John Steel.

Recall: A tower of measures is a

seq. $\vec{\mu} = \langle \mu_n : n < \omega \rangle$ s.t.

$\exists Z \ \forall n \ \mu_n(Z^n) = 1$ and

$\mu_n(A) \Rightarrow \mu_{n+1}(\{ s^\frown \langle z \rangle : s \in A \}) = 1$

$\vec{\mu}$ is ω-complete iff $\forall \langle A_n : n < \omega \rangle$ s.t.

$\mu_n(A_n) = 1 \ \forall n$

$\exists f \ \forall n \ f \restriction n \in A_n$.

$\Longleftrightarrow$ $\mathrm{ult}(V; \vec{\mu})$ is w.f.

Lem (Woodin) Supp. $A \subset X^\omega$.

Supp. $f : X^{<\omega} \to$ measures, say $f(s)$

is a measure on $Z^{\mathrm{dom}(s)}$, some Z,

is s.t. each $f(s)$ is $|X|^+$-complete,

and $x \in A \Longleftrightarrow \langle f(x \restriction n) : n < \omega \rangle$

is a well-founded tower.

then A is homogeneously Suslin.

Pf.: pick for each $x \notin A$ sets

B^x_n of measure 1 ~~of~~ for $f(x \upharpoonright n)$

s.t. $(B^x_n : n < \omega)$ witness non-

ω-completeness.

the tree T for homogeneity is

$$(s, (\alpha_0, \dots, \alpha_{n-1})) \in T \quad \text{iff}$$

$$(\alpha_0, \dots, \alpha_{n-1}) \in \bigcap_{x \supset s} B^x_n$$

T is homogeneous via $s \mapsto f(s)$,

$p[T]$ ~~al~~ $= A$. $\dashv$

let κ be supercompact.

set, for any σ:

$$TW(\sigma) = \{\vec{\mu} \in \sigma^\omega : \mu \text{ is a tower}\}$$

$$WF(\sigma) = \{\vec{\mu} \in TW(\sigma) : \vec{\mu} \text{ is w.f.}\}$$

$$IF(\sigma) = \{\vec{\mu} \in TW(\sigma) ; \vec{\mu} \text{ is ill-fdd.}\}$$

TW_α, WF_α, IF_α $:=$

$TW(\sigma)$, $WF(\sigma)$, $IF(\sigma)$ for $\sigma =$
$\{\nu : \nu$ is a α-complete
measure on $V_\alpha\}$, call that meas_α.

<u>lem.</u> (Woodin) let $j : V \longrightarrow M$,
$\text{crit}(j) = \kappa$, $V_{j(\kappa)+2} \subset M$. then
WF_κ is $j(\kappa)$-homogeneously Souslin.

Pf.: use $f = j \restriction \text{meas}_\kappa$. ⊣

fix a j as in the prev. lem, but with $j : V \longrightarrow N$.
consider $E_j \restriction 2^{2^\kappa}$ (a short extender)

$$E_j \restriction 2^{2^\kappa} = E$$

$$V_{\kappa+2} \subset \text{Ult}(V; E) = M$$

$$\begin{tikzcd}
V \arrow[rr, "j"] \arrow[dr, "i_E"'] & & N \\
& M \arrow[ur, "k_0"'] &
\end{tikzcd}$$

M is closed under ω-sequences,

~~all towers for~~ $TW(\text{meas}_\kappa) \subset M$.

$M \models$ "$i_E(\kappa)$ is a limit of woodin cardinals, and

WF_κ is $i_E(\kappa)$ – hom. souslin."

write $i = i_E$.

working in M, let δ be the least woodin $> \kappa$.

we have $M \models$ "WF_κ is δ^+ – hom. souslin,"
by martin–steel.

in M, we have "lipschitz" functions

F, G from TW_κ into some set σ of δ^+-complete measures s.t.

$$\vec{\mu} \in WF_\kappa \text{ iff } F(\vec{\mu}) \in WF(\sigma)$$
$$\vec{\mu} \in IF_\kappa \text{ iff } G(\vec{\mu}) \in WF(\sigma).$$

$$F(\vec{\mu}) = (f(\vec{\mu}\restriction n) : n<\omega)$$

$$G(\vec{\mu}) = (g(\vec{\mu}\restriction n) : n<\omega),$$

some f, g.

again by martin-steel, in M we have a lipschitz fcns

$\vec{\iota} \longmapsto b_{\vec{\iota}}$, $\vec{\iota} \longmapsto c_{\vec{\iota}}$,

where $b_{\vec{\iota}}$, $c_{\vec{\iota}}$ are the odd/even branches ~~branches~~ of an alternating chain s.t. on V_δ, crits $> \kappa$,

$\vec{\iota} \in WF_\sigma$ iff $b_{\vec{\iota}}$ is w.f.

iff $c_{\vec{\iota}}$ is ill-fdd.

