Determinacy of Refinements to the Difference Hierarchy of Co-analytic sets

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Goal

Theorems of the form:

$$\exists M(M \models T, M \text{ is iterable}) \implies \mathrm{Det}(\omega^2\text{-}\Pi^1_1 + \Gamma)$$

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Consider: $T = KP + \Sigma_1\text{-Sep}$, $\Gamma = \Sigma^0_2$.

Picture

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Π^1_1

Δ^1_1

$\vdots$

Σ^0_2

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Borel, Projective hierarchy

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$(\omega^2 + 1)\text{-}\Pi^1_1$

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Difference hierarchy on Π^1_1

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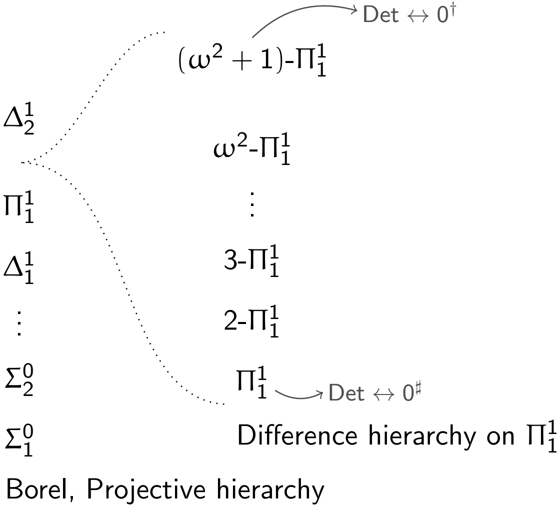
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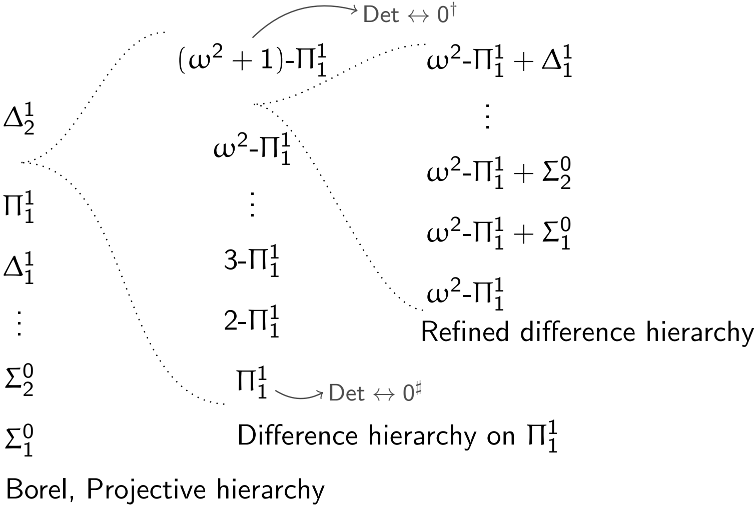
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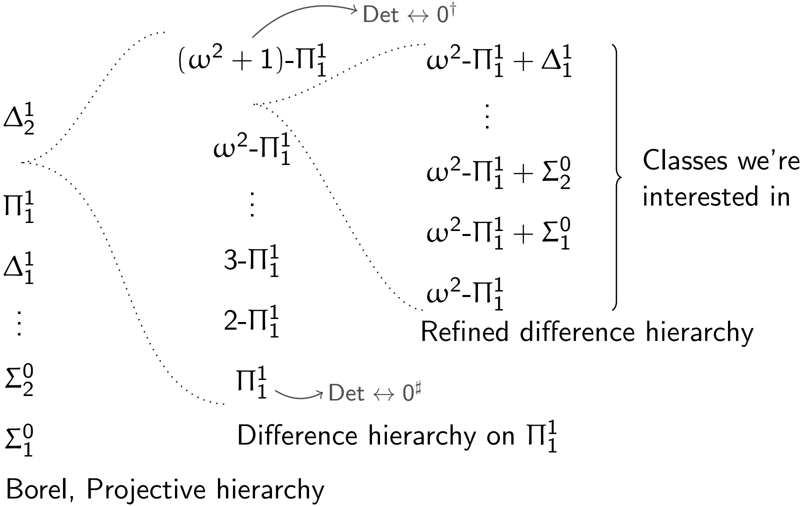
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Difference Hierarchy

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Fact

If $\alpha > 1$ is a computable ordinal then

$$\Pi^1_1 \subsetneq \alpha\text{-}\Pi^1_1 \subsetneq (\alpha+1)\text{-}\Pi^1_1 \subsetneq \Delta^1_2$$

Refining the Difference Hierarchy

We can refine the difference hierarchy by restricting the final set in the sequence.

