Farmer Schlutzenberg 3

assume π is a k-embedding

(if $\sup \pi'' \rho_k^H < \rho_k^M$ then

$H \in M$ easily)

want solidity for $\gamma \in p^M$

$\pi : H \longrightarrow M$, $\quad H = \mathrm{cHull}^M(q^M \cup \gamma)$

$q^M = p^M \setminus (\gamma+1)$

$\mathcal{B} = (H, M, \gamma, \lambda, F)$

assume $F = F^{M|\lambda}$

$M|\lambda$ is active, $\gamma^{+H} = \lambda < \gamma^{+M}$

$\delta = \gamma$.

Clai: $H \triangleleft M$

Pf.: Supp. not. Compare B with $\mathcal{B}$

$U = ult(B, E)$ for E weakly amenable (= semi-close) to B (i.e., $M|\kappa^{+M} = U|\kappa^{+U}$)

$U = B' = (M', H', \delta', \lambda', F')$

$M' = ult(M; E)$

$j : M \longrightarrow M'$

$H' = cHull^{M'}(\delta' \cup \{q^{M'}\})$

$\pi' : H' \longrightarrow M'$.

$i : H \longrightarrow H'$ commutes:

$$\begin{tikzcd} H' \arrow[r, "\pi'"] & M' \\ H \arrow[u, "i"] \arrow[r, "\pi"'] & M \arrow[u] \end{tikzcd}$$

facts. • $i \restriction \lambda = j \restriction \lambda$

• j is continuous at λ

index of F

• $\lambda' = ((\delta')^+)^{H'}$

• i is a k-embedding

i, j, π, π'

• $q^{H'} = i(q^H)$

• q^H is solid for H

$\Rightarrow$ $q^{H'}$ — ″ — H'

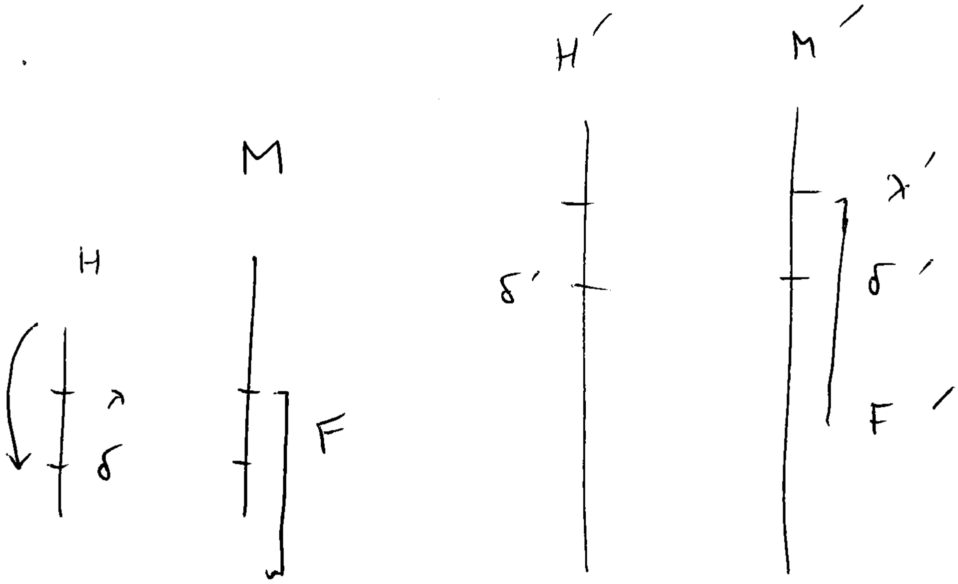
$\Rightarrow$ $q^{H'} = p^{H'} \restriction lh(q^{H'})$.

(have $p^H \setminus \gamma \leq q^H$ as $H = \mathrm{Hull}^H(q^H \cup \gamma)$)

$H <^* M$.

