

Join Theorem

Recall: A tower of measures is a

sq.  $\vec{\mu} = (\mu_n : n < \omega)$  s.t.

$\exists Z \forall n \mu_n(Z^n) = 1$  and

$\mu_n(A) \Rightarrow \mu_{n+1}(\{s \hat{\langle z \rangle} : s \in A\}) = 1$

$\vec{\mu}$  is  $\omega$ -complete iff  $\forall (A_n : n < \omega)$  s.t.

$\mu_n(A_n) = 1 \forall n$

$\exists f \forall n f \upharpoonright n \in A_n$ .

$\Rightarrow \mu_n(V; \vec{\mu})$  is w.t.

Levy (Woodin) type.  $A \subset X^\omega$ .

type.  $f: X^{<\omega} \rightarrow \text{meas}$ , say  $f(s)$

is a measure on  $Z^{\text{dom}(s)}$ , for  $Z$ ,

is s.t. each  $f(s)$  is  $|X|^+$ -complete,

and  $x \in A \iff (f(x \upharpoonright n) : n < \omega)$

is a well-founded tower.

then  $A$  is homogeneously Suslin.

77. : pick for each  $x \notin A$  sets

$B_n^x$  of mean 1  $\forall$  for  $f(x \upharpoonright n)$

s.t.  $(B_n^x : n < \omega)$  witness non-  
 $\omega$ -cylinder.

the tree  $T$  for homogeneity is

$$(s, (\alpha_0, \dots, \alpha_{n-1})) \in T \iff$$

$$(\alpha_0, \dots, \alpha_{n-1}) \in \bigcap_{x \supset s} B_n^x$$

$T$  is homogeneous via  $s \mapsto f(s)$ ,

$$p[T] \cap A = A. \quad \dagger$$

let  $\kappa$  be supercompact.

set, for any  $\sigma$ :

$$TW(\sigma) = \{ \vec{\mu} \in \sigma^\omega : \mu \text{ is a tower} \}$$

$$WF(\sigma) = \{ \vec{\mu} \in TW(\sigma) : \vec{\mu} \text{ is w.f.} \}$$

$$IF(\sigma) = \{ \vec{\mu} \in TW(\sigma) : \vec{\mu} \text{ is ill-fd.} \}$$

$$TW_\alpha, WF_\alpha, IF_\alpha =$$

$TW(\sigma), WF(\sigma), IF(\sigma)$  for  $\sigma =$

$\{ \nu : \nu \text{ is a } \alpha\text{-cyclic} \\ \text{meas on } V_\alpha \}$ , call these  $\text{meas}_\alpha$ .

lea. (woodin) let  $j: V \rightarrow M$ ,

$\text{crit}(j) = \kappa$ ,  $V_{j(\kappa)+2} \subset M$ . then

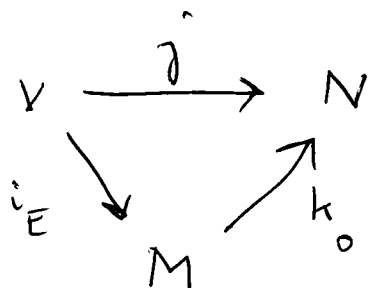
$WF_\kappa$  is  $j(\kappa)$ -homogeneously partition.

$M \therefore$  use  $f = j \upharpoonright \text{meas}_\kappa$   $\rightarrow$

fix a  $j$  as in the prev. lea, but with  $j: V \rightarrow N$ .

consider  $\begin{array}{c} \dot{j} \\ \uparrow \\ E_j \uparrow 2^{2^\kappa} \\ \parallel \\ E \end{array}$  (a short extend)

$$V_{\kappa+2} \subset \text{un}(V; E) = M$$



$M$  is closed under  $\kappa$ -sequences,  
all  $\lambda$ -sequences for  $TW(\text{meas}_\kappa) \subset M$ .

$M \models$  " $i_E(\kappa)$  is a limit of Woodin  
cardinals, and  
 $WF_\kappa$  is  $i_E(\kappa)$ -hom. solution."

with  $i = i_E$ .

working in  $M$ , let  $\delta$  be the least  
Woodin  $> \kappa$ .

we have  $M \models$  " $\mathbb{I}F_\kappa$  is  $\delta^+$ -hom. solution,"  
by matrix-steele.

in  $M$ , we have "lipschitz" functions

$F, G$  from  $TW_\kappa$  into some set  $\sigma$  of  
 $\delta^+$ -complete measures s.t.

$$\vec{\mu} \in WF_\kappa \iff F(\vec{\mu}) \in WF(\sigma)$$

$$\vec{\mu} \in IF_\kappa \iff G(\vec{\mu}) \in WF(\sigma).$$

$$F(\vec{\mu}) = (f(\vec{\mu}|n) : n < \omega)$$

$$G(\vec{\mu}) = (g(\vec{\mu}|n) : n < \omega),$$

on  $f, g$ .

again by markov-skeel, in  $M$  we have a lipschitz fct's

$$\vec{\mu} \mapsto b_{\vec{\mu}}, \quad \vec{\mu} \mapsto c_{\vec{\mu}},$$

odd/even branches

where  $b_{\vec{\mu}}, c_{\vec{\mu}}$  are the ~~branches~~ branches of an alternating chain ~~etc~~ on  $V_f$ ,  $\text{crits} > \kappa$ ,

$$\vec{\mu} \in WF_{\sigma} \iff b_{\vec{\mu}} \text{ is w.f.}$$
$$\iff c_{\vec{\mu}} \text{ is ill-fd.}$$

clai. Then  $\exists$  a  $\vec{\mu} \in WF_{\kappa}$ , and a countable set  $C$  of generators for  $E$  s.t. letting  $k: \text{wt}(V; E/C) \rightarrow M$

(i)  $f, g, \delta, \pi \in \text{ran}(k)$

(with  $f^c, g^c, \delta^c, \pi^c$  for predecessors)

then,  $\vec{z} \xrightarrow{\pi} (b_{\vec{z}}, c_{\vec{z}})$ .

