Martin's Maximum is Σ_2 complete

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Abstract

We show that Martin's Maximum proves that every Ω-consistent $\Pi_2^{H_{\omega_2}}$ statement is true. While this follows from the argument of [2] (via [5]), we here produce a direct proof of this fact which does not go through Woodin's $\mathbb{P}_{\max}$ machinery.

1 Introduction.

Assume Martin's Maximum^{++}. By [2, Theorem 1.2] we have that the $\mathbb{P}_{\max}$ axiom $(*)$ is true, so that by [5, Theorem 10.150] every Ω-consistent $\Pi_2^{H_{\omega_2}}$ statement is actually true. The current paper shows that this conclusion may be reached without going through the $\mathbb{P}_{\max}$ machinery.

2 Two consistency notions.

Ω-logic got defined in [5, Section 10.4], while the notion of honest consistency comes from [1, Definition 1.8] (see also [4, Deefinition 2.8]). We aim to introduce natural weakenings of the concepts of being Ω-consistent and of being honestly consistent which exactly fit our purposes.

Lemma 2.4 will clarify the connection between these two notions of consistency.

Definition 2.1 *Let φ be a statement in the language of set theory. We say that φ is* 1-Ω-consistent *iff for every real x there is a transitive model M such that*

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(a) $M \models \mathsf{ZFC}$,

(b) $M \models \varphi$,

(c) $x \in M$, *and*

(d) M *is closed under the operator* $z \mapsto M_1^{\#}(z)$.

One can verify that for a given φ, the statement "φ is 1-Ω-consistent" can be written in a Π^1_4 fashion. Therefore, the following holds true. (Cf. [5, Theorem 10.146].)

Lemma 2.2 *Assume that V is closed under $z \mapsto M_2^{\#}(z)$. Let φ be a statement in the language of set theory. Then "φ is 1-Ω-consistent" is absolute between V and $V^{\mathbb{P}}$ for every partially ordered set $\mathbb{P}$.*

Definition 2.3 *Let $A \subset H_{\omega_2}$, and let φ be a statement in the language of set theory. We say that $\varphi(A)$ is* 1-honestly consistent *iff inside $V^{\mathrm{Col}(\omega, 2^{\aleph_1})}$ there is a transitive model M such that*

(a) $M \models \mathsf{ZFC}$,

(b) $H^V_{\omega_2} \in M$,

(c) $M \models \varphi(A)$,

(d) M *is closed under the operator* $z \mapsto M_1^{\#}(z)$, *and*

(e) $(\mathsf{NS}_{\omega_1})^M \cap V = (\mathsf{NS}_{\omega_1})^V$.

Lemma 2.4 *Assume that V is closed under $z \mapsto M_2^{\#}(z)$ and that NS_{ω_1} is saturated. Let $\varphi \equiv \forall X \exists Y \psi(X, Y)$ be a statement in the language of set theory, where ψ is Σ_0. Assume "$H_{\omega_2} \models \varphi$" to be 1-$\Omega$-consistent. Let $A \in H_{\omega_2}$. Then*

$$\exists Y \psi(A, Y) \wedge \mathrm{cf}(\omega_2^V) = \omega \tag{1}$$

is 1-honestly consistent.

Proof. By Lemma 2.2, in $V^{\mathrm{Col}(\omega, \aleph_2)}$ there is a transitive model M such that

(a) $M \models \mathsf{ZFC}$,

(b) $H^M_{\omega_2} \models \varphi$,

(c) $(H^V_{\omega_2}; \in, \mathsf{NS}^V_{\omega_1}) \in M$, and

(d) M is closed under the operator $z \mapsto M_1^{\#}(z)$.

Write $\alpha = \omega_1^M$. Inside M, let

$$(N_i, \pi_{ij} \colon i \leq j \leq \alpha) \tag{2}$$

be a generic iteration of $N_0 = (H_{\omega_2}^V; \in, \mathsf{NS}_{\omega_1}^V)$ of length $\alpha + 1$ such that

$$\mathsf{NS}_{\omega_1}^M \cap N_\alpha = \mathsf{NS}_{\omega_1}^{N_\alpha}. \tag{3}$$

We may lift the iteration (2) to an iteration

$$(N_i^+, \pi_{ij}^+ \colon i \leq j \leq \alpha) \tag{4}$$

of V. We have that

(a') $M \models \mathsf{ZFC}$,

(b') $H_{\omega_2}^{N_\alpha^+} \in M$,

(c') $M \models \exists Y \psi(\pi_{0\alpha}^+(A), Y) \wedge \mathrm{cf}(\omega_2^{N_\alpha}) = \omega$,

(d') M is closed under the operator $z \mapsto M_1^\#(z)$, and

(e') $(\mathsf{NS}_{\omega_1})^M \cap N_\alpha^+ = (\mathsf{NS}_{\omega_1})^{N_\alpha^+}$.