$H' \not< M'$.

pf.:

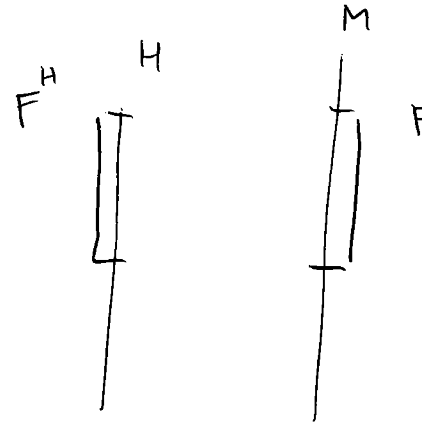


if $H' \lhd M'$, then

$H' = M' | \lambda'$, as

$\mathcal{P}(\delta')^{H'} = \mathcal{P}(\delta')^{M|\lambda'}$

So $\lambda = OR^{H}$, H is active.



$F^H \neq F$.

but then $F^{H'} \neq F'$.

<u>rules of comparison.</u> from trees $\mathcal{T}, \mathcal{U}$.

At stage α, have available models

$\mathcal{A}^{\mathcal{T}}_{\alpha}, \mathcal{A}^{\mathcal{U}}_{\alpha}$

models of $\mathcal{T}(, \mathcal{U})$ either $B^{\mathcal{T}}_{\alpha}$, bicephalus

$(H^{\mathcal{T}}_{\alpha}, M^{\mathcal{T}}_{\alpha}, \dots)$ or a premouse

$B^{\mathcal{T}}_{\alpha} = H^{\mathcal{T}}_{\alpha}$ or $M^{\mathcal{T}}_{\alpha}$.

$\mathcal{A}^{\mathcal{T}}_{\alpha} \subset \{ H^{\mathcal{T}}_{\alpha}, M^{\mathcal{T}}_{\alpha} \}$

if $E^{\mathcal{I}}_\alpha \in \mathbb{E}^{H^{\mathcal{I}}_\alpha}$ and $\alpha \in \mathcal{B}^{\mathcal{I}}$

(i.e. $B^{\mathcal{I}}_\alpha$ is a bicephalus), if $\alpha = \mathrm{pred}^{\mathcal{I}}(\beta+1)$

and if $\beta+1 \notin \mathcal{B}^{\mathcal{I}}$

then $\kappa = \mathrm{crit}(E^{\mathcal{I}}_\beta) \geq \delta^{\mathcal{I}}_\alpha$

and we form $B^{\mathcal{I}}_{\beta+1} = H^{\mathcal{I}}_{\beta+1}$

(no $M^{\mathcal{I}}_{\beta+1}$ defined)

- if $\mathcal{A}^{\mathcal{I}}_\alpha = \{N\}$, $\mathcal{A}^{\mathcal{U}}_\alpha = \{P\}$

  just iterate as lean disagreement

- if $\mathcal{A}^{\mathcal{I}}_\alpha = \{H', M'\}$, $\mathcal{A}^{\mathcal{U}}_\alpha = \{N\}$

(a) (A1) assume $N \ntrianglelefteq M'$ [o.w. stop with "exception 1"]

(b) (A2) Supp. that if $M' \trianglelefteq N$, then $M' = N$

and $\mathrm{root}^{\mathcal{U}}(\alpha) \neq \alpha$ and $\mathrm{root}^{\mathcal{U}}(\alpha) \in \mathcal{M}^{\mathcal{U}}$.

and $[0,\alpha]_{\mathcal{U}}$ does not drop (in any way)

[o.w. stop with "exception 2"]

implies N is "above M" in $\mathcal{U}$

then, $\mathrm{root}^{\mathcal{U}}(\alpha) = $ least $\beta \leq_{\mathcal{U}} \alpha$ s.t. $\beta \in \mathcal{B}^{\mathcal{U}}$

$H^{\mathcal{J}}_\alpha$ ... not bicephali

$B \rightsquigarrow B' = B^{\mathcal{U}}_\beta$

bicephalus

extender of $M^{\mathcal{U}}_\alpha$, incl. top ext.

$$\mathcal{M}^{\mathcal{U}} = \{ \alpha < lh(\mathcal{U}) : E^{\mathcal{U}}_\alpha \in \mathbb{E}_+(M^{\mathcal{U}}_\alpha) \}$$

$$\mathcal{H}^{\mathcal{U}} = \{ \alpha < lh(\mathcal{U}) : E^{\mathcal{U}}_\alpha \in \mathbb{E}_+(H^{\mathcal{U}}_\alpha) \}$$

<u>example.</u> $M' = N$

M'