Definition

For $\Lambda \subseteq \Gamma$, we say

$$A \in \alpha\text{-}\Gamma + \Lambda$$

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Let $A \in \omega^2\text{-}\Pi^1_1 + \Gamma$. In order to win the game for A, both players are trying not to be the first one to go out of an A_β for $\beta < \omega^2$, and if they both succeed then I wins if he gets into A_{ω^2}.

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- The auxiliary game is constructed so as to be determined
- Using a winning strategy for the auxiliary game, a winning strategy for the original game is defined
- In moving from the auxiliary strategy to that for the original game, the players must "imagine" the auxiliary moves being played by their opponent; indiscernibility ensures that this is possible.

Details of the Auxiliary Game

I	$\langle \alpha_0, \eta_0 \rangle$		$\langle \alpha_2, \eta_2 \rangle$		$\dots$
II		$\langle \alpha_1, \eta_1 \rangle$		$\langle \alpha_3, \eta_3 \rangle$	

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$$\begin{array}{llllllll} \mathrm{I} & \langle a_0, \eta_0 \rangle & & & \langle a_2, \eta_2 \rangle & & & \ldots \\ & & \searrow & \nearrow & & \searrow & \nearrow & \\ \mathrm{II} & & \langle a_1, \eta_1 \rangle & & & \langle a_3, \eta_3 \rangle & & \end{array}$$

- The ordinal components $\eta_i \in \aleph_\omega$ are partitioned so as to create ω^2 many countable orderings. Each should witness that $x = \langle a_0, a_1, a_2, \ldots \rangle \in A_\beta$ for some $\beta < \omega^2$.

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- II wins the auxiliary game if the play is not badly lost for either player; I wins if it is badly lost for II.

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We will need a lightface condition, so the first task is to work out what "lightface Σ^0_n" should mean for a subset of $(\omega \times \aleph_\omega)^\omega$.

Effective Descriptive Set Theory on κ^ω

Definition

Call a subset R of κ^ω *generalised lightface open* if there is a $\Sigma_1(L_\kappa)$ set $X \subseteq \kappa^{<\omega}$ such that:

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If we replace L_κ with $\langle L_\kappa[\vec{c}], \in, \vec{c}\rangle$ for some countable set of ordinals $\vec{c}$, then we can make the same definition to get the *lightface in* $\vec{c}$ open sets.

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If we replace "lightface" with "lightface in $\vec{c}$" then we get the $\Sigma^0_n(\vec{c})$ hierarchy on κ^ω.

- We can prove the analogue of the Kleene Basis theorem in this context: If $X \subseteq \kappa^\omega$ is Σ^1_1 and non-empty, it has an element definable over any admissible set M with $L_\kappa \in M$.

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- The idea is that Σ^1_1 relations are Π_1 over any admissible containing L_κ. The leftmost path through the the corresponding tree is then a definable element.
- This allows us to reduce the complexities of properties in the determinacy arguments, and hence prove determinacy of the auxiliary games in weak models.

Determinacy of the Auxiliary Game

A basic fact is that if $A \subseteq \omega^\omega$ is Σ^0_n, it is also Σ^0_n in this sense, considered as a subset of each κ^ω. A quick calculation then shows that, if the main game is ω^2-$\Pi^1_1 + \Sigma^0_n$ for $n > 1$ then the auxiliary game is $\Sigma^0_n(\langle \aleph_i \mid i < \omega \rangle)$ on $(\omega \times \aleph_\omega)^\omega$, a pointclass we abbreviate to $\widehat{\Sigma}^0_n$.

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Example

If $A \in \omega^2\text{-}\Pi^1_1 + \Sigma^0_2$ then the auxiliary winning set A^* is $\widehat{\Sigma}^0_2$ and, if M is a transitive model of $\mathsf{KP} + \Sigma_1\text{-Sep}$ containing $\langle \aleph_i \rangle$ then there is a Σ_1-definable winning strategy for A^* in M.

