Determinacy from strong compactness of ω_1

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Outline

Background

- AD and large cardinal properties of ω_1
- $\mathrm{AD}_{\mathbb{R}} + \mathrm{DC}$ and more large cardinal properties of ω_1
- Combinatorial consequences of strong compactness

Results

- Compactness properties equiconsistent with AD
- Compactness properties equiconsistent with $\mathrm{AD}_{\mathbb{R}} + \mathrm{DC}$

Work in ZF + DC.

- Without AC, "large cardinals" may not be large in the usual sense.
- For example, measurable cardinals can be successors.
- In particular, ω_1 can be measurable.
- We investigate large cardinal properties of ω_1 and their relationship to determinacy.

Definition

ω_1 is measurable if there is a countably complete nonprincipal measure (ultrafilter) on ω_1.

Remark

If μ is a countably complete nonprincipal measure on ω_1, then we can define the ultrapower map $j = j_\mu : V \to \mathrm{Ult}(V, \mu)$.

- $\mathrm{crit}(j) = \omega_1$.
- j is not elementary: $\mathrm{Ult}(V, \mu) \not\models$ "every ordinal less than $j(\omega_1)$ is countable."
- If $M \models \mathrm{ZFC}$ then $j \upharpoonright M$ is an elementary embedding from M to $\mathrm{Ult}(M, \mu)$ (using all functions $\omega_1 \to M$ in V.)

The following theories are equiconsistent:

- ZFC + "there is a measurable cardinal."
- ZF + DC + "ω_1 is measurable."

Proof.

- If ZFC holds and κ is measurable, take a V-generic filter $G \subset \mathrm{Col}(\omega, {<}\kappa)$.
- The symmetric model $V(\mathbb{R}^{V[G]})$ satisfies ZF + DC + "κ is ω_1 and is measurable."
- Conversely, if ZF + DC holds and ω_1 is measurable by μ, then $L[\mu] \models$ ZFC + "ω_1^V is measurable." □

Another route to measurability of ω_1:

Theorem (Solovay)

Assume ZF + AD. Then ω_1 is measurable (by the club filter.)

Remark

AD has higher consistency strength and proves much more:

- There are many measurable cardinals
- ω_1 has stronger large cardinal properties.

We focus on stronger large cardinal properties of ω_1, and obtain equiconsistencies with determinacy theories.

Definition

Let X be an uncountable set and let μ be a measure (ultrafilter) on $\mathcal{P}_{\omega_1}(X)$. We say that μ is:

- countably complete if it is closed under countable intersections.
- fine if $\{\sigma \in \mathcal{P}_{\omega_1}(X) : x \in \sigma\} \in \mu$ for every $x \in X$.

Definition

For X an uncountable set, we say ω_1 is X-strongly compact if there is a countably complete fine measure on $\mathcal{P}_{\omega_1}(X)$.

Remark

ω_1 is ω_1-strongly compact if and only if it is measurable.

Remark

Let X and Y be uncountable sets.
If ω_1 is X-strongly compact and there is a surjection from X to Y, then ω_1 is Y-strongly compact.

Corollary

If ω_1 is $\mathbb{R}$-strongly compact then it is measurable.
(I don't know about the converse.)

Remark

The following theories are equiconsistent:

- ZFC + "there is a measurable cardinal."
- ZF + DC + "ω_1 is $\mathbb{R}$-strongly compact."

(The proof is similar to that for "ω_1 is measurable.")

Another route to $\mathbb{R}$-strong compactness of ω_1:

Theorem (Martin)

Asssume ZF + AD. Then ω_1 is $\mathbb{R}$-strongly compact.

Proof.

Let $A \subset \mathcal{P}_{\omega_1}(\mathbb{R})$. Then the set of Turing degrees d such that $\{x \in \mathbb{R} : x \leq_{\mathrm{T}} d\} \in A$ contains or is disjoint from a cone. □

Definition

Θ is the least ordinal that is not a surjective image of $\mathbb{R}$.