(ii) each  $\vec{\mu} \upharpoonright n \in \text{ult}(V; E \upharpoonright C)$ ,  $\vec{\mu} \xrightarrow{\text{new}}$

(iii) let  $\vec{\rho} = \bigcup_n f^c(\vec{\mu} \upharpoonright n)$

$\vec{z} = \bigcup_n g^c(\vec{\mu} \upharpoonright n)$

we have that

$\text{ult}(V; \vec{\mu}) = \text{ult}(\text{ult}(V; E \upharpoonright C); \vec{z})$

$(\vec{\mu} = \text{tower } \upharpoonright \text{EIC} - \text{then} - \vec{z})$ .

pf.: construct  $\vec{\mu} \upharpoonright n, C$

~~say~~ simultaneously.  $G$  is  $\omega$ -power!

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def.  $\vec{\mu}$  is well-founded.

pf.: if not, let  $\vec{z}^+ = G(\vec{\mu}^+)$ .

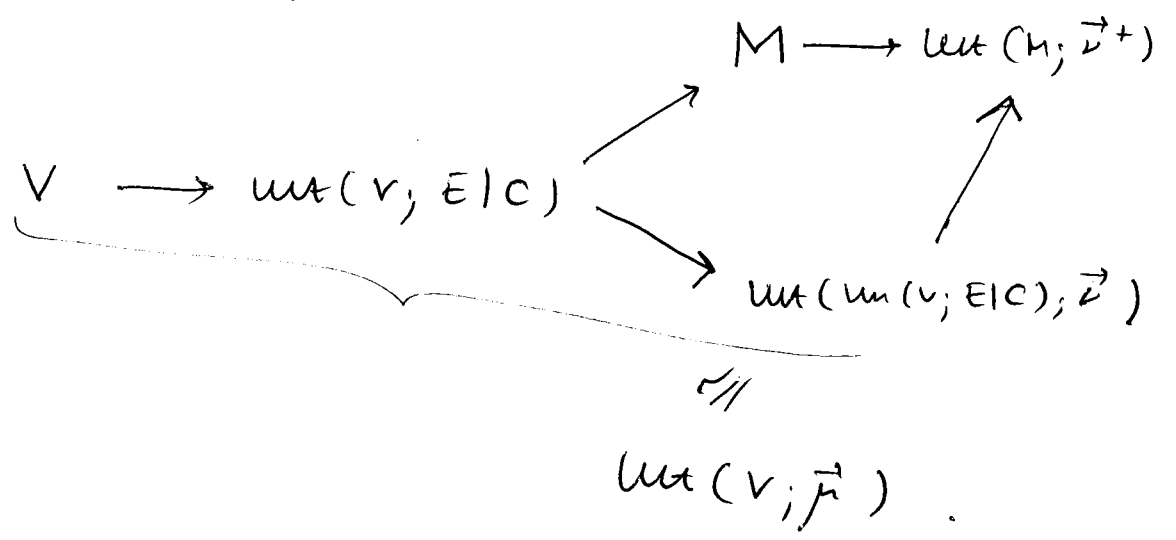
$\vec{z}^+$  is wellfounded

but  $k^c : \text{un}(V; E|C) \rightarrow M$

maps  $\vec{z}$  into  $\vec{z}^+$ . (initial seq. wise)

[note:  $\vec{z} \notin \text{un}(V; E|C)$ ]

so  $\vec{z}$  is w.f.



so  $\text{un}(V; \vec{z})$  is well-fdd.  $\rightarrow$

notation.  $M^c = \text{un}(V; E|C)$ .

so  $\text{un}(M^c; \vec{z})$  is w.fdd.

cli.  $\text{un}(M^c; \vec{\rho})$  is well-founded.

$\downarrow$   
 well-fdd.  $\rightarrow \text{un}(M; \vec{\rho}^+)$

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$\vec{\rho}^+$  is well-fdd., as  $\vec{\mu}$  is.

$$(\vec{\rho}^+ = F(\vec{\rho}) .)$$

let  $b^+ = b_{\vec{\mu}}^+$ ,  $c^+ = c_{\vec{\mu}}^+$ , where  $b = k^{-1}(b^+)$ ,  $c = k^{-1}(c^+)$

clm.  $\text{ult}(M^C, E_b)$  is well-founded,

and  $\text{ult}(M^C, E_c)$  is well-founded.

pf: for  $b$ , we  $\text{ult}(M, E_{b^+})$  is well-fdd.

for  $c$ ;

let  $T$  where  $\text{IF}_k$  is  $\sigma^+$ -hom.

so that

$U =$  tree in  $M$  that tries to build a tower  $\vec{\eta} \in \text{TW}_k$ , a path thru

$T \vec{\eta}$ , and an inf. desc. chain thru

$\text{ult}(M, E_{c^+})$ .

$U$  is well-founded!



may assume:  $T, u \in \text{ran}(k^G)$ .

wh  $T^C, u^C$  for the preimages.

subcl<sup>e</sup>.  $\text{ult}(\text{ult}(M^C, \vec{u}), E_c)$

is well-founded.

[ this is  $\text{ult}(M^C, E_c)$  is w.f., as  
 $M^C \rightarrow \text{ult}(M^C, \vec{u})$  . ]

Pf.  $\therefore \text{ult}(M^C, \vec{u})$  is well-fdd.

if the subcl<sup>e</sup> is false,

$i_{\vec{u}}(u^C)$  has an infinite path.

first coordinate fin by  $\vec{u}$ .



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