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So, taking $T = \mathsf{KP} + \Sigma_1$-Sep, $n = 1$, this principle would imply that the winning strategy for the $\widehat{\Sigma}^0_2$ auxiliary game behaves the same when defined over any of these models.

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$$\mathbb{P} = \{\langle p, X\rangle \mid p \in [\kappa_\lambda]^{<\omega}, X \in F_\lambda \cap M_\lambda\}$$

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Theorem (L.S.)

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This requires us to show that, although $\mathbb{P}$ is a class forcing over M_λ, $\mathsf{KP} + \Sigma_n\text{-}\mathsf{Sep}$ holds in the generic extension.

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- Define a set of "names," which can be thought of as $L_\theta[\dot{\vec{c}}]$ where $\dot{\vec{c}}$ is a constant symbol for a Prikry-generic sequence. We give each name a rank according to where it appears in $L_\theta[\dot{\vec{c}}]$.

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- Define a ranked language $\mathcal{L}_\mathbb{P}$ which in addition to everything from $\mathcal{L}_{\{\in\}}$ contains *ranked variables* v_i^α for $\alpha < \theta$ and all the names from $L_\theta[\dot{\vec{c}}]$. A sentence of $\mathcal{L}_\mathbb{P}$ is *ranked* if all variables in it are ranked.

Ramified Forcing

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1. $\mathbf{p} \Vdash^* x \in y$ iff $x \in L_\alpha[\dot{\vec{c}}], y \in L_\beta[\dot{\vec{c}}]$ and:
    1. $\alpha = \beta = 0$ and either $x \in y \in \kappa$ or $x \in p \wedge y = \dot{\vec{c}}$; or
    2. $\alpha < \beta$, $y = \{z^\gamma \mid \varphi(z^\gamma)\}$ and $\mathbf{p} \Vdash^* \varphi(x)$; or else
    3. $\alpha \geqslant \beta$ and $\exists z \in L_\gamma[\dot{\vec{c}}]$ for some γ, either $\beta > \gamma$ or $\beta = \gamma = 0$ and
    $$\mathbf{p} \Vdash^* z = x \wedge z \in y$$
2. $\mathbf{p} \Vdash^* x = y$ iff $\mathbf{p} \Vdash^* \forall z^\alpha (z \in x \leftrightarrow z \in y)$ for α the maximum of the ranks of x and y.
3. $\mathbf{p} \Vdash^* \varphi \wedge \psi$ iff $\mathbf{p} \Vdash^* \varphi$ and $\mathbf{p} \Vdash^* \psi$.
4. $\mathbf{p} \Vdash^* \neg\varphi$ iff $\forall \mathbf{q} \in \mathbb{P}(\mathbf{q} \leqslant \mathbf{p} \implies \mathbf{q} \nVdash^* \varphi)$.
5. $\mathbf{p} \Vdash^* \exists x^\alpha(\varphi(x^\alpha))$ iff there is some $t \in L_\alpha[\dot{\vec{c}}]$ such that $\mathbf{p} \Vdash^* \varphi(t)$.
6. $\mathbf{p} \Vdash^* \exists x(\varphi(x))$ iff there is some $t \in \bigcup_\alpha L_\alpha[\dot{\vec{c}}]$ such that $\mathbf{p} \Vdash^* \varphi(t)$.

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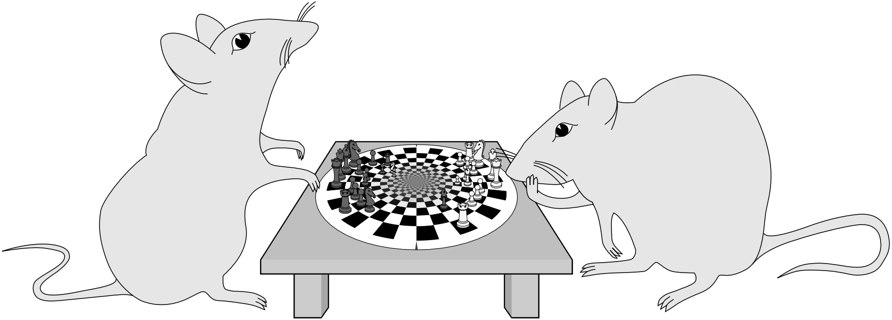
- This is proved first for Δ_1 formulæ simultaneously with the Prikry lemma, and relies on the fact that we don't quantify over $\mathbb{P}$ in the definition of $\Vdash^*$ for atomic formulæ.
- This slide was a bit empty, so here's a picture of mice playing a game:



Forcing Theorems

The definition of genericity is odd due to the weak setting:

Definition

$G \subseteq \mathbb{P}$ is M_λ-generic if it *both* meets all $\Sigma_n^{M_\lambda}$ dense subclasses of $\mathbb{P}$ *and* decides every Σ_n sentence of $\mathcal{L}_{\mathbb{P}}$.