Remark

If ω_1 is $\mathbb{R}$-strongly compact, then it is $<\Theta$-strongly compact (λ-strongly compact for every uncountable cardinal $\lambda < \Theta$.)

- $\Theta = \omega_2$ in the symmetric model obtained from a measurable cardinal by the Levy collapse, in which case $<\Theta$-strongly compact just means ω_1-strongly compact.
- Θ is a limit cardinal under AD by Moschovakis's coding lemma, in which case $<\Theta$-strongly compact implies ω_2-strongly compact, *etc.*

To get more strong compactness of ω_1, we need stronger determinacy axioms.

Definition

$\mathsf{AD}_{\mathbb{R}}$ is the *Axiom of Real Determinacy*, which strengthens AD by allowing moves to be reals instead of integers.

Remark

- $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ has higher consistency strength than $\mathsf{ZF} + \mathsf{AD}$.
- It cannot hold in $L(\mathbb{R})$.

Remark

The consistency strength of $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ is increased by adding DC (Solovay) unlike that of $\mathsf{ZF} + \mathsf{AD}$ (Kechris).

Theorem (Solovay)

$\mathrm{Con}(\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}) \implies \mathrm{Con}(\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}} + \mathrm{cf}(\Theta) = \omega)$.
Moreover $\mathrm{cf}(\Theta) = \omega$ in any minimal model of $\mathsf{AD}_{\mathbb{R}}$.

Theorem (Solovay)

$\mathrm{Con}(\mathsf{ZF}+\mathsf{DC}+\mathsf{AD}_{\mathbb{R}}) \implies \mathrm{Con}(\mathsf{ZF}+\mathsf{DC}+\mathsf{AD}_{\mathbb{R}}+\mathrm{cf}(\Theta) = \omega_1)$.
In fact $\mathrm{cf}(\Theta) = \omega_1$ in any minimal model of $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$.

Corollary

$\mathrm{Con}(\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}_{\mathbb{R}}) \implies$
$\mathrm{Con}(\mathrm{ZF} + \mathrm{DC} +$ "ω_1 is $\mathcal{P}(\mathbb{R})$-strongly compact"$)$.

Proof.

- Assume WLOG that $\mathrm{cf}(\Theta) = \omega_1$.
- Write $\mathcal{P}(\mathbb{R}) = \bigcup_{\alpha<\omega_1} \Gamma_\alpha$ (Wadge initial segments).
- Combine measures on $\mathcal{P}_{\omega_1}(\Gamma_\alpha)$ with measure on ω_1.[2] □

Corollary

$\mathrm{Con}(\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}_{\mathbb{R}}) \implies$
$\mathrm{Con}(\mathrm{ZF} + \mathrm{DC} +$ "ω_1 is Θ-strongly compact"$)$.

[2] To avoid choice, use unique normal measures. (Woodin)

Some natural questions so far:

Questions

What is the consistency strength of the theory
ZF + DC + "ω_1 is ω_2-strongly compact"?

- It follows from ZF + DC + AD.
- Is it equiconsistent with it?

What is the consistency strength of the theory
ZF + DC + "ω_1 is Θ-strongly compact"?

- It follows from ZF + DC + $\mathrm{AD}_{\mathbb{R}}$.
- Is it equiconsistent with it?

Like strong compactness in ZFC, strong compactness of ω_1 implies useful combinatorial principles.

Definition

Let λ be an ordinal. A coherent sequence of length λ is a sequence $(C_\alpha : \alpha \in \lim(\lambda))$ such that for all $\alpha \in \lim(\lambda)$,

- C_α is club in α, and
- $C_\alpha \cap \gamma = C_\gamma$ for all $\gamma \in \lim(C_\alpha)$.

Definition

A limit ordinal λ of uncountable cofinality is threadable[3] if every coherent sequence of length λ can be extended (by adding a thread C_λ) to a coherent sequence of length $\lambda + 1$.

[3] Also denoted by $\neg\square(\lambda)$

Remark

λ is threadable if and only if $\mathrm{cf}(\lambda)$ is threadable.

