Dominik Adolf

Let $(\kappa_i : i < \lambda)$ seq. of regular cardinals,

each $S_i \subset \kappa_i$ stationary.

$(S_i : i < \lambda)$ is mutually stationary iff

$$\{ A \subset \sup_i \kappa_i : \forall i < \lambda \; \kappa_i \in A \Rightarrow \sup (A \cap \kappa_i) \in S_i \}$$

is stationary.

consider:

(A) $(S^3_1, S^4_2, S^5_1, S^6_2, \ldots)$ is m. stat.

(B) $(S^3_1, S^4_1, S^5_2, S^6_2, S^7_1, S^8_1, S^9_2, \ldots)$ is m. stat.

(C) $(S^3_1, S^4_2, S^5_3, \ldots)$ is m. stat.

th. (Adolf - Cox) $(\neg 0^\P)$

(a) assume (A). then $K | \aleph'_\omega$ has infinitely many measures of Mitchell order $\geq \omega_1$

(case I), or

in K there is a measure on $\aleph_\omega$ concentrating on V- cardinals $\alpha < \aleph_\omega$ s.t. $\forall \gamma < \omega_1$ $\{\beta < \alpha : o^K(\beta) \geq \gamma\}$ is stat. in V (case II)

(b) assume (B) then in $K / \aleph_\omega$ there are infinitely many V- cardinals $\alpha < \aleph_\omega$ s.t. $\exists \beta < \alpha$ in $o^k(\beta) \geq (\alpha^+)^V$

(c) assume (C). then $0^{\P}$ ex.

<u>th.</u> (liu - shelah)

assume that $\exists$ inf. many cardinals κ s.t. $o(\kappa) \geq \omega_1 + 1$. then $\exists$ gen. extension in which (A) holds.

<u>th.</u> (shelah)

let $A_0 = \{\aleph_3, \aleph_4, \aleph_7, \aleph_8, \dots\}$

$A_1 = \{\aleph_5, \aleph_6, \aleph_9, \aleph_{10}, \dots\}$

$$tcf\left(\prod_{n \in A_0} \aleph_n\right) = \aleph_{\omega+1}$$

$$tcf\left(\prod_{n \in A_1} \aleph_n\right) = \aleph_{\omega+2}.$$

idea: (B) holds and also

$(S_1^3, S_1^4, S_1^5, S_1^6, S_2^7, S_2^8, S_2^9, S_2^{10}, S_1^{11}, \dots)$

$\Rightarrow$ $\Box$ holds.

<u>lem.</u> (folklore)

let M be a pm, $\kappa \in M$ an M-regular cardinal s.t.

$$\rho_{n+1}^M = \alpha < \kappa \leq \rho_n^M, \quad \text{and}$$

M is sound above α.

then $\mathrm{cf}^V(\kappa) = \mathrm{cf}^V(\rho_n^M)$.

Pf.: w.l.o.g., $n = 1$ (o.w. work w/ the n^{th} reduct)

have for all $\beta \in M \cap \mathrm{OR}$

$\mathrm{Hull}_1^{M|\beta}(\alpha \cup \{p_{n+1}^M\}) \in M \Rightarrow$ bounded below κ

and $\bigcup_{\beta \in M \cap \mathrm{OR}} \mathrm{Hull}_1^{M|\beta}(\alpha \cup \{p_{n+1}^M\}) = M$.

$\beta \mapsto \sup(\mathrm{Hull}_1^{M|\beta}(\alpha \cup \{p_{n+1}^M\}) \cap \kappa)$ $\dashv$

<u>lem.</u> (cox) $(\neg 0^{\P})$

let $S \subset \mathcal{P}(\aleph'_\omega)$ be stat. s.t. for all $X \in S$, $X \cap \aleph'_\omega$ is ω-closed.

then there is some $T \subset S$ stat. s.t. $\forall X \in T$

(a) ~~t~~ in the coit. of K_X (coll. of $K \cap X$) with K, K_X does not move.

(b) let $\alpha = \mathrm{crit}(\pi)$, $\pi : H \longrightarrow X$ then $(\alpha^+)^{K_X}$ is not a K-cardinal.

(c) if E is the 1^{st} extender applied to K and there is no drop at this stage, then $(lh(E))^{+K_X}$ is not an $ult(K; E)$-cardinal.

