John Steel 3.

background induced strategies normalize well.

def. M is uniquely normally iterable above κ iff whenever $\mathcal{T}$ is a nice normal tree on M with all crits $> \kappa$, then $\mathcal{T}$ has a unique cof. w.f. branch.

def. M is strongly uniquely iterable above κ iff this is true for finite stacks of normal trees.

"nice": $M_\alpha^{\mathcal{T}} \models$ "$\mathrm{lh}(E_\alpha^{\mathcal{T}}) = \mathrm{strength}(E_\alpha^{\mathcal{T}})$ is inaccessible"

rmk: enough probably to assume

$M_\alpha^{\mathcal{T}} \models$ "$\mathrm{ult}(V, E_\alpha^{\mathcal{T}})$ is ω-closed"

need some restriction on $E_\alpha^{\mathcal{T}}$:

woodin counterexample.

UBH can fail o.w.; CBH can fail, too.

<u>Thm.</u> (Woodin) Let κ be supercompact, and assume UBH holds for nice, normal $\mathcal{T}$ above κ. Then V is uniquely it. th. above κ, in all $V[G]$

Pf.: Let $\mathcal{T}$ on V above κ, lim lh. need to see $\exists$ cof. well-fdd. branch b.

Let $j : V \longrightarrow M$, $\text{crit}(j) = \kappa$.

$$j'' V_\eta \in M, \text{ where } \mathcal{T} \in V_\eta .$$

then $j''\mathcal{T} \in M$, and $j''\mathcal{T} \cong j\mathcal{T}$.

working in M, $j\mathcal{T}$ has a cof. w.fdd branch : $|j\mathcal{T}| < j(\kappa)$, so existence theorem of Martin–Steel work:

(make $j\mathcal{T}$ ctbl. via G, think of it as being in $V[G]$, use UBH in $V[G]$.)

<u>folk thm.</u> (AD^+) if (N^*, Σ^*) is a coarse iterable Γ^+-woodin model ($\Gamma \subset \Gamma^+$ some nice pointclass) then for any $\eta < \delta^{N^*}$, the restriction of Σ^* to trees on $V_\eta^{N^*}$ witnesses unique iterability of $V_\eta^{N^*}$ by Σ^* (even for trees in V).

the 1st Γ-woodin

<u>"pf."</u> $\Sigma^*(\vec{\mathcal{T}} \frown \mathcal{U})$ = unique cof b s.t. $C_\Gamma(m(\mathcal{U})) \subset M_b^{\mathcal{T}}$.

e.g. if δ is least s.t. $L(V_\delta) \models$ "δ is woodin" and $\forall$ sn x, $x^\#$ ex., then $L(V_\delta)$ is uniquely it. ble above ω.

for trees on V, we have a counterexample to UBH for ~~st~~ stacks of normal trees, the stack being of length 2.

(Neeman, Steel: counterexamples to UBH + CBH)

<u>Sketch:</u>

$L_\alpha(V_\delta, F) \models$ "KP + δ is Woodin + $\mathrm{ult}(V, F) \supset V_\delta$"

min such satisfying

$$\mathrm{cf}(\delta) = \omega.$$

Can build an 'alternating chain' $(E_n : n<\omega)$ on V_δ strongly non-overlapping, nice extenders, such that

$$\delta(\mathcal{U}) = \delta = i_b^{\mathcal{U}}(\delta) = i_c^{\mathcal{U}}(\delta)$$

and $M_n^{\bar{d}} \models$ "E_n is $i_{0n}(F)$-strong" ~~+~~

get $i_b(F) = i_c(F)$.

let $\mathcal{I} = \langle F \rangle \cap \mathcal{U}$.

then $M_b^{\mathcal{I}} = M_c^{\mathcal{I}}$.

$$\begin{tikzcd} V \arrow[r, "F"] \arrow[dr] & \mathrm{ult}(V; F) \arrow[d, "i_b"] \\ & M_b^{\mathcal{I}} \end{tikzcd}$$

$$\mathrm{ult}(V; i_c(F)) = \mathrm{ult}(V; i_b(F))$$
$$\| $$
$$M_c^{\mathcal{I}}$$

$\dashv$

example with $\mathcal{I}$ nice (woodin)

assume δ is a measurable woodin

(drop $\mathrm{cf}(\delta) = \omega$)

• μ on δ.