Remark

Every measurable cardinal is threadable, even ω_1, which cannot be threadable in ZFC:

- Given a measure μ and a coherent sequence $\vec{C}$, use the elementarity of $j_\mu \restriction L[\vec{C}]$.

More generally, threadability of larger cardinals can be obtained from more strong compactness of ω_1.

Proposition

Assume ZF + DC + "ω_1 is λ-strongly compact" where λ is an ordinal of uncountable cofinality. Then λ is threadable.

Proof.

- Let μ be a countably complete fine measure on $\mathcal{P}_{\omega_1}(\lambda)$.
- let $\vec{C}$ be a coherent sequence of length λ.
- $j = j_\mu$ is discontinuous at λ.
- $j \restriction L[\vec{C}]$ (using all functions in V) is elementary.
- As usual, define the club

$$C_\lambda = \bigcup \{ C_\alpha : j(\alpha) \in \lim(j(\vec{C})_{\sup j[\lambda]})\}.$$

□

A further consequence:

Proposition

Assume ZF + "ω_2 is threadable or singular." Then $\neg\Box_{\omega_1}$.

Proof.

If not, we have a $\Box_{\omega_1}$ sequence $(C_\alpha : \alpha \in \lim(\omega_2))$.

- If we have a thread C_{ω_2} then its order type is at most $\omega_1 + \omega$ by the usual argument, so ω_2 is singular.
- If ω_2 is singular, take a club C_{ω_2} in ω_2 of order type $\leq \omega_1$.
  Recursively define surjections $f_\alpha : \omega_1 \to \alpha$ for $\alpha \in [\omega_1, \omega_2]$, using C_α at limit stages. Contradiction.

(Coherence was not needed in the singular case.) □

Some natural questions:

Questions

What is the consistency strength of the theory
ZF + DC + "ω_1 is threadable and $\neg\Box_{\omega_1}$"?

- It follows from ZF + DC + AD.
- Is it equiconsistent with it?

What is the consistency strength of the theory
ZF + DC + "every uncountable regular cardinal $\leq \Theta$ is threadable"?

- It follows from ZF + DC + $\mathrm{AD}_{\mathbb{R}}$.
- Is it equiconsistent with it?

Theorem (Trang–W.)

The following theories are equiconsistent modulo $\mathsf{ZF} + \mathsf{DC}$.

1. AD.
2. ω_1 is $\mathcal{P}(\omega_1)$-strongly compact.
3. ω_1 is $\mathbb{R}$-strongly compact and ω_2-strongly compact.
4. ω_1 is $\mathbb{R}$-strongly compact and $\neg\square_{\omega_1}$.

Proof of (1) $\implies$ (2).

Assume AD. Then ω_1 is $\mathbb{R}$-strongly compact by Martin's cone theorem, and there is a surjection from $\mathbb{R}$ onto $\mathcal{P}(\omega_1)$ by the coding lemma. So ω_1 is $\mathcal{P}(\omega_1)$-strongly compact. □

Proof of (2) $\implies$ (3).

Assume that ω_1 is $\mathcal{P}(\omega_1)$-strongly compact.
There are surjections from $\mathcal{P}(\omega_1)$ onto $\mathbb{R}$ and ω_2,
so ω_1 is $\mathbb{R}$-strongly compact and ω_2-strongly compact. □

Proof of (3) $\implies$ (4).

If ω_1 is ω_2-strongly compact then we saw that $\square_{\omega_1}$ fails. □

Remark

In the rest of this subsection we prove $\text{Con}(4) \implies \text{Con}(1)$.
Assume that ω_1 is $\mathbb{R}$-strongly compact and $\neg\square_{\omega_1}$.
We will prove $L(\mathbb{R}) \models \text{AD}$ by a core model induction.

Definition

A mouse operator assigns to each set a in its domain the least mouse over a that is sound, projects to a, and satisfies a given first-order property.