<u>claim</u>. there is a $\vec{\mu} \in WF_\kappa$, and a countable set C of generators for E s.t. letting

$$k : \mathrm{ult}(V; E\restriction C) \longrightarrow M$$

(i) $f, g, \delta, \pi \in \mathrm{ran}(k)$

(write $f^C, g^C, \delta^C, \pi^C$ for preimages)

here, $\vec{\nu} \xmapsto{\pi} (b_{\vec{\nu}}, c_{\vec{\nu}})$.

(ii) each $\vec{\mu} \restriction n \in \mathrm{Ult}(V; E \restriction C)$, new, and

(iii) letting $\vec{\rho} = \bigcup_n f^C(\vec{\mu} \restriction n)$

$\vec{\nu} = \bigcup_n g^C(\vec{\mu} \restriction n)$

we have that

$$\mathrm{ult}(V; \vec{\mu}) = \mathrm{Ult}(\mathrm{Ult}(V; E \restriction C); \vec{\nu})$$

$(\vec{\mu} =$ tower for $E \restriction C$ – then – $\vec{\nu})$.

pf.: Construct $\vec{\mu} \restriction n$, C

~~s~~ simultaneously. G is Lipschitz!

$\dashv$

claim. $\vec{\mu}$ is well-founded.

pf.: if not, then $\vec{\nu}^+ = G(\vec{\mu})$.

$\vec{\nu}^+$ is ~~wf~~ wellfounded.

but $k^C : \mathrm{Ult}(V; E|C) \longrightarrow M$

maps $\vec{\nu}$ into $\vec{\nu}^+$. (initial seg. wise)

[note : $\vec{\nu} \notin \mathrm{Ult}(V; E|C)$]

so $\vec{\nu}$ is w.f.

$$\begin{tikzcd}
 & & M \arrow[r] & \mathrm{Ult}(M; \vec{\nu}^+) \\
V \arrow[r] & \mathrm{Ult}(V; E|C) \arrow[ur] \arrow[dr] & & \\
 & & \mathrm{Ult}(\mathrm{Ult}(V; E|C); \vec{\nu}) \arrow[uur] &
\end{tikzcd}$$

$$\underbrace{V \longrightarrow \mathrm{Ult}(V; E|C) \longrightarrow \mathrm{Ult}(\mathrm{Ult}(V; E|C); \vec{\nu})} \cong \mathrm{Ult}(V; \vec{\mu}).$$

so $\mathrm{Ult}(V; \vec{\mu})$ is well-fdd. $\dashv$

~~note~~

<u>notation.</u> $M^C = \mathrm{Ult}(V; E|C)$.

so $\mathrm{Ult}(M^C; \vec{\nu})$ is w.fdd.

<u>claim.</u> $\mathrm{Ult}(M^C; \vec{\rho})$ is well-founded.

$$\begin{array}{c} \mathrm{Ult}(M^C; \vec{\rho}) \\ \downarrow \\ \mathrm{Ult}(M; \vec{\rho}^+) \end{array}$$

well-fdd. $\nearrow$

$\vec{\rho}^{\,+}$ is well-fdd., as $\vec{\mu}$ is.

$(\vec{\rho}^{\,+} = F(\vec{\rho}).)$

let $b^+ = b^+_{\vec{\mu}}$, $c^+ = c^+_{\vec{\mu}}$, where $b = k^{-1}(b^+)$, $c = k^{-1}(c^+)$

<u>claim</u>. $ult(M^C, E_b)$ is well-founded,

and $ult(M^C, E_c)$ is well-founded.

pf.: for b, use $ult(M, E_{b^+})$ is

well-fdd.

for c;

let T when IF_κ is δ^+-hom.

soln.

$\mathcal{U}$ = tree in M that tries to build a

tower $\vec{\eta} \in TW_\kappa$, a path thru

$T_{\vec{\eta}}$, and an inf. desc. chain thru

$ult(M, E_{c^+_{\vec{\eta}}})$.

$\mathcal{U}$ is well-founded!

may assume : $T, \mathcal{U} \in \operatorname{ran}(k^C)$.

write T^C, $\mathcal{U}^C$ for the preimages.

<u>subclaim</u>. $\operatorname{ult}\left(\operatorname{ult}^{M^C}(M^C, \vec{\iota}), E_c\right)$

is well-founded.

[this since $\operatorname{ult}(M^C, E_c)$ is w.f., as $M^C \to \operatorname{ult}(M^C, \vec{\iota})$.]

pf.: $\operatorname{ult}(M^C, \vec{\iota})$ is well-fdd.

if the subclaim is false, $i_{\vec{\iota}}(\mathcal{U}^C)$ has an infinite path.

first coordinate given by $\vec{\mu}$.

$\not\downarrow$ $\dashv$