$B' = (H', M', -)$

$B^* = (H^*, M^*, -)$

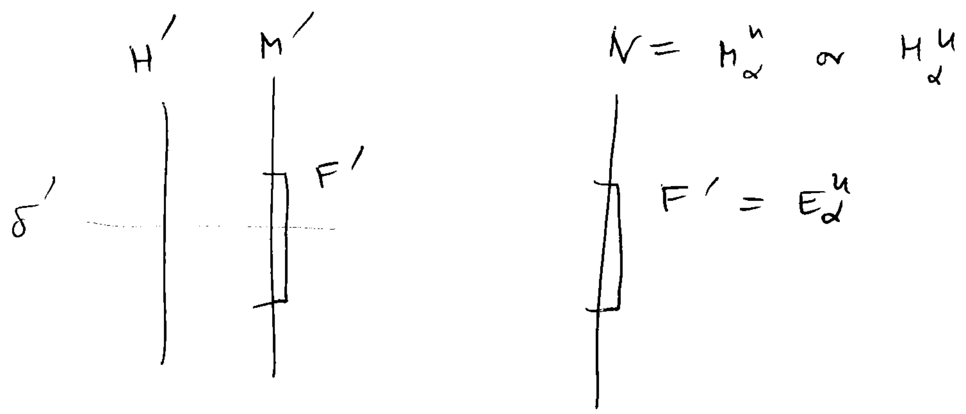
$\mathcal{B}$ $\qquad$ $\mathcal{B}$

$\mathcal{J}$ $\qquad$ $\mathcal{U}$

(c) if (i) $M' = N$, or

(ii) ...

<u>then</u> in $\mathcal{J}$ we move into H', and set $E^u_\alpha = F'$.



iterate least disagreement between H', N.

i.e. thus $E^u_\alpha = F'$

+ either $H' | \lambda'$ is active with

$E = E^{\mathcal{J}}_\alpha$

or o.w., the set $A^{\mathcal{J}}_{\alpha+1} = \{H\}$ + $E^{\mathcal{J}}_\alpha = \emptyset$.

(d) o.w. $M' \not\leq N \not\leq M'$ (by assumption)

iterate at least disagreement

if $lh(E^u_\alpha) > \delta'$, in $\mathcal{J}$ move into M'.

if $\mathcal{A}^{\mathcal{T}}_{\alpha} = \{H_0, M_0\}$, $\mathcal{A}^{\mathcal{U}}_{\alpha} = \{H_1, M_1\}$.

(a) if $M_0 \trianglelefteq M_1$ or $M_1 \trianglelefteq M_0$, stop with "exception 3."

(b) assume (A3) if $M_0 = M_1$ then $\mathcal{B}_0 = \mathcal{B}_1$.
[o.w. stop with "exception 4"]

(c) if (i) $M_0 = M_1$, or (ii) ...
then in $\mathcal{T}$ move into H_0
and set $E^{\mathcal{U}}_{\alpha} = F_0 = F_1$
(with $\mathcal{B}_0 = (H_0, M_0, \delta_0, \lambda_0, F_0)$)

(d) o.w. $M_0 \ntrianglelefteq M_1 \ntrianglelefteq M_0$, initiate least disagreement
and if $lh(E^{\mathcal{U}}_{\alpha}) > \delta_0$, then in $\mathcal{T}$ move into M_0.

notes :

① supp. at stage α, in $\mathcal{J}$ we move into R, $\alpha \in \mathcal{B}^{\mathcal{J}}$, $\mathcal{A}^{\mathcal{J}}_{\alpha} = \{H', M'\}$

then $\forall \beta < \alpha$ $\quad lh(E^{\mathcal{J}}_{\beta})$, $lh(E^{\mathcal{U}}_{\beta}) \leq \delta(B^{\mathcal{J}}_{\alpha})$

and $lh(E^{\mathcal{U}}_{\alpha}) > \delta(B^{\mathcal{J}}_{\alpha})$
$\quad\neq \emptyset$