In our situation:

- The domain will always be a cone in HC.
- "Mouse" means an ω_1-iterable premouse, which implies $(\omega_1 + 1)$-iterability because ω_1 is measurable.

Example (the $\mathcal{M}_n^\sharp$ operator)

$\mathcal{M}_n^\sharp(a)$ is the least mouse over a that is sound, projects to a, is active, and has n Woodin cardinals.

We will show PD by showing that the $\mathcal{M}_n^\sharp$ operator is total on HC for all $n < \omega$ (by induction on n.)

What's next?

- *Not* $\mathcal{M}_\omega^\sharp$: this corresponds roughly to $AD^{L(\mathbb{R})}$, which is too big a leap.
- Consider Woodinness with respect to more complicated operators, rather than greater numbers of Woodins.
- For example, the least Woodin cardinal of $\mathcal{M}_{n+1}^\sharp(a)$ is Woodin with respect to the $\mathcal{M}_n^\sharp$ operator.
- Complexity of operators is measured in terms of the Jensen hierarchy of $L(\mathbb{R})$.

Let $\mathcal{F}$ be a *CMI operator* (a mouse operator for now, but later a strategy operator or a strategy-hybrid-mouse operator.)

Definition (the $\mathcal{M}_1^{\sharp,\mathcal{F}}$ operator)

$\mathcal{M}_1^{\sharp,\mathcal{F}}(a)$ is the least mouse over a that is sound, projects to a, is active, has one Woodin cardinal, and is closed under $\mathcal{F}$.

For AD in $J_{\alpha+1}(\mathbb{R})$, start with appropriate CMI operator $\mathcal{F}$ and obtain operators $\mathcal{F}' = \mathcal{M}_1^{\sharp,\mathcal{F}}$, $\mathcal{F}'' = \mathcal{M}_1^{\sharp,\mathcal{F}'}$, ...

Example

- For AD in $J_2(\mathbb{R})$ (i.e. PD), start with $\mathcal{F} = \text{rud}$.[4]
- For AD in $J_3(\mathbb{R})$, start with $\mathcal{F}$ s.t. $\mathcal{F}(a) = \bigcup_{i<\omega} \mathcal{M}_i^\sharp(a)$.

[4] Note that $\mathcal{M}_1^{\sharp,\mathcal{M}_n^\sharp}(a) \rhd \mathcal{M}_{n+1}^\sharp(a)$.

Lemma

Assume that ω_1 is $\mathbb{R}$-strongly compact and $\neg\Box_{\omega_1}$. Let $\mathcal{F}$ be a CMI operator defined on the cone in HC over some $a \in \mathrm{HC}$. Then the $\mathcal{M}_1^{\sharp,\mathcal{F}}$ operator is defined on the same cone in HC.

Proof sketch

- Define operators $\mathcal{F}^\sharp = \mathcal{M}_0^{\sharp,\mathcal{F}}$ and $\mathcal{F}^{\sharp\sharp} = \mathcal{M}_0^{\sharp,\mathcal{F}^\sharp}$.
- $\mathcal{F}^\sharp$ and $\mathcal{F}^{\sharp\sharp}$ are total on the cone over a because ω_1 is measurable (or use ω_1 is threadable and $\neg\Box_{\omega_1}$.)
- Let $x \in \mathrm{HC}$ be in cone over a. We show $\mathcal{M}_1^{\sharp,\mathcal{F}}(x)$ exists.
- For simplicity, first assume that $\mathcal{F}^{\sharp\sharp}(\mathbb{R})$ exists. (E.g., take ultraproduct of $\mathcal{F}^{\sharp\sharp}(\sigma)$ if ω_1 is $\mathbb{R}$-supercompact.)