(e') is given by (3). By absolutess, there is a transitive model M with these properties (a') through (e') inside $(N_\alpha^+)^{\mathrm{Col}(\omega, \aleph_2^{N_\alpha})}$. This is statement which is expressible over N_α^+ and which may thus be pulled back to V using $\pi_{0\alpha}^+$. Therefore we get that inside $V^{\mathrm{Col}(\omega, \aleph_2)}$ there is a model witnessing that

$$\exists Y \psi(A, Y) \wedge \mathrm{cf}(\omega_2^V) = \omega$$

is 1-honestly consistent. □ (Lemma 2.4)

The same argument may be used to show that in the presence of Woodin cardinals, if $\varphi \equiv \forall X \exists Y \psi(X, Y)$ is a statement in the language of set theory, where ψ is Σ_0, "$H_{\omega_2} \models \varphi$" is Ω-consistent (in the original sense of [5]), and $A \in H_{\omega_2}$, then

$$\exists Y \psi(A, Y) \wedge \mathrm{cf}(\omega_2^V) = \omega$$

is honestly consistent (in the original sense of [1]).

There is also an abstract argument to show that one can get "$\mathrm{cf}(\omega_2^V) = \omega$" for free in (1):

Lemma 2.5 *Let $A \in H_{\omega_2}$, and let "$\exists Y \psi(A, Y)$" be 1-honestly consistent, where ψ is Σ_0. Then*

$$\exists Y \psi(A, Y) \wedge \mathrm{cf}(\omega_2^V) = \omega$$

is 1-honestly consistent.

Proof. Let θ be some large enough regular cardinal. Inside V, pick

$$\sigma\colon H \to H_\theta$$

such that H is transitive, $\mathrm{Card}(H) = \aleph_1$, $\mathrm{crit}(\sigma) = \sigma^{-1}(\omega_2)$, and $\mathrm{cf}(\mathrm{crit}(\sigma)) = \omega$. Let $M \in V^{\mathrm{Col}(\omega,2^{\aleph_1})}$ witness that "$\exists Y\, \psi(A, Y)$" is 1-honestly consistent, so that M also witnesses that "$\exists Y\, \psi(A, Y) \wedge \mathrm{cf}(\omega_2^H) = \omega$" is 1-honestly consistent. By absoluteness, in $H^{\mathrm{Col}(\omega,(2^{\aleph_1})^H)}$ there is some M' witnessing that "$\exists Y\, \psi(A, Y) \wedge \mathrm{cf}(\omega_2^H) = \omega$" is 1-honestly consistent. But then we may apply σ to get that there is some $M'' \in V^{\mathrm{Col}(\omega,2^{\aleph_1})}$ witnessing that "$\exists Y\, \psi(A, Y) \wedge \mathrm{cf}(\omega_2^V) = \omega$" is 1-honestly consistent. (Lemma 2.5)

The role of "$\mathrm{cf}(\omega_2^V) = \omega$" in (1) is explained by the following.

Lemma 2.6 *Assume that V is closed under $z \mapsto M_1^\#(z)$ and that NS_{ω_1} is saturated. Let $A \in H_{\omega_2}$, and suppose*

$$\exists Y\, \psi(A, Y) \wedge \mathrm{cf}(\omega_2^V) = \omega \tag{5}$$

to be 1*-honestly consistent. Then inside $V^{\mathrm{Col}(\omega,\aleph_2)}$ there are a, y, e,*

(i) $(N_i, \pi_{ij}\colon i \le j \le \omega_1^V)$, *and*

(ii) $(P_i, \sigma_{ij}\colon i \le j \le \omega_1^V)$

such that

(a) $P_0 = M_1^\#(a, y, e, N_{\omega_1^V})$ *is countable,*

(b) $(N_i, \pi_{ij}\colon i \le j \le \omega_1^V)$ *is a generic iteration with* $N_{\omega_1^V} = (H_{\omega_2}; \in, \mathsf{NS}_{\omega_1})^V$ *via the nonstationary ideal on* $\omega_1^{N_0}$ *and its images,*

(c) $(N_i, \pi_{ij}\colon i \le j \le \omega_1^{P_0}) \in P_0$,

(d) $(P_i, \sigma_{ij}\colon i \le j \le \omega_1^V)$ *is a generic iteration of* P_0 *via the countable stationary tower given by the Woodin cardinal of* P_0 *and its images,*

(e) $(N_i, \pi_{ij}\colon i \le j \le \omega_1^V) = \sigma_{0\omega_1^V}((N_i, \pi_{ij}\colon i \le j \le \omega_1^{P_0}))$,

(f) $\mathsf{NS}_{\omega_1^V}^{P_{\omega_1^V}} \cap H_{\omega_2}^V = \mathsf{NS}_{\omega_1}^V$, *and*

(g) $A = \sigma_{0\omega_1^V}(a)$, *and* $P_{\omega_1^V} \models \psi(A, \sigma_{0\omega_1^V}(y)) \wedge \mathrm{cf}(\omega_2^V) = \omega$ *as being witnessed by* $\sigma_{0\omega_1^V}(e)$.