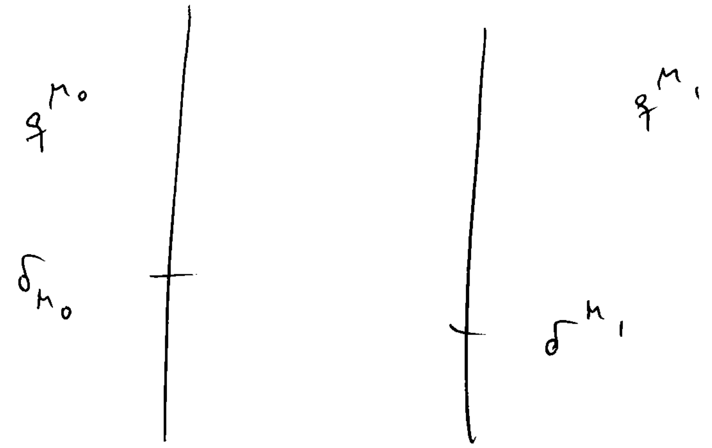
and for all $\beta > \alpha$ $\quad R \ntrianglelefteq M^{\mathcal{U}}_{\beta}$ or

$R \ntrianglelefteq H^{\mathcal{U}}_{\beta}$.

(can't have $R \lhd M^{\mathcal{U}}_{\beta}$ for $\beta > \alpha$
because R projects to $\delta(B^{\mathcal{J}}_{\alpha})$)

this gives (A1) .

② (A3) : $M_0 = M_1$



$(q^{M_0}, \delta^{M_0}) =$ lex-least

(q, δ) s.t. $M_0 = \mathrm{Hull}^{M_0}(q \cup (\delta+1))$.

[pf.: if $q^{M_0} \cup \{\delta^{M_0}\} \in$
$\mathrm{Hull}^{M_0}(q \cup (\delta+1))$, then $(q,\delta) <$
(q^{M_0}, δ^{M_0}), this pulls back to
M.]

Analyze comparison.

1) all extenders applied to $\underline{1}$ model
or close to it.

—

if $\alpha \in \mathcal{B}^{\mathcal{I}}$ then $\mathcal{A}^{\mathcal{I}}_{\alpha} = \{N\}$.

let $\mathrm{movein}^{\mathcal{I}}(\alpha) =$ larger $\beta \leq_{\mathcal{I}} \alpha$ s.t.
$\mathrm{card}(\mathcal{A}^{\mathcal{I}}_{\beta}) = 2$ $\quad(\mathrm{root}^{\mathcal{I}}(\alpha) = \alpha)$

<u>clm.</u> if comparison stops with an exception, then M is solid.

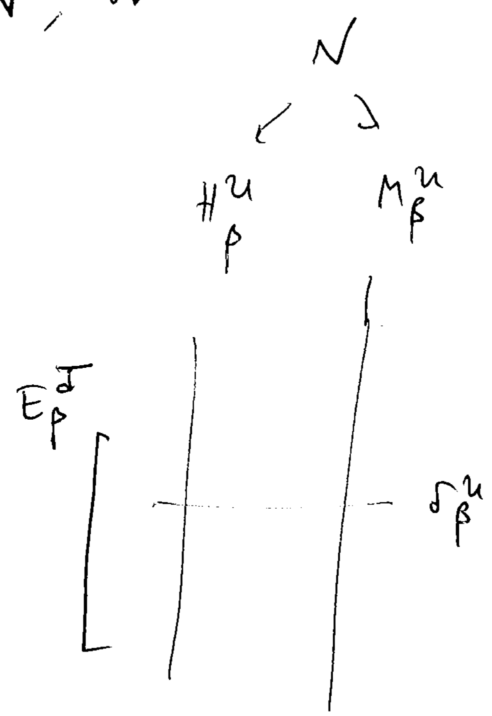
pf.: suff. to see M' is solid. (by the lem for last time, then M is solid.)

Supp. exception 2 happens.

$\mathcal{A}_\alpha^{\mathcal{T}} = \{H', M'\}$ $\quad A_\alpha^{\mathcal{U}} = \{N'\}$.

either $M' \lhd N \Rightarrow M'$ is solid

or $M' = N$, and in $\mathcal{U}$, there's a drop leading to N, or N is above it.



Suppose $\mathrm{root}^{\mathcal{U}}(\alpha) = \alpha$.

let $\beta = \mathrm{movein}^{\mathcal{U}}(\alpha) < \alpha$.

then $\mathrm{lh}(E_\beta^{\mathcal{T}}) > \delta_\beta^{\mathcal{U}}$.