Proof sketch (continued)

- Let $H = \mathrm{HOD}_{\{\mathcal{F},x\}}^{\mathcal{F}^{\sharp\sharp}(\mathbb{R})}$ and let Ξ be the critical point of the top extender of $\mathcal{F}^{\sharp\sharp}(\mathbb{R})$.[5]
- Do the $K^{c,\mathcal{F}}(x)$ construction in H up to Ξ. (Like K^c construction but relativized to $\mathcal{F}$ and over x.)
- If it reaches $\mathcal{M}_1^{\sharp,\mathcal{F}}(x)$, we are done.
- Otherwise the core model $K = (K^{\mathcal{F}}(x))^H$ exists and has no Woodin cardinals.
- Why work in H? We need a ZFC model and H is big enough: every real is $<\Xi$-generic over H by Vopěnka.[6]

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[5] If ω_1 is $\mathbb{R}$-strongly compact, let H be ultraproduct of $\mathrm{HOD}_{\{\mathcal{F},x\}}^{\mathcal{F}^{\sharp\sharp}(\sigma)}$.
[6] *cf.* Schindler, *Successive weakly compact or singular cardinals.*

Proof sketch (continued)

- Define $\kappa = \omega_1^V$ and $j = j_\mu$ for μ a measure on κ.
- $\square_\kappa$ holds in $j(K)$ (Schimmerling–Zeman) but not V so

$$(\kappa^+)^{j(K)} < \kappa^+. \qquad (*)$$

- Take $A \subset \kappa$ coding wellordering of $(\kappa^+)^{j(K)}$ of length κ.
- A is in a $<j(\Xi)$-generic extension $j(H)[g]$ of $j(H)$.
- $j(H)[g]$ sees the failure of covering $(*)$ for its core model.
- The $(\kappa, j(\kappa))$-extender from $j \restriction j(K)$ is in $j(j(H))[j(g)]$ by Kunen's argument.
- Its initial segments are on the sequence of $j(j(K))$ and witness that κ is Shelah in $j(j(K))$. Contradiction. □

Now say we have AD in $J_\alpha(\mathbb{R})$ and we want AD in $J_{\alpha+1}(\mathbb{R})$. We need to start with the appropriate CMI operator $\mathcal{F}$. This is standard.

Key points

- If α is successor or has countable cofinality, $\mathcal{F}$ is a mouse operator given by unions of mice already constructed.
- If α has uncountable cofinality and $J_\alpha(\mathbb{R})$ is inadmissible, this is witnessed by a $\Delta_1(z)$ function for some z. Then $\mathcal{F}$ is a diagonal mouse operator defined on cone over z.
- If α is admissible, then $\mathcal{F}$ is a strategy operator that feeds in branches for iteration trees on a suitable premouse $\mathcal{P}$. It is defined on the cone over $\mathcal{P}$.

Theorem (Trang–W.)

The following theories are equiconsistent modulo $\mathsf{ZF} + \mathsf{DC}$.

1. $\mathsf{AD}_{\mathbb{R}}$ (plus DC).
2. ω_1 is $\mathcal{P}(\mathbb{R})$-strongly compact.
3. ω_1 is $\mathbb{R}$-strongly compact and Θ-strongly compact.

Proof of Con (1) $\implies$ Con (2)

Recall (2) holds in any minimal model of $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$.

Proof of (2) $\implies$ (3)

Use surjections from $\mathcal{P}(\mathbb{R})$ onto $\mathbb{R}$ and Θ.

In the rest of this subsection we prove $\mathrm{Con}\,(3) \implies \mathrm{Con}\,(1)$.

Strong hypothesis:

$\mathrm{ZF} + \mathrm{DC} +$ "*ω_1 is $\mathbb{R}$- and Θ-strongly compact.*"

Goal

Find a pointclass Ω such that $L(\Omega, \mathbb{R}) \models \mathrm{AD}_{\mathbb{R}} + \mathrm{DC}$.

Smallness assumption:

There is no model M of $\mathrm{ZF} + \mathrm{AD}$ containing all reals and ordinals and with a pointclass $\Gamma \subsetneq \mathcal{P}(\mathbb{R})^M$ such that $L(\Gamma, \mathbb{R}) \models \mathrm{AD}_{\mathbb{R}} + \mathrm{DC}$.

(If this fails, we are done.)