$I =$ iterate μ ω times. gives $i : V \to N$

$F^* = i(F)$.

the tree is

$I ^\frown \langle F^* \rangle ^\frown$ an alt. chain as constr. above.

def. M is uniquely iterable for ~~stacks~~ 2-stacks above κ iff (b)

whenever $(\mathcal{I}, \mathcal{U})$ is a stack on M above κ, $\mathcal{U}$ has lim length, and $W(\mathcal{I}, \mathcal{U})$ is by Σ_0, then ~~[illegible]~~ $\exists$ unique cofinal b of $\mathcal{U}$ s.t. $W(\mathcal{I}, \mathcal{U}^\frown b)$ is by Σ_0

(a) M is uniquely iterable above κ, say by Σ_0.

def. for $\mathcal{U}_0, \dots, \mathcal{U}_n$ a finite stack of normal trees,

$$W(\mathcal{U}_1, \dots, \mathcal{U}_n) = W(W(\mathcal{U}_1, \dots, \mathcal{U}_{n-1}), \pi \mathcal{U}_n),$$

where $\pi : M_\infty^{(\mathcal{U}_1, \dots, \mathcal{U}_{n-1})} \longrightarrow M_\infty^{W(\mathcal{U}_1, \dots, \mathcal{U}_{n-1})}$

cones for normalizing.

$$W(\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3) \underset{?}{\neq} W(\mathcal{U}_1, W(\mathcal{U}_2, \mathcal{U}_3)).$$

<u>def.</u> M is uniquely it. th for finite stacks iff

(a) M is uniquely iterable ~~for~~ ~~by~~ w.r.t. normal trees, via Σ_0, and

(b) when $(\mathcal{U}_1, \ldots, \mathcal{U}_n)$ is a stack s.t. $W(\mathcal{U}_1, \ldots, \mathcal{U}_n)$ is by Σ_0, then $\exists$ unique cof. b s.t. $W(\mathcal{U}_1, \ldots, \mathcal{U}_n{}^\frown b)$ is by Σ_0.

So have a unique completion Σ of Σ_0 for finite stacks.

for $M = V$,
write $\Omega^{uBH}_{\text{normal}}$ for the partial strategy of choosing unique w.fdd. branches,
Ω^{uBH} = its completion.

proposition. Suppose $\mathcal{U}_1, \ldots, \mathcal{U}_{n-1}$ is a stack by Ω^{uBH}, last model M_{n-1}, and $M_{n-1} \models$ "$\mathcal{U}_n$ is by $\Omega^{uBH}_{\text{normal}}$" then $(\mathcal{U}_1, \ldots, \mathcal{U}_n)$ is by Ω^{uBH}.

proof: exercise, with the iterability pf of martin-steel.

thm. ~~[illegible]~~ assume V is uniquely iterable above κ. (i.e., Ω^{uBH} is total on stacks above κ) let $\mathbb{C}$ be a W-construction above κ. let

$\Omega_{\nu,k}$ = strategy for $M^{\mathbb{C}}_{\nu,k}$ induced by Ω^{uBH}.

then $\Omega_{\nu,k}$ normalizes well.