$\Rightarrow \delta' > \delta_\beta^{\mathcal{U}}$

(then $\alpha > \beta$,

$\mathcal{B}_\alpha^{\mathcal{T}} = B' = (H', M', —)$

$\Rightarrow M' \neq M_\beta^{\mathcal{U}}$

$\Rightarrow M' = H_\beta^{\mathcal{U}} \Rightarrow$

$q^{M'} = q^{H^u_\beta}$

but $\delta' \notin \mathrm{Hull}^{M'}(\delta' \cup q^{M'})$

but $\delta' \in \mathrm{Hull}^{H^u_\beta}(\delta^u_\beta \cup q^{H^u_\beta})$.

$\subsetneq$)

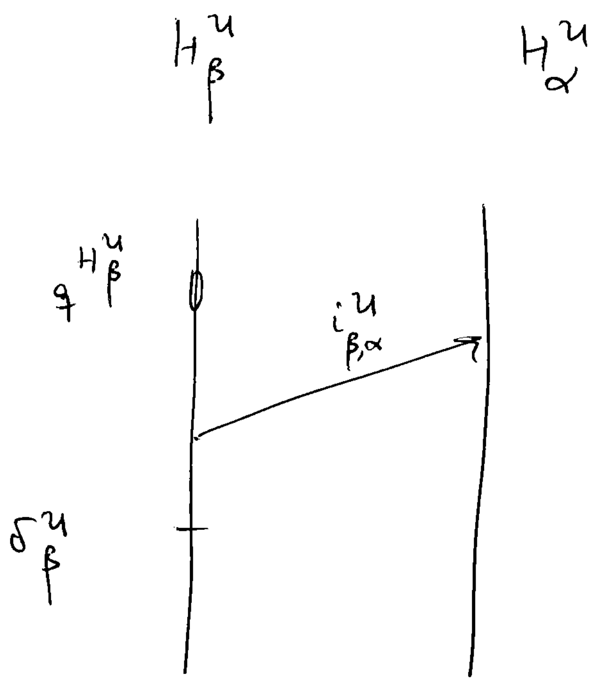
so $\mathrm{root}^u(\alpha) < \alpha$.

if $[0,\alpha]_u$ drops then N is solid

$\Rightarrow$ M' is solid.

so no drop.

since exception 2 holds, $B^u_\alpha = H^u_\alpha$ (abn H).



$i^u_{\beta,\alpha}$ preserves

P_{k+1}

(as $q^{H^u_\beta}$ is solid

and $\mathrm{crit}(i^u_{\beta,\alpha}) \geq \delta^u_\beta$)

so $i^{\mathcal{U}}_{\beta,\alpha}(\mathfrak{q}^{H^{\mathcal{U}}_\beta}) = \mathfrak{q}^{M'} + \delta' \in \mathfrak{p}^{M'}$

$\Rightarrow \delta' < \delta^{\mathcal{U}}_\beta$

but $i^{\mathcal{U}}_{\beta,\alpha}$ adds generators in $[\delta^{\mathcal{U}}_\beta, OR^{M'})$.

contradiction !

<u>now supp. no exception</u>

let $\xi + 1 = lh(\mathcal{T}, \mathcal{U})$

terminates :

$\mathcal{A}^{\mathcal{T}}_\xi = \{P\}$, $\mathcal{A}^{\mathcal{U}}_\xi = \{N\}$.

$P \trianglelefteq N$ or $N \trianglelefteq P$.

$\Rightarrow P = N$.

supp. $\mathrm{root}^{\mathcal{T}}(\xi) \neq \xi \neq \mathrm{root}^{\mathcal{U}}(\xi)$

- standard args gives contradiction,

unless $\mathrm{root}^{\mathcal{T}}(\xi) \in \mathcal{M}^{\mathcal{T}}$ ~~and~~ and $[0,\xi]_{\mathcal{T}}$ no drop

$\mathrm{root}^{\mathcal{U}}(\xi) \in \mathcal{M}^{\mathcal{U}}$ and $[0,\xi]_{\mathcal{T}}$ no drop.