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From now on we denote $M[G]$ as $M_\lambda[\vec{c}]$ as above.

Note that, as in modern forcing, the generic extension is the class of names interpreted by the generic.

Forcing Theorems

Theorem (L.S.)

The Truth Lemma holds for any such generic, and we can always find one by taking the Prikry sequence $\vec{c}$ a countable sequence of critical points cofinal in κ_λ.

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Theorem (L.S.)

For any $p = \{c_0, \ldots, c_l\}$ and y an arbitrary constant Σ_n-definable in $M_\lambda[\vec{c}]$ (without indiscernible parameters above c_l). Suppose ψ is Π_{n-1}. Then:

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Now we can show that $M_\lambda[\vec{c}] \models \mathsf{KP} + \Sigma_n$-Sep.

Indiscernibility

- Now it's easy to see that all $M_\lambda[\vec{c}]$ models are Σ_n elementary equivalent, minimal and models of $\mathsf{KP} + \Sigma_n$-Sep with $\vec{c} \in M_\lambda[\vec{c}]$.

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- I.e. $M_\lambda[\vec{c}] = \mathcal{A}_T[\vec{c}]$.
- We thus have the ingredients required to mimic Martin's proof: definable winning strategies for the auxiliary game, and suitable indiscernibles.

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- To win the original game, the player must be able to ignore the components of the auxiliary game that are not played in the original.
- They "imagine" their opponent has played indiscernibles $\vec{c}$.
- They then move according to the auxiliary strategy's output, as computed by any model $M_\lambda[\vec{c}] = \mathcal{A}_T[\vec{c}]$.
- This strategy is winning in V because any counterexample would be a real existing by Shoenfield absoluteness in a suitable $M_\lambda[\vec{c}]$.

Results

Theorem (L.S.)

The implication:

$$\exists M(M \vDash T, M \textit{ is iterable}) \implies \mathrm{Det}(\omega^2\text{-}\Pi^1_1 + \Gamma)$$

holds for the following values of T *and* Γ*:*

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holds for the following values of T *and* Γ*:*

T	Γ
" *'cleverness'* $+ \exists$ *a 'clever mouse'* "	Σ^0_1
$\text{KP} + \Sigma_1\text{-Sep}$	Σ^0_2
$\text{KP} + \Sigma_2\text{-Sep}$	Σ^0_3
$\text{KP} + \Sigma_{n+1}\text{-Sep}$	$n\text{-}\Pi^0_3$
$\text{ZFC}^- + \mathcal{P}^\alpha(\kappa)$ *exists*	$\Sigma^0_{1+\alpha+3}$ $(\alpha < \omega_1^{CK})$
ZFC	Δ^1_1.

Open Questions

1. What other combinations of T, Γ can we find proofs of?

C. M. Le Sueur.
Determinacy of refinements to the difference hierarchy of co-analytic sets.
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1. What other combinations of T, Γ can we find proofs of?
2. Are there reverse implications, or at least limitations?
3. Does the generalised lightface hierarchy generate interesting effective descriptive set theory?
4. Is the specialised forcing useful for anything else?

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Thanks

Definition

Let M be a mouse, Q^M_κ the Q-structure of M at κ and $\theta = On \cap Q^M_\kappa$. M is *clever* if, for any Σ_1 formula $\varphi(x, y)$ and parameter $p \in [\kappa]^{<\omega}$,

$$\{\xi < \kappa \mid Q^M_\kappa \models \varphi(\xi, p)\} \in F^M \implies$$
$$\exists \tau < \theta \left(\{\xi < \tau \mid J^{\vec{F}}_\tau \models \varphi(\xi, p)\} \in F^\kappa \cap Q^M_\kappa \right)$$

This implies Rowbottom's theorem holds for partitions Σ_1 definable over M.