(d) if the coiteration is longer than appl. E as in c), then K drops.

let $S \subset \mathcal{P}(\aleph'_\omega)$ where (A), (B), or (C), then S is as in (a) — (d).

for $X \in S$ write

$$\pi_X : H_X \xrightarrow{\sim} X \,, \quad K_X = \pi_X^{-1} {}'' K \cap X$$

$$\beta_n^X = \pi_X^{-1}(\aleph_n) \,, \quad \beta_\omega^X = \sup_{n<\omega} \beta_n^X \,.$$

$$(M_\alpha^X \,,\ E_\alpha^X \,,\ n_\alpha^X \,,\ i_{\alpha\beta}^X : \alpha \le \beta \le \theta_X)$$

$$\kappa_\alpha^X = \mathrm{crit}(E_\alpha^X) \,, \quad \nu_\alpha^X = \mathrm{lh}(E_\alpha^X) \,.$$

<u>Claim</u>. $\forall X \in S$ θ_X is a limit ordinal.

Pf.: deny. let K^* be the final iterate on the K-side.

$\rho_{n+1}^{K^*} \le \beta_{n^*}$, some $n^* < \omega$,

$\cancel{\Rightarrow cf^V(\beta_n) = cf^V(\beta_{n+1})}$

$\Rightarrow$ $cf^V(\beta_n)$ constant for a tail end of n. $\dashv$

so there is a largest α^*_X at which the iteration drops.

let $m^*_X = n^X_{\alpha^*_X}$.

assume $\mathrm{cof}^V(\rho_{m^*_X}((M^X_{\alpha^*_X+1})^*)) = \omega_2$.

then f.a. $\alpha^*_X < \alpha \leq \theta_X$

$$\mathrm{cf}^V(\rho_{m^*_X}(M^X_\alpha)) = \omega_2 .$$

let n^*_X be min. with $\alpha > \alpha^*_X \Rightarrow$

$$\kappa^X_\alpha > \beta^X_{n^*_X} .$$

now assume (A).

<u>claim</u>. $\forall n > n^*$ odd

$\{ \kappa^X_\alpha : \alpha < \theta_X \} \cap \beta^X_n$ is unbounded in β^X_n.

pf.:

$M^X_{\alpha+1} \models \beta^X_n$ is reg.

$M^X_{\alpha+1}$ sound above $(\kappa^X_\alpha + 1) < \beta^X_n$.

$\Rightarrow \mathrm{cf}(\beta^X_n) = \mathrm{cf}(\rho_{m^*}(M^X_{\alpha+1})) = \omega_2$ ↯

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let $\theta^n_X = \sup\{\alpha < \theta_X : \kappa^X_\alpha < \kappa^X_n\}$

for $n > n^*_X$, n odd.

<u>claim</u>. f.a. $n > n^*$, n odd

$\exists\, \alpha^X_n < \theta^n_X$ s.t. f.a.

$\alpha^X_n \le \alpha \le \beta \le \theta_X$ $\quad i^X_{\alpha\beta}(\kappa^X_\alpha) = \kappa^X_\beta$.

(at least $\exists$ club of such α, β)

proof: deny. we have $\beta^X_n \in M^X_{\theta^n_X}$.

let $\alpha < \theta^n_X$ be s.t. $\bar\beta$ ~~has~~ is a preimage of β^X_n in M^X_α.

by assumption, $\bar\beta > \kappa^X_\alpha$.

$M^X_\alpha \models \bar\beta$ is regular.

$\Rightarrow \mathrm{cf}^V(\bar\beta) = \omega_2$, ~~...~~

but $i_{\alpha\theta^n_X}$ is cont. at $\bar\beta$ $\Rightarrow$

$\mathrm{cf}^V(\beta) = \omega_1$ ↯ $\dashv$

let $C^n_X = \{ \kappa^X_\alpha : \cancel{\ldots}_n \le \alpha < \theta^n_X$

$i_{\alpha\theta_X}(\kappa^X_\alpha) = \beta_n \}$.

contains a club.

<u>claim</u>. for all $n > n^*$ odd, for all

$\alpha^*_X < \alpha \le \theta^*_X$.

$M^X_\alpha \models o(\kappa^X_\alpha) \ge \omega_1$.

pf.: assume not.

$\exists S^* \subset C^n_X$ stat., concentrating

a points of cof ω

s.t. $\forall \alpha \in S^*$ $o(E^X_\alpha) = \gamma$,

some fixed γ.

let $\beta^* \in S^* \cap \lim(S^*)$.

we can cover $\beta^* \cap S$ by a

cdth. set in X. this allows us

to generate a mean on $\pi_X(\beta^*)$

in K of mitchell order γ.

then $K_X \models o(\beta^*) \geq \gamma$.

but $M_{\theta^n_X} \models o(\beta^*) < \gamma$. ↯

<u>Corollary.</u> $\forall \gamma < \omega_1 \ \exists C \subset C^n_X$ club

$\forall \alpha \in C \quad K_X \models o(\alpha) \geq \gamma$.

two cases:

<u>case I:</u> for infinitely many pairs $k, \ell > n^*$ odd there ex. $\alpha \in [\theta^k_X, \theta^\ell_X)$ s.t. $\kappa^X_\alpha > i_{\theta^k_X \alpha}(\kappa^X_{\theta^k_X})$

$\Rightarrow K_X \models o(i_{\theta^k_X \alpha}(\kappa^X_{\theta^k_X})) \geq \omega_1$.