Definition

The *maximal AD^+ pointclass* is

$$\Omega = \{A \subset \mathbb{R} : L(A, \mathbb{R}) \models \mathrm{AD}^+\}.$$

From weaker assumptions we proved $\mathrm{AD}^{L(\mathbb{R})}$, so $\Omega \neq \emptyset$.
From current assumptions we will prove:

- $L(\Omega, \mathbb{R}) \cap \mathcal{P}(\mathbb{R}) = \Omega$, which implies
  - $L(\Omega, \mathbb{R}) \models \mathrm{AD}^+$
  - $L(\Omega, \mathbb{R})$ is the *maximal model of $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$*.
- $L(\Omega, \mathbb{R}) \models \mathrm{AD}_{\mathbb{R}} + \mathrm{DC}$.

No divergent models of AD^+, by smallness assumption (Woodin).[7]

Definition

Θ^Ω is the Wadge rank of Ω.

Definition

The *Solovay sequence of* Ω:

- $\theta^\Omega_{-1} = 0$.
- $\theta^\Omega_{\alpha+1}$ is the least ordinal not the surjective image of $\mathbb{R}$ by any $\mathrm{OD}_A^{L(B,\mathbb{R})}$ function where $A, B \in \Omega$ and $|A|^\Omega_W = \theta^\Omega_\alpha$.
- $\theta^\Omega_\lambda = \sup_{\alpha<\lambda} \theta^\Omega_\alpha$ if λ is limit.

[7] Sargsyan proved this under a weaker smallness assumption.

Remark

$L(\Omega, \mathbb{R}) \cap \mathcal{P}(\mathbb{R}) = \Omega$ will imply $\theta^{\Omega}_{\alpha} = \theta^{L(\Omega,\mathbb{R})}_{\alpha}$ (the usual Solovay sequence.) Meanwhile we use this local definition.

Definition

The *length of the Solovay sequence of* Ω is the least α such that $\theta^{\Omega}_{\alpha} = \Theta^{\Omega}$.

Remark

By our smallness assumption, the length is $\leq \omega_1$.
We want to show the length is ω_1, because $L(\Omega, \mathbb{R})$ satisfies:

- $\mathrm{AD}_{\mathbb{R}}$ iff length is limit.
- DC iff length is not countable cofinality limit.

Constructible closure of Ω

If the Solovay sequence of Ω has successor length, say $\alpha + 1$:

- Take $A \subset \mathbb{R}$ of Wadge rank θ_α^Ω in Ω.
- We may assume A codes Σ where $(\mathcal{P}, \Sigma)$ is a hod pair.[8,9]
- Then every set $B \in \Omega$ is in a "self-iterable" Σ-mouse over $\mathbb{R}$ satisfying AD^+. (Sargsyan and Steel)
- The union of such Σ-mice is constructibly closed by a core model induction,[10] so $L(\Omega, \mathbb{R}) \cap \mathcal{P}(\mathbb{R}) = \Omega$.

[8] Or $(\mathcal{P}, \Sigma) = (\emptyset, \emptyset)$, the base case.

[9] All iteration strategies for hod pairs are taken to have branch condensation and be Ω-fullness preserving in this talk.

[10] This uses scales in Σ-mice over $\mathbb{R}$ (Schlutzenberg and Trang). We use Θ-strong compactness to extend Σ to an $(\Theta + 1)$-iteration strategy.

Constructible closure of Ω (continued)

If the Solovay sequence of Ω has limit length $\leq \omega_1$:

- Let $\mathcal{H}$ be direct limit of all hod pairs $(\mathcal{P}, \Sigma)$ with $\Sigma \in \Omega$.
- $\mathcal{H}$ is a hod premouse of height Θ^Ω, and its Woodin cardinals have the form $\theta^\Omega_{\alpha+1}$.
- $\mathcal{H}$ is full in $L[\mathcal{H}]$; otherwise we can take a countable hull to get an anomalous hod pair $(\mathcal{Q}, \Lambda)$ with $\Lambda \notin \Omega$. But $L(\Lambda, \mathbb{R}) \models \mathsf{AD}^+$ by a CMI so $\Lambda \in \Omega$, a contradiction.
- We can add Ω back to $L[\mathcal{H}]$ by a Vopěnka-like forcing (Woodin) to get $L(\Omega, \mathbb{R})$, showing again that $L(\Omega, \mathbb{R}) \cap \mathcal{P}(\mathbb{R}) = \Omega$. Also $\mathcal{H} = (V_\Theta^{\mathsf{HOD}})^{L(\Omega,\mathbb{R})}$.

