans

$Q = Hull^M(\kappa \cup X)$

$\tau : \underset{X}{N} \to Q$.

then $Q \trianglelefteq G | \theta_m$ and there is a unique Ψ

s.t. (Q, Ψ) is a lbr hod pair and

$\tau \upharpoonright \pi_X^{-1}(\theta_m) =$ iteration map by Ψ.

rmk: $\operatorname{cof}(\theta_{max})^{\text{AD-world}} > \omega$.

rmk: so $\operatorname{cof}(\overbrace{\theta_m(\sigma)^{+G}}^{=\alpha}) = \omega$

in $L(A_1(\sigma), \sigma)$.

(has cf ω in $N^*[g]$, also $\alpha < \theta^{L(A_1(\sigma),\sigma)}$; ordinals $< \theta^{L(A_1(\sigma),\sigma)}$ are closed ~~for~~ ω-seq. in $N^*[g]$.)

remark: can show: if $\alpha < \delta_\eta$ and a cardinal in G, ~~α~~ and $G|\alpha \in L(A_1(\sigma), \sigma)$, then

(1) $\operatorname{cf}(\alpha) = \omega$ in $N^*[g]$ $\Rightarrow$ $G|\alpha^+ \in L(A_1(\sigma), \sigma)$.

(2) $\operatorname{cf}(\alpha) > \omega$ and G-def. conjecture true "below images of α," then using (2)

$$G|\alpha^{+G} \in L(A_1(\sigma), \sigma).$$

lem. with N^*, etc. as above, $\kappa_0 \leq \eta < \xi$, inacc. limits of Γ_0 woodins

$\sigma = \mathbb{R} \cap N^*[g]$ in $\operatorname{Col}(\omega, <\eta)$,

$\tau = \mathbb{R} \cap N^*[g^\frown h]$ on $\mathrm{Cor}(\omega, <\xi)$,

then $\pi_{\eta\xi} \restriction \theta_m(\sigma)^+ = \psi_{\sigma,\tau} \restriction \theta_m(\sigma)^+$.

pf.:

(1) $\pi_{\eta,\xi}$ agrees with $\psi_{\sigma,\tau}$ on $\theta_m(\tau)$.

$\alpha < \theta_m(\tau)$, α has an $\mathcal{I}^*(\sigma)$ code,

$\alpha = \pi_{K,G}(\bar\alpha)$, $K \in \mathcal{I}^*(\sigma)$,

$\alpha < \beta_K = 1^{\text{st}}$ strong of K.

then $\psi_{\sigma,\tau}(\alpha) = \pi^{\mathcal{I}(\tau)}_{(K|\beta_K, \Phi),\infty}(\bar\alpha)$

$\Phi = \Omega|_{K|\beta_K}$

$$\begin{array}{ccc} \psi_{\sigma,\tau} : \Phi_\sigma & \longrightarrow & \Phi_\tau \\ \| & & \| \\ \Phi \cap L(A_1(\sigma),\sigma) & & \Phi \cap L(A_1(\tau),\tau) \end{array}$$

$= \pi^{\mathcal{I}^*(\xi)}_{K,G_\xi}(\bar\alpha) = \pi_{\eta,\xi} \circ \pi_{K,G_\eta}(\bar\alpha) = \pi_{\eta,\xi}(\alpha)$.

enough to show: $\Psi_{\sigma,\tau}$ agrees with $\pi_{\eta,\xi}$ on some copinal subset of $\Theta_m(\sigma)^+ G_\eta$.

e.g. for $i_0 : H \longrightarrow G_\eta$, on $\mathrm{ran}(i_0)$.

special case: $\eta = \kappa_0$.

let $j : N^* \longrightarrow N^{**}$, $\mathrm{crit}(j) = \kappa_0$, $\xi \ll \mathrm{strength}(j)$. we have the diagram:

$K = j(H)$

$$\begin{tikzcd}
K \arrow[r, "i_1"] & L \arrow[r, "\pi_1"] & M \\
\triangledown & \triangledown \; i_0(B) & \triangledown \\
H/\xi \arrow[r, "i_0"] & P \arrow[r, "\pi_0"] & Q \\
 & \vartriangle & \\
 & G \arrow[r, "\pi_{\kappa_0,\xi}"] & G_\xi
\end{tikzcd}$$

$B \subset \kappa_0$

$(A_\sigma(o))_x$

$j(i_0) : K \to j(G_{\kappa_0})$

$$\begin{tikzcd}
M \arrow[r, "k"] & j(G_{\kappa_0}) \\
\triangledown & \triangledown \\
Q \arrow[r, "l"] & N
\end{tikzcd}$$

$$j(i_0(B)) = A_0(\rho)_x$$
$$\parallel$$
$$j(i_0)(j(B))$$

$$B = j(B) \cap \Theta_m(\sigma)$$

john steel, july 26.