Proof. Let us fix $M \in V^{\mathrm{Col}(\omega,\aleph_2)}$ witnessing that (5) is 1-honestly consistent. Let $Y, E \in M$ be such that

$$M_1^{\#}(A, Y, E, H_{\omega_2}^V) \vDash \psi(A, Y) \wedge E \text{ is a set of ordinals cofinal in } \omega_2^V \text{ with } \mathrm{otp}(E) = \omega.$$

Write $P_0 = M_1^{\#}(A, Y, E, H_{\omega_2}^V)$ and $N^* = (H_{\omega_2}; \in, \mathsf{NS}_{\omega_1})^V$. Inside P_0, let

$$\pi \colon N \cong \mathrm{Hull}^{N^*}(E) \prec N^*,$$

where $\mathrm{Hull}^{N^*}(E)$ denotes the countable Skolem hull of N^* generated from E, N is transitive, and π is the inverse of the transitive collapse. By [5, Lemma 3.12], the fact that E is cofinal in ω_2^V, and $\mathrm{Card}(H_{\omega_2}) = \aleph_2$ in V, we have that

$$\pi = \pi_{0\omega_1^V},$$

where

$$(N_i, \pi_{ij} \colon i \leq j \leq \omega_1^V) \tag{6}$$

is a generic iteration with $N_0 = N$ and $N_{\omega_1^V} = N^*$ via the nonstationary ideal on $\omega_1^{N_0}$ and its images. As M witnesses that (5) is 1-honestly consistent, we clearly that that

$$\mathsf{NS}_{\omega_1}^{P_0} \cap H_{\omega_2}^V = \mathsf{NS}_{\omega_1}^V. \tag{7}$$

Inside $V^{\mathrm{Col}(\omega,\aleph_2)}$, let us now produce $(P_i, \sigma_{ij} \colon i \leq j \leq \omega_3^V)$, a generic iteration of P_0 via the countable stationary tower given by the Woodin cardinal of P_0 and its images. We may apply $\sigma_{0\omega_3^V}$ to (6) to get a stretch

$$(N_i, \pi_{ij} \colon i \leq j \leq \omega_3^V)$$

of (6). The tail end

$$(N_i, \pi_{ij} \colon \omega_1^V \leq i \leq j \leq \omega_3^V)$$

of this stretch may be lifted to produce a generic iteration

$$(N_i^+, \pi_{ij}^+ \colon \omega_1^V \leq i \leq j \leq \omega_3^V)$$

of $N_0^+ = V$. The rest is an absoluteness and pulling back argument as in the proof of Lemma 2.4. (Lemma 2.6)

3 Σ_2 completeness from Martin's Maximum.

Theorem 3.1 *Asssume Martin's Maximum. Let φ be a Π_2 sentence in the language of set theory such that "$H_{\omega_2} \models \varphi$" is 1-$\Omega$-consistent. Then "$H_{\omega_2} \models \varphi$" is true.*

Proof. Let $\varphi \equiv \forall X \exists Y \psi(X, Y)$ be a statement in the language of set theory, where ψ is Σ_0. Let $A \in H_{\omega_2}$. Applying Lemmas 2.4 and 2.6, we get the conclusion of 2.6. This conclusion in turn may serve as the starting point for forcing the existence of objects as in (a) through (g) of the statament of Lemma 2.6 by a stationary set preserving forcing, in much the same way as in [1] (having P_0 replacing the $\mathbb{P}_{\max}$ condition in the given dense set). But then $\exists Y \psi(A, Y)$ holds true by MM. As A was arbitrary, we showed that φ holds true. □ (Theorem 3.1)

References

[1] D. Asperó and R. Schindler, *Bounded Martin's Maximum with an asterisk*, Notre Dame J. Formal Logic 55 (2014), pp. 333-348.

[2] D. Asperó and R. Schindler, *Martin's Maximum^{++} implies Woodin's Axiom* $(*)$, Annals of Mathematics 193 (3) (2021), pp. 793-835.

[3] R. Schindler, *Woodin's axiom $(*)$, or Martin's Maximum, or both?*, in: Foundations of mathematics, essays in honor of W. Hugh Woodin's 60th birthday, Harvard University (Caicedo et al., eds.), pp. 177-204, 2017.

[4] W. H. Woodin, *The axiom of determinacy, forcing axioms, and the non-stationary ideal*, de Gruyter, Berlin-New York 1999.