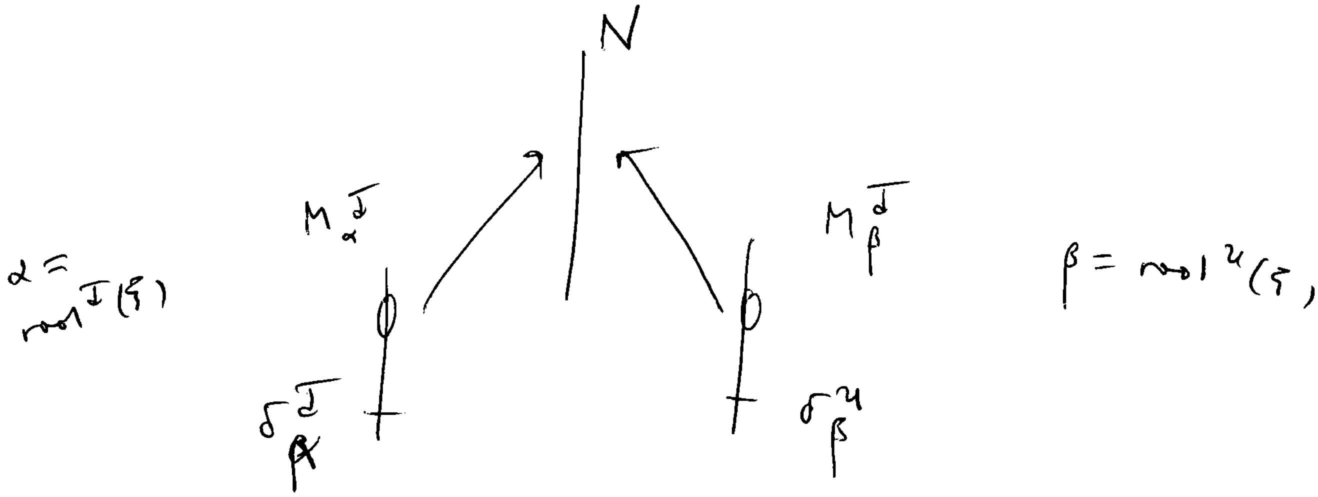
say $\mathrm{root}^{\mathcal{J}}(\xi) \neq \xi \neq \mathrm{root}^{\mathcal{U}}(\xi)$

+ $\mathrm{root}^{\mathcal{J}}(\xi) \in \mathcal{M}^{\mathcal{J}}$ + $[0,\xi]_{\mathcal{J}}$ has no drop.

if $[0,\xi]_{\mathcal{U}}$ drops $\Rightarrow$ N is solid

$\Rightarrow$ M is solid

so supp. $[0,\xi]_{\mathcal{U}}$ has no drop.



$i^{\mathcal{J}}_{\alpha\xi}$ preserves p_{k+1},

so does $i^{\mathcal{U}}_{\beta\xi}$.

use hull property arg't to show comparison extends used.

[no $E^{\mathcal{U}}_{\gamma}$ used in $[0,\xi]_{\mathcal{U}}$ overlaps $\delta^{\mathcal{J}}_{\alpha}$

because N has the (ϱ^N, σ^N)-hull

propy at $\delta^{\mathcal{J}}_\alpha$.

now supp. $\mathrm{root}^{\mathcal{J}}(\xi) = \xi$.

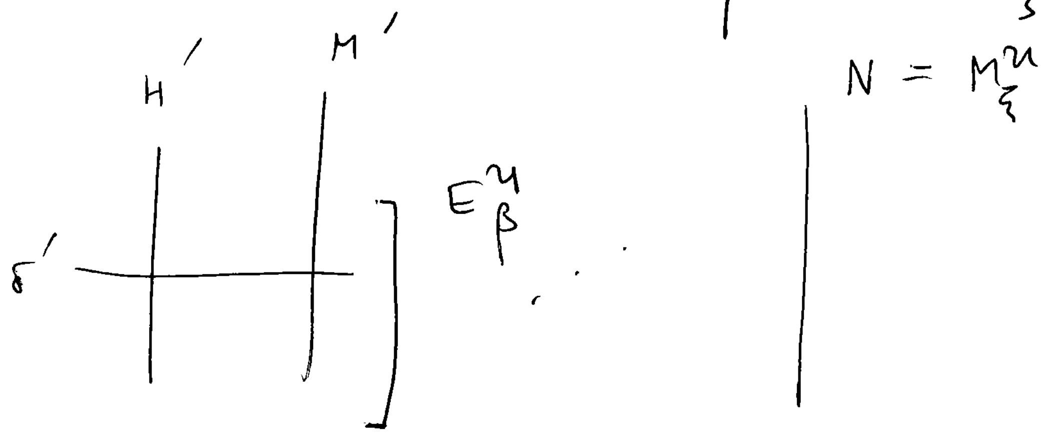
$\Rightarrow$ $\mathrm{root}^{\mathcal{U}}(\xi) < \xi$.