$\underset{\sim}{\Delta}_{max} = \{ A \subset \mathbb{R} : \exists$ lbr hod pair (P, Σ) s.t.
$A \leq_w Code(\Sigma) \}$

$C \quad S_\infty \cap \breve{S}_\infty$

and $\underset{\sim}{\Delta}_{max} \subsetneq S_\infty \cap \breve{S}_\infty$.

want: it. strategy for a pure extender model with a long extender.

let $\underset{\sim}{\Delta}_{max} \subsetneq \Gamma_0 \subsetneq \Gamma_1 \subsetneq \Gamma_2$ ✓ scale prop.

$A_0^{\#} \leq_w A_1 \leq_w A_2$

let $(N^*, \Sigma^*, \delta^*)$ be a coarse Γ_2-woodin.

let $(H, \Omega) = (M^{\mathbb{C}}_{\delta^*, 0}, \Omega^{\mathbb{C}}_{\delta^*, 0})$

like to see: some long ext can be added to an $M^{\mathbb{C}}_{\nu, k}$.

showed: $L(\underset{\sim}{\Delta}_{max}, \mathbb{R}) \models AD_{\mathbb{R}} + \theta$ reg.

let κ_0 = least strg of H.

showed $H \models \kappa_0$ is a lim of woodin cardinals.

let $\sigma = \mathbb{R}^{N^*[g]}$, g on $Col(\omega, < \eta)$,

$\kappa_0 \leq \eta$, η inacc. lim of Γ_0 - woodins in N^*.

$L(\Delta_m(\sigma), \sigma) \models AD_{\mathbb{R}} + \Theta$ reg.

given such an η,

$\mathcal{I}^*(\eta)$ = dir syst of all iterates of (H, Ω) via $\mathcal{I}$ in $N^*|\eta$.

$i_\eta : H \longrightarrow G_\eta$ iterati map, $\kappa_0 < \eta < \xi$ ← lim η

$i_\eta = \pi_{0\eta}$, $H = G_0$

$\pi_{\eta,\xi} : G_\eta \to G_\xi$.

let $\gamma_\eta = i_\eta(\eta)$.

<u>conjecture</u>. (a) $G_\eta | \gamma_\eta$ is dy. the in $L(A_1 \cap \sigma, \sigma)$ from $A_1 \cap \sigma$ (uniformly in σ), where $A_1(\sigma) = L(A_1 \cap \sigma, \sigma)$

(b) $\pi_{\eta,\xi} \restriction \gamma_\eta$ is dy. the in $L(A_1 \cap \tau, \tau)$ fr $A_1 \cap \tau$, σ, $N^*|\eta$.

rmk. $G_\eta \restriction i_\eta(\kappa_0) = \mathrm{HOD}^{L(\Delta_m(\sigma),\sigma)} \restriction \theta$.

Conjecture enough to get long extenders.

last time showed: conjecture true if

γ_η is repr. by $(\theta_m(\sigma)^+)^{G_\eta}$.

k-consistent strategies:

def. if Q is a mouse with a strong cardinal. let β_Q = the least strong of Q.

$Q_{bt} = Q \mid \beta_Q^{+Q}$.

so $\beta(G_\eta) = \theta_m(\sigma)$.

given (R, Ψ) a lbr hod pair coded in σ,

$\sigma = (\mathbb{R}^{\mathrm{Col}(\omega,<\eta)})^{N^*}$ as above.