<u>case II:</u> o.w. for a tail of $n > n^*$,

$i_{\theta^X_k \theta^X_\ell}(\beta_k) = \beta_\ell$.

this is a generating seq. for a measure on λ'_ω.

now (B).

note: if $\alpha \in [\beta_n^X, \beta_{n+1}^X)$ is a

K_X – succ. cardinal.

$\Rightarrow cf^V(\alpha) = cf^V(\beta_n^X)$.

<u>claim.</u> $\forall n > n^*$ odd $\exists \alpha < \theta_X$

$$\kappa_\alpha^X < \beta_{2n+1}^X < (\beta_{2n+1}^X)^{+K_X} < \beta_{2n+3}^X \leq lh(E_\alpha^X).$$

pf. deny.

let α be min. s.t.

$lh(E_\alpha^X) > \beta_{2n+1}^X$.

assume $\kappa_\alpha^X \geq \beta_{2n+1}^X$.

then

M_α^X is sound above β_{2n+1}^X, $(\beta_{2n+1}^X)^{+K_X}$

is regular.

$\Rightarrow cf((\beta_{2n+1}^X)^{+K_X}) = \omega_2$ ↯

$\Rightarrow \kappa_\alpha^X < \beta_{2n+1}^X$.

M_α^X is sound above κ_α^X and $(\kappa_\alpha^X)^{+K_X}$

is regular, $(\kappa_\alpha^X)^{+K_X} \leq \beta_{2n+1}^X$.

$\Rightarrow$ $cf^V((\kappa_\alpha^X)^{+K_X}) = \omega_2$.

$\Rightarrow$ $cf^V(\mathrm{lh}(E_\alpha^X)) = \omega_2$,

$\rho_{m+}(M_\alpha^X) \geq (\kappa_\alpha^X)^{+K_X}$.

on the other hand, $\mathrm{lh}(E_\alpha^X)$ is

a K_X - succ. $\Rightarrow$ $\mathrm{lh}(E_\alpha^X) \geq \beta_{2n+3}^X$. ⊣

<u>claim</u>. let $\beta < \beta_{2n+3}^X$ be a K_X - succ.

then the gen. of E_α^X are cofinal in β.

pf.: deny. let ξ = sup of the

generators $< \beta$. set

$M' = \mathrm{ult}(M_\alpha^X, E_\alpha^X \restriction \xi)$, sound above ξ

β is reg. in M'

(factor map has crit $\geq \beta$)

$E_\alpha^X \restriction \xi$ is close to M_α^X. $\Rightarrow$ $cf^V(\beta) = \omega_2$ ↯

now (C) :

let $\mathrm{cf}^V(\rho_{m_X^*}(M^X_{\alpha^*+1})) = \aleph'_k$.

let $n^{**} > n^*$ s.t. $\forall n > n^{**}$ $\mathrm{cf}^V(\beta^X_n) = \aleph'_k$.

let E^*_α be s.t. $\mathrm{lh}(E^X_\alpha) > \beta^X_{n^{**}}$.

then: M^X_α is sound above $\kappa^X_\alpha \leq \rho_{m_X^*}(M^X_\alpha)$

$\Rightarrow$ $\mathrm{cf}^V(\kappa^X_\alpha) = \aleph'_k$.

$\Rightarrow$ $\mathrm{cf}^V(\mathrm{lh}(E^X_\alpha)) = \aleph'_k$.

but α is a K_X-succ. in

$(\beta^X_{n^{**}+1}, \beta^X_\omega)$ $\Rightarrow$ $\mathrm{cf}(\alpha) > \aleph'_k$ ↯

questions :

I : Is (C) consistent? if so, can one get a woodin cardinal from (C) ?

II : what's the exact con. strength of (A), (B), resp.

III : is the following consistent : ?

"$\forall f : \omega \longrightarrow \{\omega_1, \omega_2\}$ $(S^3_{f(0)}, S^4_{f(1)}, \ldots)$ is ~~[illegible]~~."

mutually stat.

this statement gives $0^{\P}$, does it give a inner model with a Woodin cardinal ?

$\underline{\overline{\text{IV}}}$: what is the con. strength of

"$(S^2_0, S^3_0, S^4_1, S^5_1, S^6_0, S^7_0, \ldots)$ is mut. stationary" ?

(know : $(S^2_0, S^3_1, S^4_0, S^5_1, \ldots) \overset{\text{con}}{\Longleftrightarrow}$ $\underline{1}$ measurable cardinal)