So we have $L(\Omega, \mathbb{R}) \models AD^+$.

DC in $L(\Omega, \mathbb{R})$

If not, then $\Theta^{L(\Omega,\mathbb{R})}$ has countable cofinality.

- By DC in V, take a countable hull X of $L(\Omega, \mathbb{R})$ that is cofinal in Θ.
- The corresponding hull $\mathcal{Q}$ of the hod premouse $\mathcal{H}$ has a natural iteration strategy Λ.
- $\Lambda \notin \Omega$ because X is cofinal in $L(\Omega, \mathbb{R})$.
- But $L(\Lambda, \mathbb{R}) \models AD^+$ by a CMI so $\Lambda \in \Omega$, a contradiction.

$AD_{\mathbb{R}}$ in $L(\Omega, \mathbb{R})$

If not, then $L(\Omega, \mathbb{R}) \models \Theta = \theta_\Sigma$ for some hod pair $(\mathcal{P}, \Sigma)$.[11]

- Define $\Gamma = \Sigma^2_1(\text{Code}(\Sigma))^{L(\Omega,\mathbb{R})}$.
- $\boldsymbol{\Gamma}$ is the pointclass of all Suslin sets in $L(\Omega, \mathbb{R})$.
- We will get an Ω-scale on a complete $\check{\Gamma}$ set, contradiction.
- The norms of the scale will be $\text{Env}(\Gamma)$-norms where $\text{Env}(\Gamma)$ is the *envelope* of Γ.
- It turns out $\text{Env}(\Gamma) = \text{OD}^{L(\Omega,\mathbb{R})}_{\{\Sigma\}} \cap \mathcal{P}(\mathbb{R})$.
- In the meantime we must define $\text{Env}(\Gamma)$ more locally.

[11] Or $(\mathcal{P}, \Sigma) = (\emptyset, \emptyset)$, the base case.

$\mathsf{AD}_{\mathbb{R}}$ in $L(\Omega, \mathbb{R})$ (continued)

- Define $\Delta = \Delta^2_1(\mathrm{Code}(\Sigma))^{L(\Omega,\mathbb{R})}$.
- Define $\mathrm{Env}(\Gamma)$ to consist of sets that are countably approximated by Δ-in-an-ordinal sets.
- $\mathrm{Env}(\Gamma)$ has a definable wellordering of length $\leq \Theta$, so ω_1 is $\mathrm{Env}(\Gamma)$-strongly compact. (W.)
- This implies, using $\mathrm{Scale}(\Gamma)$, that every $\check{\Gamma}$ set has a scale whose norms are $\mathrm{Env}(\Gamma)$-norms. (W.)
- Get a self-justifying system $\mathcal{A} \subset \mathrm{Env}(\boldsymbol{\Gamma})$ containing a complete $\check{\Gamma}$ set, and show $L(\mathcal{A}, \mathbb{R}) \models \mathsf{AD}^+$ by a CMI. Contradiction. $\square$

Question

Are the following theories equiconsistent?

1. ZF + DC + $AD_{\mathbb{R}}$.
2. ZF + DC + "ω_1 is $\mathbb{R}$-strongly compact and Θ is threadable or singular.

Remark

- Θ is threadable *and* singular in a minimal model of $AD_{\mathbb{R}}$ + DC.
- In the case "Θ is singular" the answer is yes.
- In the case "Θ is threadable" we would need to prove that $\mathrm{Env}(\Gamma)$ is constructibly closed.