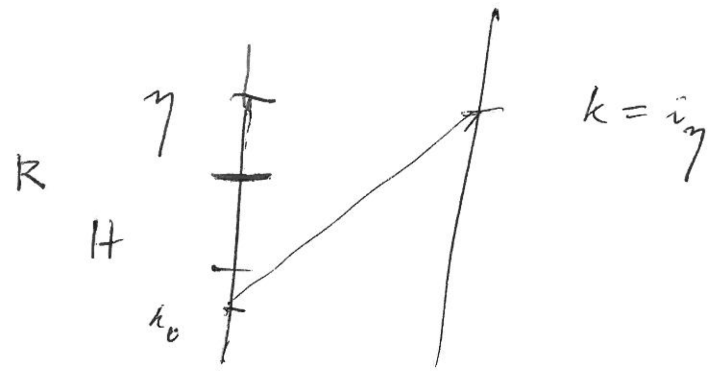
let $G = G_\eta$. given β_R exists, and

in $\mathcal{I}^*(\sigma)$ $k : R_{bt} \longrightarrow G_{bt}$.

then (R, Ψ) is <u>k-consistent</u> iff there is a

map $\tau : M_\infty(R,\Psi)_{bt} \longrightarrow G_{bt}$ s.t.

and $k = \tau \circ \pi^{\mathcal{I}^*(\sigma)}_{R,\infty} \upharpoonright R_{bt}$.



<u>Lem.</u> ~~let~~ (uniqueness of k-consistent strategies)

let (R, Φ) and (R, Ψ) be in $\mathcal{I}^*(\sigma)$,

and k-consistent, where

$$k : R_{bt} \longrightarrow G_{bt} .$$

Supp. $R \models$ "there are no woodin cardinals $> \beta_R$,"

and there is a large cardinal.

then $\Phi = \Psi$.

Pf.: let $M_\infty(R, \Psi) \trianglelefteq M_\infty(R, \Phi) \triangleleft G | \beta_G$.

have normal trees $\mathcal{I}, \mathcal{U}$ s.t.

$M_\infty^{\mathcal{I}} = M_\infty(R, \Psi)$ all cm weak hulls are by Ψ

$M_\infty^{\mathcal{U}} = M_\infty(R, \Phi)$ ... by Φ

have the picture

$$\begin{tikzcd}
 & & G_{bt} & \\
 & & & M_\infty(R,\Phi)_{bt} \arrow[ul, bend right] \\
R_{bt} \arrow[uurr, "k"] \arrow[urrr, "i^{\mathcal{U}}_{0,\alpha_1}"] \arrow[rr, "i^{\mathcal{T}}_{0,\alpha_0} = i^{\mathcal{T}_0}_{\infty}"'] & & M_\infty(R,\Psi)_{bt} \arrow[uu, bend right=80] &
\end{tikzcd}$$

can win $\mathcal{T} = \mathcal{T}_0 ^\frown \mathcal{T}_1$, $i_{\mathcal{T}_1}$ has crit $> \beta(M_\infty(R,\Psi))$.

$\mathcal{U} = \mathcal{U}_0 ^\frown \mathcal{U}_1$ $\quad i_{\mathcal{U}_1} \quad > \beta(M_\infty(R,\phi))$.

by k-consistency, $S^{\mathcal{T}}_{\alpha_0} = S^{\mathcal{U}}_{\alpha_0}$,

ext. used on main branch for α_0 to ∞ on $\mathcal{T}$

$\alpha_0 \in$ main branch of $\mathcal{U}$.

so $\mathcal{T}_1 = \mathcal{U}_1$.

$i^{\mathcal{T}}_{0\infty} = i^{\mathcal{U}}_{0\infty}$ and $M_\infty(R,\phi) = M_\infty(R,\Psi)$.

So $\phi = \Psi$. $\dashv$

$$\begin{tikzcd}
(R,\phi) \arrow[r, bend left, "i"] & {} \arrow[d, phantom, "\parallel"] & & M_\infty \; \text{common} \\
 & (S,\Lambda) \arrow[urr, "i_\Lambda"] & & \\
(R,\Psi) \arrow[ur, "j"'] & & &
\end{tikzcd}$$

$\Rightarrow$ ϕ, Ψ both pullbacks of Λ via the same map.

__Cor.__ H has a woodin cardinal $> \kappa_0$.

__sketch.__

$Q_0 \in L(T_{\Gamma_1}, N^* | \delta)$.

$(Q_{\theta_1}, \Omega_{Q_\theta}) = (Q, \Omega)$ $\trianglelefteq (H, \Omega)$ Q-str. for δ

δ a Γ_1-woodin, $M^{\Phi}_{\delta,0} = H|\delta$

inacc.,

no woodins in H in (κ_0, δ).