Sketch. let $(\mathcal{I}, \mathcal{U})$ be by ~~[illegible]~~ $\Omega_{\nu,k}$.

let $\operatorname{lift}(\mathcal{I}, M_{\nu,k}, \mathbb{C}) =$

$$\left\langle \mathcal{I}^+, (\eta^{\mathcal{I}}_\xi, \ell^{\mathcal{I}}_\xi : \xi \le \xi_0), (\psi^{\mathcal{I}}_\xi : \xi \le \xi_0) \right\rangle$$

$\mathcal{I}^+$ by ~~[illegible]~~ $\Omega^{\mathrm{UBH}}_{\mathrm{nmeal}}$.

ln

$$\operatorname{lift}\left(\psi^{\mathcal{I}}_{\xi_0}\mathcal{U}, \; M^{i^{\mathcal{I}^*}_{0,\xi_0}(\mathbb{C})}_{\langle \eta^{\mathcal{I}}_{\xi_0}, \ell^{\mathcal{I}}_{\xi_0} \rangle}, \; i^{\mathcal{I}^*}_{0,\xi_0}(\mathbb{C})\right)$$

$$= \left\langle \mathcal{U}^*, (\eta^{\mathcal{U}}_\xi, \ell^{\mathcal{U}}_\xi : \xi < \operatorname{lh}(\mathcal{U})), (\Psi^{\mathcal{U}}_\xi : \xi < \operatorname{lh}(\mathcal{U})) \right\rangle$$

assme <u>strong</u> unique iterability for V for simplicity.

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we want to see $W(\mathcal{I}, \mathcal{U})$ is by $\Omega_{\nu,k}$.
we have that $W(\mathcal{I}^*, \mathcal{U}^*)$ is by Ω^{UBH}. show $\operatorname{lift}(W(\mathcal{I},\mathcal{U}), M_{\nu,k}, \mathbb{C})$

$$= \Big(W(\mathcal{I}^*, \mathcal{U}^*), \underbrace{\ldots}_{(*)}\Big)$$

to show this, let

$$W_\gamma = W(\mathcal{I}, \mathcal{U}\restriction \gamma+1)$$

$$W^*_\gamma = W(\mathcal{I}^*, \mathcal{U}^*\restriction \gamma+1)$$

let $\text{lift}(W_\gamma, M_{\nu,h}, \mathbb{C}) = (S^*_\gamma, \underbrace{\ldots}_{(**)})$.

show by induction on γ that

$$S^*_\gamma = W^*_\gamma \restriction \text{lh}(W_\gamma),$$

and relate (*) and (**) as you go along.

idea: normalization commutes with lifting.

<u>th.</u> let P be a ctbl. pm,

Σ a strategy for P (for finite stacks of normal trees) s.t. Σ normalizes well, has strong hull condensation, and is universally baire.

let $\mathbb{C}$ be a W-construction above ω,

then f.o. ν, k, either

(a) $\exists$ Σ-iterate Q via normal $\mathcal{T}$ s.t. $M_{\nu,k} \trianglelefteq Q$ and

$$\Sigma_{\mathcal{T}, M_{\nu,k}} = \Omega_{\nu,k}, \quad \text{or}$$

(b) $\exists (\bar{\nu}, \bar{k}) <_{lex} (\nu, k)$ s.t. (a) is true with $Q = M^{\mathbb{C}}_{(\bar{\nu},\bar{k})}$.

pf.: let (ν_0, k_0) be s.t. $M^{\mathbb{C}}_{(\nu_0,k_0)}$ exists.

we have by ancient lore that for all $(\nu, k) \leq_{lex} (\nu_0, k_0)$

then is a unique normal tree

$W^*_{\nu,k}$ on P via Σ with

last model $M_{\nu,k}$.

$$P \xrightarrow[\Sigma]{\mathcal{J}} M^-_{\nu,k} \longrightarrow F^*$$

background for F

$\uparrow$

$M_{\nu,k}$ without top ext. F

look at $i_{F^*}(\mathcal{J})$

is acc. to Σ by Σ being univ. basic

now use an ext. agreeing w/ F^*

note $\mathbb{C}|k(\nu,k) \sim \mathbb{C}^{\text{Ult}(V;\, F^*)}|k(\nu,k)$

---

let $M = M_{\nu_0,k_0}$. let $\mathcal{U}$ be a

normal tree on M of lim length,

by both $\Sigma_{W^*_{\nu,k},\, M_{\nu_0,k_0}}$ and Ω_{ν_0,k_0}.