let $\beta = \mathrm{movein}^{\mathcal{J}}(\xi)$

so $\mathrm{lh}(E^{\mathcal{U}}_\beta) > \delta^{\mathcal{J}}_\xi = \delta^{\mathcal{J}}_\beta$

$E^{\mathcal{J}}_\alpha = \emptyset$ f.a. $\alpha \geq \beta$.

let $(H', M', -) = B^{\mathcal{J}}_\beta = B^{\mathcal{J}}_\xi$



Supp. $N = M'$.

no exception!

$\Rightarrow$ in $\mathcal{U}$, N is above M', no drop in $[0, \zeta]_{\mathcal{U}}$.

$$M' = \mathrm{Hull}^{M'}((\delta'+1) \cup q^{M'})$$

$$q^{M'} \cap \delta' \trianglelefteq p^{M'}.$$

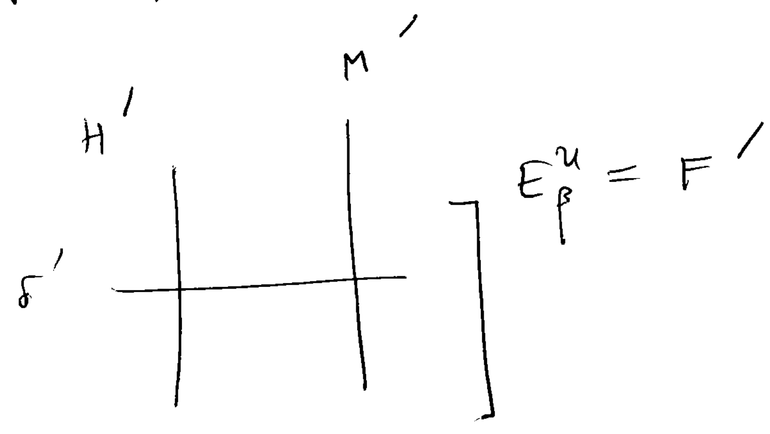
$\Rightarrow$ $\zeta = \beta + 1$ (o.w. add generators ...)

$\Rightarrow$ $E = E_\beta^{\mathcal{U}}$ is superstrong, crit$(E) = \delta_\alpha^{\mathcal{U}}$

$i_E(\delta_\alpha^{\mathcal{U}}) = \delta'$

($\neq$ if there are no superstrongs on the sequence)

So $N = H'$.



rules of comparison:
the more iter H', $E_\beta^{\mathcal{U}} = F'$.

and $M' = N = M^{\mathcal{U}}_{\beta}$

and because no exception,

have $A^{\mathcal{U}}_{\beta} = \{N\}$

(consider this case)

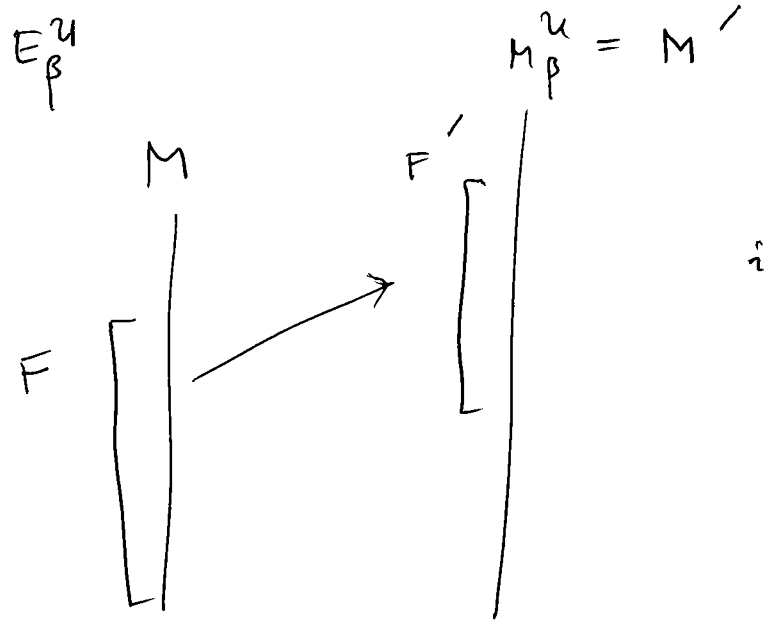
$\Rightarrow M' = N$

and in $\mathcal{U}$, N is also M without drop.