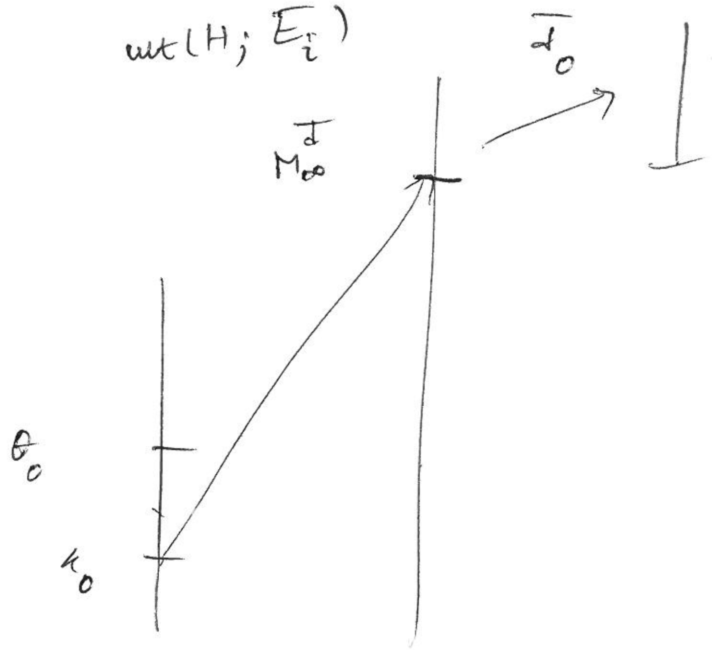
(Q_0, Ω_{Q_0}) is the unique $\Psi_{\sigma,\tau}$-consistent Q-structure for $(H|\delta, \Omega_{H|\delta})$.

Consider $H|\theta_0$, θ_0 is a succ. cardinal.

$\kappa_0 < \theta_0 < \mu_0$, μ_0 = 1st strong to the 1st woodin $> \kappa_0$.

Let $i = i_\eta \restriction H_{bt}$.

$(H|\theta_0, i)$ can be used in $L(A_1(\sigma), \sigma)$ to recover $G_\eta \restriction i(\theta_0)$.



$H/\theta_0 \in \sigma$.

<u>def.</u> (N, ψ) is a <u>gs-pair</u> iff

$\psi : N_{bt} \longrightarrow G_{bt}$ and

$\Vdash^*_{\mu_0} X \Vdash^*_{\mu} Y$ in $\mathcal{P}_{\omega_1}(G_{bt})$, letting

$\pi_X : M_X \longrightarrow G_{bt}$

$\pi_Y : M_Y \longrightarrow G_{bt}$ are the collapses

$\sigma_X = \pi_X^{-1} \circ \psi : N_{bt} \to M_X$

$\sigma_Y = \pi_Y^{-1} \circ \psi : N_{bt} \longrightarrow M_Y$

and $Q_X = \mathrm{ult}(N; E_{\sigma_X})$,

$Q_Y = \mathrm{ult}(N; E_{\sigma_Y})$, with

$\sigma_X^* : N \longrightarrow Q_X$ ext. σ_X

$\sigma_Y^* : N \longrightarrow Q_Y$ ext. σ_Y,

$\sigma_{XY}^* : Q_X \longrightarrow Q_Y$, we have

① $N \models \beta_N$ ex., $\exists$ lgm card.,
no woodin's $> \beta_N$.

② Q_X, Q_Y have (unique) π_X and π_Y consistent strategies Λ_X, Λ_Y

③ $\Lambda = \Lambda_X^{\sigma_X^*}$, $\Lambda_X = \Lambda_Y^{\sigma_{XY}^*}$.

④ σ_X^*, σ_Y^* are "M_∞-adjustable"
(there is a nat. map $\pi_{XY}: M_\infty(Q_X, \Lambda_X)$
$\longrightarrow M_\infty(Q_Y, \Lambda_Y)$. s.t.
s.t. $\pi \upharpoonright \beta(M_\infty(Q_X, \Lambda_X)) = id$).

all these ~~fle~~ $M_\infty(N, \Lambda)$ give $\trianglelefteq G$.

$\rightsquigarrow G / i(\delta_0)$?