let $\mathrm{lift}(\mathcal{U}, M, \mathbb{C}) =$

$$\langle \mathcal{U}^*, \langle \eta_\tau, \ell_\tau : \tau \leq \mathrm{lh}(\mathcal{U})\rangle, \langle \psi^{\mathcal{U}}_\tau : \tau < \mathrm{lh}(\mathcal{U})\rangle \rangle .$$

claim. if b is a cof. w.f. branch of $\mathcal{U}^*$, then $b = \Sigma_{W^*_{\nu,k}, M_{\nu_0,k_0}}(\mathcal{U})$.

rmk.: this b is unique as a w.f. branch of $\mathcal{U}^*$, if it ex.

but b is "absolutely def. ble" for Σ, so it ex. (if not, run a rep. arg., in the end, show b ex.)

so enough to prove the claim.

proof of claim. let

$$S_\xi = M^{\mathcal{U}^*}_\xi$$

$$N_\xi = M^{i^{\mathcal{U}^*}_{0,\xi}(\mathbb{C})}_{\eta_\xi, \ell_\xi} = M^{S_\xi}_{\eta_\xi, \ell_\xi}$$

so $\psi^{\mathcal{U}}_\xi : M^{\mathcal{U}}_\xi \longrightarrow N_\xi$.

$P = P_0$.

we write

$$W^{* \, S_\tau}_{\nu,k} = i^{\mathcal{U}^*}_{0\tau}(\langle \eta, \ell \rangle \mapsto W^*_{\eta,\ell})_{\nu,k}$$

note $\quad i^{\mathcal{U}^*}_{0\tau}(\Sigma) \cap S_\tau = \Sigma \cap S_\tau$,

and $\quad i^{\mathcal{T}}_{0,\tau}(P_\delta) = P_0$.

so $\quad W^{* \, S_\tau}_{\nu,k}$ is by Σ.

all this makes sense with b:

$$S_b = M^{\mathcal{U}^*}_b .$$

$$N_b = M^{S_b}_{\eta_b, \ell_b} ,$$

$$\varphi^{\mathcal{U}}_b : M^{\mathcal{U}}_b \longrightarrow N_b , \quad \text{etc.}$$

let $\quad W^*_\gamma = W^{* \, S_\gamma}_{\eta_\gamma, \ell_\gamma}$ for $\gamma < lh(\mathcal{U})$ or $\gamma = b$.

so $\quad W^*_0$ is our normal tree for P_0

to $\quad M_{\nu_0, k_0} = N_0$.

the last model of W_γ^* is N_γ.

look at $W(W_0^*, \mathcal{U})$.

set $W_\gamma = W(W_0^*, \mathcal{U} \restriction \gamma+1)$.

$W_0 = W_0^*$.

$$W_b = W(W_0^*, \mathcal{U}^\frown b).$$

the W_γ's are all by Σ, since Σ normalizes well.

suff. to show $\boxed{W_b \text{ is by } \Sigma.}$

(because if $\Sigma(\langle W_0, \mathcal{U}\rangle) = c$, then W_c is by Σ by normalizing well; so $\underbrace{br(b, W_0, \mathcal{U})} = br(c, W_0, \mathcal{U})$, so $b = c$.

the branch ~~det~~. thru $\mathcal{U}$ det. by b thru $W(W_0^*, \mathcal{U})$.

$\boxed{W_b \text{ is by } \Sigma :}$

enough to see:

<u>subclaim.</u> W_b is a pseudo hull of W_b^*.

note. W_b^* is by $i_b^{u^*}(\Sigma)$, so by Σ, and Σ has strong hull condensation.