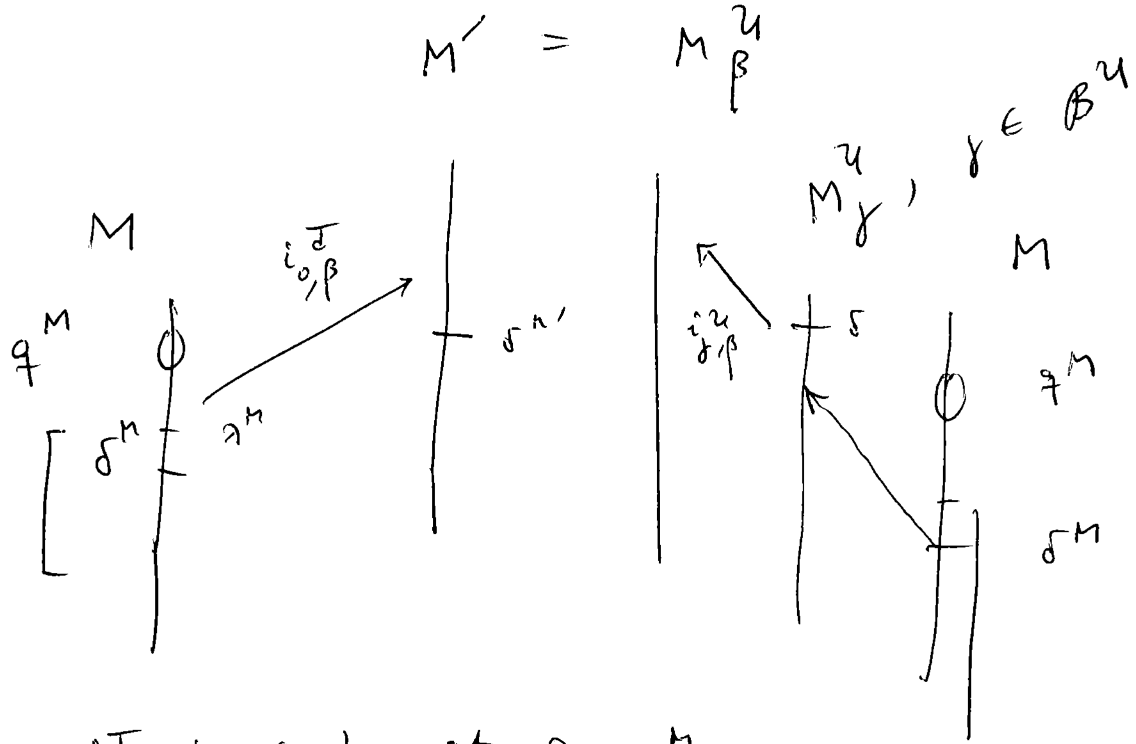
then $F' \in \operatorname{ran}(i^{\mathcal{U}}_{0,\beta})$

$\quad\; \|$

$\quad E^{\mathcal{U}}_{\beta}$



in fact, $F' = i^{\mathcal{U}}_{0,\beta}(F)$.



$i^{\mathcal{T}}_{0,\beta}$ is cont. at $\lambda = \lambda^M$

$i^{\mathcal{T}}_{0,\beta}(F) = F'$

$i^{\mathcal{U}}_{0,\beta}$ cont. at λ

$i^{\mathcal{T}}_{0,\beta}(F) = i^{\mathcal{U}}_{0\beta}(F)$.

$$\mathrm{crit}(i^{\mathcal{U}}_{\gamma,\beta}) > i^{\mathcal{U}}_{0\gamma}(\kappa) = \mathrm{crit}(F)$$

$\Rightarrow \gamma = \mathrm{pred}^{\mathcal{U}}(\beta+1)$, and $\beta+1 \in \mathcal{B}^{\mathcal{U}}$.

so $\beta+1 < \xi$, as $\xi \notin \mathcal{B}^{u}$.

but then $H' \neq N = M^{u}_{\xi}$, as M^{u}_{ξ} would have too many generators to exclude E^{u}_{η}, some $\eta > \beta$.

contradiction!