<u>pf. of subclaim:</u> Construct by induction on γ a pseudo hull embedding Φ_γ from W_γ into W_γ^*.

we write
$$z(\gamma) = lh(W_\gamma) - 1$$
$$z^*(\gamma) = lh(W_\gamma^*) - 1$$

$$\Phi_\gamma = (\underset{\substack{\uparrow \\ \text{indices} \to \text{indices}}}{u^\gamma}, \langle t_\beta^{0,\gamma} : \beta \le z(\gamma)\rangle, \langle t_\beta^{1,\gamma} : \beta < z(\gamma)\rangle, p^\gamma \rangle$$

$$\Big(\ \mathrm{dom}(u^\gamma) = z(\gamma)$$
$$p^\gamma(E_\alpha^{W_\gamma}) = t_\alpha^{1,\gamma}(E_\alpha^{W_\gamma}) = E_{u^\gamma(\alpha)}^{W_\gamma^*} \ . \Big)$$

let $\Phi_\gamma \restriction \xi$ be the "initial segment" of Φ_γ that is a pseudo hull embedding of $W_\gamma \restriction \xi$ into $W_\gamma^* \restriction \xi$.

we also have $\hat{p}^\gamma$, $\hat{p}^\gamma(S_\alpha^{W_\gamma}) =$ downward ν^γ close of $\langle p^\gamma(S_\alpha^{W_\gamma}(i)) : i \in \mathrm{dom}\, S_\alpha^{W_\gamma} \rangle$ $= S_{\nu^\gamma(\alpha)}^{W_\gamma^*}$.

$M_\alpha^{W_\gamma}, E_\alpha^\gamma$

$$\begin{tikzcd}
M_\beta^{W_\gamma} \arrow[r, "E_\alpha^{W_\gamma}"] \arrow[d, "t_\beta^{0,\gamma}"'] & M_{\alpha+1}^{W_\gamma} \arrow[drr] & & \\
& & & M_{u(\alpha)+1}^{W_\gamma^*} = M_{\nu^\gamma(\alpha+1)}^{W_\gamma^*} \\
M_{\nu^\gamma(\beta)}^{W_\gamma^*} \arrow[r] & M_{\beta^*}^{W_\gamma^*} \arrow[r] \arrow[urr] & M_{u(\beta)}^{W_\gamma^*} &
\end{tikzcd}$$

$$E_{u(\alpha)}^{W_\gamma^*} = p^\gamma(E_\alpha^{W_\gamma})$$

inductive hypotheses :

assume Φ_η, $\eta \le \gamma$ given, satisfying

$(\dagger)_\gamma$ (a) for $\xi < \eta < \gamma$,

$$\Phi_\xi \upharpoonright \alpha_\xi + 1 = \Phi_\eta \upharpoonright \alpha_\xi + 1$$

here : α_ξ = least α s.t. F_ξ is a the $M_\alpha^{N_\xi}$- seq.

$$W_{\xi+1} = W(W_\nu, F_\xi) .$$

(b) for $\nu < \eta \le \gamma$,

$$t^\eta \upharpoonright lh(F_\nu) = res_\nu \circ t^\nu \upharpoonright lh(F_\nu) .$$

$\uparrow$

resurrects map of F_ν in S_ν .

$$t^\eta = t^{0,\eta}_{z(\eta)} , \text{ or rather}$$

$$= \hat{\imath}_{\nu'(z(\eta)), z^*(\eta)} \circ t^{0,\eta}_{z(\eta)} .$$

(c) for $\xi \le \gamma$, $\Psi_\xi^{\mathcal{U}} = t^\xi \circ \sigma_\xi$

$$\begin{tikzcd} M_\xi^{\mathcal{U}} \arrow[r, "\sigma_\xi"] \arrow[rr, bend right, "\Psi_\xi^{\mathcal{U}}"'] & M_{z(\xi)}^{W_\xi} \arrow[r, "t^\xi"] & N_\xi = M_{z^*(\xi)}^{W_\xi^*} \end{tikzcd}$$